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REDUCTION THEORY AND K_3 OF THE GAUSSIAN INTEGERS

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Introduction. This paper represents an attempt to obtain new information about the abelian group $K_3(\mathbb{Z}[i])$ as defined by D. Quillen [10]. The rank, which is one, has been computed by A. Borel [2], and the fact that the torsion part of $K_3(\mathbb{Z}[i])$ has a direct summand which is $\mathbb{Z}_{24}$ may be derived from G. Segal and B. Harris [12]. The main result of the present work is that this result is sharp for odd torsion. That is, the odd torsion subgroup of $K_3(\mathbb{Z}[i])$ is $\mathbb{Z}_3$.

The K-theory result follows from computations of certain homology groups of $\mathrm{SL}(n, \mathbb{Z}[i])$ and $\mathrm{GL}(n, \mathbb{Z}[i])$ for $n = 2$ and 3. What is needed for K-theory is stated in Chapter I; the complete results are given in Chapter IV. The method is modeled on that of R. Lee and R. H. Szczarba [9]. Chapter II is a précis of the reduction theory of quadratic forms, the key to analysing the group homology. A detailed exposition of the theory may be found in P. Humbert [5] and M. Koecher [6]. The basic construction is due to Voronoi [15]. Chapter III is the construction applied to the case of positive definite hermitian forms over the Gaussian integers.

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Chapter I. In this chapter $\mathcal{O}$ will denote the ring of Gaussian integers and Λ will denote the ring $\mathbb{Z}[\frac{1}{2}]$, the integers localized away from 2. $\mathrm{St}(n)$ will denote the Steinberg module of the vector space $\mathbb{Q}(i)^n$. We assume these parts of Theorems IV.1.3. and IV.1.4.

$$H_p(\mathrm{SL}(2,\mathcal{O}); \mathrm{St}(2) \otimes \Lambda) = H_p(\mathrm{GL}(2,\mathcal{O}); \mathrm{St}(2) \otimes \Lambda) = \begin{cases} 0 & p = 0, 1, \\ \Lambda \oplus \mathbb{Z}_3 & p = 2 \end{cases} \quad (1)$$

$$H_p(\mathrm{SL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \Lambda) = H_p(\mathrm{GL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \Lambda) = 0, \quad p = 0, 1. \quad (2)$$

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These data are applied to K-theory as follows. Let BQ (BQ_n) denote the classifying space of the Q-category of projective $\mathcal{O}$-modules (of rank less than or equal to n) as in Quillen [10], [11]. By definition $K_p(\mathcal{O}) = \pi_{p+1}(BQ)$. There is also an exact sequence [11]

$$\to H_q(BQ_{n-1};\Lambda) \to H_q(BQ_n;\Lambda) \to H_{q-n}(\mathrm{GL}(n,\mathcal{O});\mathrm{St}(n)\otimes\Lambda)$$
$$\to H_{q-1}(BQ_{n-1};\Lambda) \to$$

We are interested in $H_4(BQ;\Lambda)$. The sequence above implies

$$H_4(BQ_5;\Lambda) \overset{\cong}{\to} H_4(BQ_6;\Lambda) \overset{\cong}{\to} \cdots \overset{\cong}{\to} H_4(BQ;\Lambda).$$

Applying the result of [7] that $H_0(\mathrm{GL}(n,\mathcal{O});\mathrm{St}(n)) = 0$ ($n \geqslant 3$), one derives an isomorphism

$$H_4(BQ_4;\Lambda) \overset{\cong}{\to} H_4(BQ_5;\Lambda)$$

and a surjection

$$H_4(BQ_3;\Lambda) \to H_4(BQ_4;\Lambda) \to 0.$$

This is the general theory behind

THEOREM I.1.1.

$$H_4(BQ;\Lambda) = \Lambda \oplus \mathbf{Z}_3;$$

$$K_3(\mathbf{Z}[i]) \otimes \Lambda = \Lambda \oplus \mathbf{Z}_3.$$

The proof is the concatenation of the following sequence of lemmas.

LEMMA I.1.2.

$$H_p(BQ_1;\Lambda) = \begin{cases} \Lambda, & p = 0,1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The Q-category of projective modules of rank less than or equal to one has two objects, 0 and $\mathcal{O}$, and morphism sets

$$\mathrm{Hom}_{Q_1}(0,0) = \{\mathrm{Id}\}$$

$$\mathrm{Hom}_{Q_1}(0,\mathcal{O}) = \{a,b\},$$

and

$$\mathrm{Hom}_{Q_1}(\mathcal{O},\mathcal{O}) \cong \mathcal{O}^* = \mathbf{Z}_4,$$

the units in $\mathcal{O}$. There are also rules of composition $ua = a$, $ub = b$ for $u \in \mathrm{Hom}_{Q_1}(\mathcal{O}, \mathcal{O})$.

It follows that BQ_1 may be identified with the unreduced suspension $\Sigma B\mathcal{O}^*$ of $B\mathcal{O}^*$ with the additional identification of the "north pole" to the "south pole." Hence, $BQ_1 \simeq S^1 \vee \Sigma B\mathcal{O}^*$ and the homology is readily calculated.

LEMMA I.1.3.

$$H_p(BQ_2; \Lambda) = \begin{cases} \Lambda & p = 0, 1 \\ 0 & p = 2, 3 \\ \Lambda \oplus \mathbf{Z}_3 & p = 4. \end{cases}$$

Proof. Apply I.1.2., (1), and the Quillen exact sequence.

LEMMA I.1.4. ([8], p. 35.) *BQ is homotopy equivalent to $S^1 \times \widetilde{BQ}$, S^1 the circle and $\widetilde{BQ}$ the universal cover of BQ.*

This lemma permits us to argue modulo the Serre class $\mathcal{C}$ of abelian 2-groups. It is well known that $\pi_2(\widetilde{BQ}) = K_1(\mathcal{O}) = 0 \bmod \mathcal{C}$, and by [1], p. 436, that $\pi_3(\widetilde{BQ}) = K_2(\mathcal{O}) = 0 \bmod \mathcal{C}$. Therefore $\widetilde{BQ}$ is 3-connected mod $\mathcal{C}$, and by [2] and [12] (cf. Introduction) $\Lambda \oplus \mathbf{Z}_3 \rightarrowtail K_3(\mathcal{O}) \otimes \Lambda \cong H_4(BQ; \Lambda)$. On the other hand, further use of the Quillen sequence shows there is a surjection

$$\Lambda \oplus \mathbf{Z}_3 = H_4(BQ_2; \Lambda) \twoheadrightarrow H_4(BQ; \Lambda).$$

The theorem follows.

Chapter II. Here we describe briefly the algorithm of Voronoi's reduction theory, since we will have to describe in detail results of this procedure in Chapters III and IV. For a complete discussion of a generalization of Voronoi's original theory see M. Koecher [6]. Other references are P. Humbert [5] and Voronoi's original paper [15].

Let $H(n)$ denote the vector space of hermitian forms on $\mathbf{C}^n$, identified with the space of hermitian matrices in the obvious way. Let $\mathrm{PH}(n)$ denote the cone of positive definite hermitian forms (matrices). Note that such a form is determined by its associated quadratic form. For any form φ let $M_L(\varphi)$ denote the minimum value of φ on the complement of $\{0\}$ in some lattice L in $\mathbf{C}^n$. One usually takes $L = \mathcal{O}^n$, $\mathcal{O}$ a ring of integers in a quadratic imaginary number field. Let $\tilde{m}_L(\varphi)$ denote the set of vectors in L at which this value is attained. $\tilde{m}_L(\varphi)$ is called the set of minimal vectors of φ in L. If there is no possibility of confusion, the L will be omitted from the notation.

Definition. Consider a form φ on $\mathbf{C}^n$ given by

$$\varphi(v) = {}^t\bar{v}Av,$$

where A is a positive definite, hermitian matrix. Let

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} + iy_{1n} \\ \vdots & & \vdots \\ x_{1n} - iy_{1n} & \cdots & x_{nn} \end{bmatrix}$$

denote an hermitian matrix of unknowns and solve the system of equations

$${}^t\bar{l}Xl = M_L(\varphi), \qquad l \in \tilde{m}_L(\varphi).$$

If $X = A$ is the unique solution then, with respect to L, φ is a *perfect* form, one determined by its minimal vectors.

If the solution set is infinite, let B be a nonzero solution to the set of homogeneous equations

$${}^t\bar{l}Xl = 0, \qquad l \in \tilde{m}_L(\varphi)$$

and put

$$\psi(v) = {}^t\bar{v}Bv.$$

For an appropriate $t \in \mathrm{R}$ the form $\varphi_t = \varphi + t\psi$ will have a larger set of minimal vectors than φ. Since the cardinality of $\tilde{m}_L(\varphi)$ is finite, one can prove

PROPOSITION II.1.1. *Let φ be any positive definite hermitian form. Then there exists a form φ' such that*
(1) $M_L(\varphi') = M_L(\varphi)$
(2) $\tilde{m}_L(\varphi') \supset \tilde{m}_L(\varphi)$
(3) *φ' is perfect with respect to L.*

Denote by $P(L)$ the set of forms on C^n perfect with respect to L and having minimum value one on $L - \{0\}$. For each $\varphi \in P(L)$ define a convex cone in $\overline{\mathrm{PH}(n)}$, the closure of $\mathrm{PH}(n)$, as follows:

Definition. The *domain associated to the perfect form* φ is the set

$$D(\varphi) = \left\{ \sum_l v(l) l {}^t\bar{l} : l \in \tilde{m}_L(\varphi);\ v(l) \geqslant 0 \right\}.$$

Note that we could also write

$$D(\varphi) = \left\{ \sum v(\lambda)\lambda : \lambda \in m_L(\varphi),\ v(\lambda) \geqslant 0 \right\}$$

if we define

$$m_L(\varphi) = \left\{ \lambda = l {}^t\bar{l} : l \in \tilde{m}_L(\varphi) \right\}.$$

We will take this second description more often since two vectors in $\tilde{m}_L(\varphi)$ differing by a unit in $\mathcal{O}^*$ determine the same *vertex* $\lambda \in m_L(\varphi)$.

Since $D(\varphi)$ is convex it may also be described as an intersection of half spaces of $H(n): D(\varphi) = \{X : (\psi, X) \geq 0\}$ for some family of ψ's. (The inner product $(\psi, X) = \operatorname{trace} \psi X$ where ψ and X are represented as matrices.) Let us say $\{\psi_i\}_{i \in I}$ defines $D(\varphi)$ if $D(\varphi) = \{X : (\psi_i, X) \geq 0;\ i \in I\}$ and if the hyperplane $H(\psi_i) = \{X : (\psi_i, X) = 0\}$ intersects $D(\varphi)$ in a codimension one face.

One can show that

(1) $\mathrm{PH}(n) \subset \bigcup_{\varphi \in P(L)} D(\varphi) =_{\mathrm{def.}} \mathrm{PH}(n)^* \subset \overline{\mathrm{PH}(n)}$.

(2) $D(\varphi) \cap D(\varphi')$ is a proper face of each if $\varphi \neq \varphi'$.

(3) A neighborhood of a point of $\mathrm{PH}(n)$ meets only finitely many $D(\varphi)$'s.

(4) The action of $\mathrm{GL}(n, \mathcal{O})$, $X \mapsto {}^t\bar{g} X g$, is cellular and carries $D(\varphi)$ to $D(\varphi')$ where $\varphi' = g^{-1} \varphi \, {}^t\bar{g}^{-1}$.

(5) A fundamental domain for this action is contained in the union of finitely many $D(\varphi)$'s.

Last of all, if φ is perfect and ψ is a defining vector of $D(\varphi)$ the form φ' whose domain is across the face of $D(\varphi)$ determined by ψ may be expressed as $\varphi' = \varphi + t_0 \psi$ for some $t_0 > 0$. It follows that one may study the automorphism group of φ as it acts on the faces of $D(\varphi)$, on the $\{\psi_i\}$ defining $D(\varphi)$, or on the forms of domains adjacent to $D(\varphi)$.

Chapter III. Now we determine the forms in two and three variables which are perfect with respect to $L = \mathbf{Z}[i]^n$, $n = 2$, or 3, and their groups of automorphisms. $\mathcal{O} = \mathbf{Z}[i]$ throughout the rest of the paper.

1. PROPOSITION III.1.1. *Let φ be given by the matrix*

$$A = \begin{pmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 1 \end{pmatrix}.$$

Then φ is perfect with respect to L and a set of representatives for $\tilde{m}_L(\varphi)$ modulo $\mathcal{O}^*$ is

$$\left\{ l_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, l_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, l_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, l_4 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}, l_5 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}, l_6 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Proof. First note that ${}^t\bar{v} A v = \frac{1}{2}(|x+y|^2 + |x+iy|^2)$ for $v = {}^t(x, y)$. With ${}^t\bar{v} A v$ written in this form it is easy to check that the minimum value of φ is 1 on $L - \{0\}$ and that all minimal vectors are represented above. Then one checks that l_1, l_2, l_3, and l_6 do in fact determine A.

For later use write $A = {}^t\overline{M} A' M$:

$$\begin{pmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}.$$

PROPOSITION III.1.2. *Let* $G(\varphi)$ *denote the automorphism group of* φ *in* $\mathrm{GL}(2,\mathcal{O})$ *and let* $\mathrm{SG}(\varphi) = G(\varphi) \cap \mathrm{SL}(2,\mathcal{O})$. *Then there are central extensions*

$$1 \to Z_4 \to G(\varphi) \to S_4 \to 1$$
$$1 \to Z_2 \to \mathrm{SG}(\varphi) \to A_4 \to 1$$

where S_4 *and* A_4 *are the symmetric and alternating groups on four letters.*

PROPOSITION III.1.3. *Any form in* $P(L)$ *can be written* φh^* *for some* $h \in \mathrm{SL}(2,\mathcal{O})$.

Proposition III.1.2. will be proved by studying the permutation representation of $G(\varphi)$ on the faces of the domain $D(\varphi)$. To prove III.1.3. we will need III.1.2. and the forms whose domains share codimension one faces with $D(\varphi)$. Therefore we will describe $D(\varphi)$ and a set of defining vectors.

Change coordinates by the matrix M exhibited above. Then the vertices of $D(\varphi)$ will have coordinates $\lambda_i' = M l_i {}^t\bar{l}_i {}^t\overline{M}$, $1 \leq i \leq 6$, and the new coordinates of defining vectors ψ' will be related to the old coordinates by the formula $\psi' = {}^t\overline{M}^{-1}\psi M^{-1}$. Let

$$\psi' = \begin{pmatrix} p_{11}' & p_{12}' + iq_{12}' \\ p_{12}' - iq_{12}' & p_{22}' \end{pmatrix}$$

be a defining vector. Computing $(\psi', \lambda_i') \geq 0$ for $1 \leq i \leq 6$ gives the following set of inequalities:

$$\begin{array}{llll} (1) & 2p_{11}' \geq 0 & (2) & 2p_{22}' \geq 0 \\ (3) & p_{11}' + 2p_{12}' + p_{22}' \geq 0 & (4) & p_{11}' - 2p_{12}' + p_{22}' \geq 0 \\ (5) & p_{11}' + 2q_{12}' + p_{22}' \geq 0 & (6) & p_{11}' - 2q_{12}' + p_{22}' \geq 0 \end{array}$$

Hypothesizing that $F(\psi') = \{X \in D(\varphi) \mid (\psi', X) = 0\}$ is a codimension one face of $D(\varphi)$ is the same as saying three of these inequalities are actually equalities, since such a face must contain at least three vertices.

Which triples of vertices can belong to a codimension one face of $D(\varphi)$? For example, suppose λ_1 and λ_2 were to belong to one face. Then (1) and (2) become equalities and force $\psi = 0$, a contradiction. Similarly, neither $\{\lambda_3, \lambda_4\}$ nor $\{\lambda_5, \lambda_6\}$ are subsets of spanning sets of vertices. It follows that $D(\varphi)$ has eight faces, each spanned by three vertices, as displayed in Table III.1.1.

Proof of III.1.2. Consider the action of $G(\varphi)$ on the set of unordered pairs $S = \{(\psi_3, \psi_6) = a, (\psi_4, \psi_5) = b, (\psi_2, \psi_7) = c, (\psi_1, \psi_8) = d\}$. Geometrically, $G(\varphi)$ is acting on the set of unordered pairs of opposite faces of $D(\varphi)$. Recall that $g \in \mathrm{GL}(2,\mathcal{O})$ takes φ to $g^{-1}\varphi {}^t\bar{g}^{-1} = \varphi g^*$, ψ_i to $\psi_i g^*$, and F_i to ${}^t\bar{g} F_i g = F_i g$. Let

$$g_1 = \begin{pmatrix} -1+i & 1 \\ i & 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 1 & -1-i \\ 1 & -1 \end{pmatrix}.$$

TABLE III.1.1.

Face F_i	Vertices	Normal vector ψ_i
F_1	$\{\lambda_1,\lambda_3,\lambda_5\}$	$\begin{pmatrix} 0 & 2i \\ -2i & 4 \end{pmatrix}$
F_2	$\{\lambda_1,\lambda_3,\lambda_6\}$	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$
F_3	$\{\lambda_1,\lambda_4,\lambda_5\}$	$\begin{pmatrix} 4 & 2+4i \\ 2-4i & 4 \end{pmatrix}$
F_4	$\{\lambda_1,\lambda_4,\lambda_6\}$	$\begin{pmatrix} 4 & 2i \\ -2i & 0 \end{pmatrix}$
F_5	$\{\lambda_2,\lambda_3,\lambda_5\}$	$\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$
F_6	$\{\lambda_2,\lambda_3,\lambda_6\}$	$\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$
F_7	$\{\lambda_2,\lambda_4,\lambda_5\}$	$\begin{pmatrix} 4 & 4+2i \\ 4-2i & 4 \end{pmatrix}$
F_8	$\{\lambda_2,\lambda_4,\lambda_6\}$	$\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$

Then both g_1 and g_2 are in $G(\varphi)$ and g_1 induces the four-cycle $(abcd)$ while g_2 induces the transposition (ab) in the permutation group of S. Surjectivity of the map $G(\varphi) \to S_4$ now follows from the well-known fact that an n-cycle and a transposition generate S_n.

Now we claim that the kernel of this representation consists of the four diagonal matrices $\{\pm I, \pm iI\}$. To begin to see this, observe that $G(\varphi)$ permutes the vertices of $D(\varphi)$ preserving the sums $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = \lambda_5 + \lambda_6$ which are proportional to the barycenter $\sum_{1 \leqslant i \leqslant 6} \lambda_i$ of $D(\varphi)$, a fixed point for $G(\varphi)$. Assuming some matrix $g \in G(\varphi)$ leaves fixed the elements of S, we observe also that g preserves partitions of the set of vertices in a way consistent with the geometry of $D(\varphi)$. One can use these observations to show that such a g fixing the set $\{F_1, F_8\}$ mnst fix F_1 and F_8 pointwise, and that therefore such a $g \in \{\pm I, \pm iI\}$. This completes the proof of the first part of the proposition.

For the second, introduce

$$g_3 = \begin{pmatrix} -1 & 1+i \\ -1+i & 1 \end{pmatrix} \quad \text{and} \quad g_4 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

These two matrices induce the permutations $(ac)(bd)$ and (bcd) respectively, which proves that $\mathrm{SG}(\varphi)$ maps onto A_4, the subgroup of S_4 generated by these two elements. Since $\det g_2 = i$ and $\det h = \pm 1$ for any scalar matrix in $G(\varphi)$, the transposition (ab) cannot be in the image of $\mathrm{SG}(\varphi)$. The proof is completed by the trivial observation that $\{\pm I, \pm iI\} \cap \mathrm{SL}(2, \mathcal{O}) = \{\pm I\}$.

Proof of III.1.3. Using g_3 and g_4 one sees immediately that there are two orbits of $\mathrm{SG}(\varphi)$ in the set of faces of $D(\varphi)$, one represented by F_2, the other by F_6. By this remark and the last one in Chapter II, if there are perfect forms inequivalent to φ, one will be either $\varphi_2 = \varphi + t_2\psi_2$ or $\varphi_6 = \varphi + t_6\psi_6$ where $t_i \in \mathbf{R}$ is chosen so that $D(\varphi_i)$ is adjacent to $D(\varphi)$ along the face F_i. Taking $t_i = \frac{1}{2}$ produces perfect forms φ_i having this property. But if

$$h_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad h_6 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix},$$

$\varphi h_2^* = \varphi_2$ and $\varphi h_6^* = \varphi_6$. From this, we draw the conclusion of the proposition.

2. Turning to the case of forms in three variables, we have

PROPOSITION III.2.1. *Let φ be given by the matrix*

$$A = \begin{bmatrix} 1 & 1/2 & (1+i)/2 \\ 1/2 & 1 & (1+i)/2 \\ (1-i)/2 & (1-i)/2 & 1 \end{bmatrix}$$

Then $M(\varphi) = 1$ and φ is perfect. A set of representatives for $\tilde{m}(\varphi)$ modulo $\mathcal{O}^*$ is

$$\left\{ l_1 = \begin{bmatrix} 1+i \\ 0 \\ -1 \end{bmatrix}, l_2 = \begin{bmatrix} 0 \\ 1+i \\ -1 \end{bmatrix}, l_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, l_4 = \begin{bmatrix} 1 \\ 1 \\ -1+i \end{bmatrix}, l_5 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \right.$$

$$l_6 = \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}, l_7 = \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}, l_8 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, l_9 = \begin{bmatrix} 0 \\ 1 \\ -1+i \end{bmatrix}, l_{10} = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix},$$

$$\left. l_{11} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, l_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, l_{13} = \begin{bmatrix} 1 \\ 0 \\ -1+i \end{bmatrix}, l_{14} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, l_{15} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Proof. Observe that, if ${}^t v = (x, y, z)$, then ${}^t\bar{v}Av = \frac{1}{2}(|x|^2 + |y|^2 + |x + y + (1+i)z|^2)$. Then it is easy to check that the minimum value of φ on $\mathcal{O}^3 - \{0\}$ is 1, and that the fifteen vectors above are essentially all the solutions to $\varphi(l) = 1$. Finally, the l_i for $i = 3, 5, 6, 8, 10, 11, 12, 14$, and 15 give a set of equations which do determine A.

As in section one, let us write $A = {}^t\overline{M}A'M$:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1-i \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1+i \end{bmatrix}.$$

At this point we suggest that the reader make tables of vertices of $D(\varphi)$ in both prime and unprime coordinates. Recall a vertex $\lambda_i = l_i{}^t\bar{l}_i$ and $\lambda_i' = M\lambda_i{}^t\overline{M}$. We will prove two more propositions in this section.

PROPOSITION III.2.2. *Let* $G(\varphi)$ *denote the automorphism group of* φ *in* $\mathrm{GL}(3,\mathcal{O})$ *and let* $\mathrm{SG}(\varphi) = G(\varphi) \cap \mathrm{SL}(3,\mathcal{O})$. *Then there are split extensions*

$$1 \to H \to G(\varphi) \to S_3 \to 1$$
$$1 \to \mathrm{SH} \to \mathrm{SG}(\varphi) \to S_3 \to 1.$$

S_3 *is the symmetric group on three letters,* $\mathrm{SH} = H \cap \mathrm{SL}(3,\mathcal{O})$ *and* $H \cong (\mathcal{O}^*)^3$ *in such a way that the action of* S_3 *on* H *is the permutation action.*

PROPOSITION III.2.3. *Any form in* $P(L)$ *can be written* φh^* *for some* h *in* $\mathrm{SL}(3,\mathcal{O})$.

To prove these statements we follow roughly the steps of section one. Again we change coordinates by the matrix M defined above. The vertices of $D(\varphi)$ will have coordinates $\lambda_i' = Ml_i{}^t\bar{l}_i{}^t\overline{M} = M\lambda_i{}^t\overline{M}$ and we will determine defining vectors ψ' for $D(\varphi)$. These are again related to defining vectors ψ in the old coordinates by ${}^t\overline{M}\psi' M = \psi$.

Let

$$\psi' = \begin{bmatrix} p'_{11} & p'_{12} + iq'_{12} & p'_{13} + iq'_{13} \\ p'_{12} - iq'_{12} & p'_{22} & p'_{23} + iq'_{23} \\ p'_{13} - iq'_{13} & p'_{23} - iq'_{23} & p'_{33} \end{bmatrix}.$$

Calculating (h) $(\psi',\lambda_h') \geqslant 0$ we obtain a table of inequalities:

A: (1) $2p'_{11} \geqslant 0$ (2) $2p'_{22} \geqslant 0$ (3) $2p'_{33} \geqslant 0$

B:	(4)	$p'_{11} + 2p'_{12} + p'_{22} \geqslant 0$	$-B$:	(6)	$p'_{11} - 2q'_{12} + p'_{22} \geqslant 0$
	(5)	$p'_{11} - 2p'_{12} + p'_{22} \geqslant 0$		(7)	$p'_{11} + 2q'_{12} + p'_{22} \geqslant 0$
C:	(8)	$p'_{22} + 2p'_{23} + p'_{33} \geqslant 0$	$-C$:	(10)	$p'_{22} - 2q'_{23} + p'_{33} \geqslant 0$
	(9)	$p'_{22} - 2p'_{23} + p'_{33} \geqslant 0$		(11)	$p'_{22} + 2q'_{23} + p'_{33} \geqslant 0$
D:	(12)	$p'_{11} + 2p'_{13} + p'_{33} \geqslant 0$	$-D$:	(14)	$p'_{11} - 2q'_{13} + p'_{33} \geqslant 0$
	(13)	$p'_{11} - 2p'_{13} + p'_{33} \geqslant 0$		(15)	$p'_{11} + 2q'_{13} + p'_{33} \geqslant 0$

Assuming that $F(\psi') = \{x \in D(\varphi) \mid (\psi', X) = 0\}$ is a codimension one face of $D(\varphi)$ implies that at least eight of the inequalities will be equalities. More can be said. Since each p'_{ij} and each q'_{ij} $(i \neq j)$ is to be determined up to scalar multiple by the vertices of the face, we must have at least one equality from the two inequalities involving p'_{ij} or q'_{ij}. Secondly, it must not be possible to deduce $p'_{11} = p'_{22} = p'_{33} = 0$ from the equalities, or we find $\psi' = 0$. Thirdly, it is easy to see that hypothesizing equality in group X ($X = \pm B, \pm C, \pm D$) implies there are two equalities in A and equalities in $-X$. (Hence four entries in ψ' are zero.) It is now possible to deduce: There are ten vertices in any codimension-one face, corresponding to the following choice of equalities. Choose two from A and the two pairs X and $-X$ consistent with the two from A. Then choose one equality from each of the four remaining pairs of inequalities. One may then solve the systems and obtain a set of forty-eight defining vectors

$$\left\{\begin{bmatrix} 2 & \pm(1\pm i) & +(1\pm i) \\ \pm(1\mp i) & 0 & 0 \\ \pm(1\mp i) & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pm(1\pm i) & 0 \\ \pm(1\mp i) & 2 & \pm(1\pm i) \\ 0 & \pm(1\mp i) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm(1\pm i) \\ 0 & 0 & \pm(1\pm i) \\ \pm(1\mp i) & \pm(1\mp i) & 2 \end{bmatrix}\right\}$$

(Each $\pm$ above the diagonal may be chosen independently of the others.)

Proof of III.2.2. The group of automorphisms of $D(\varphi)$ described in prime coordinates is the group $G' = \{g \in \mathrm{GL}(3,\mathbf{C}) : A'g^* = A'$ and ${}^t\overline{M}g{}^t\overline{M}^{-1} \in \mathrm{GL}(3,\mathcal{O})\}$ where A' and M are as at the beginning of the section. This group obviously contains the group G generated by diagonal matrices with entries in $\mathcal{O}^*$ and permutation matrices. If G' contains no other elements, Proposition III.2.2. follows immediately.

Note that if $g \in G'$ permutes the set $\{\lambda'_1, \lambda'_2, \lambda'_3\} = S$ there exists a permutation matrix $h \in G$ such that $\lambda'_i gh = \lambda'_i$ for $i = 1, 2$ and 3. Therefore gh is diagonal and $g \in G$. The proof will be concluded by showing that any $g \in G'$ must permute the set S.

So, let $g \in G'$ and suppose $Sg \neq S$. Since $\lambda'_1 + \lambda'_2 + \lambda'_3$ is proportional to the "barycenter" (= the sum of all the vertices) of $D(\varphi')$, it must be true that $\lambda'_1 g + \lambda'_2 g + \lambda'_3 g = \lambda'_1 + \lambda'_2 + \lambda'_3$. In view of the forms of the λ'_i, $4 \leqslant i \leqslant 15$ it is clear that $Sg \cap S \neq \emptyset$. Using known elements of G to change g if necessary, it may be assumed that $\lambda'_3 g = \lambda'_3$ and $\lambda'_1 g = \lambda'_4$. Then

$$\begin{aligned} \lambda'_4 + \lambda'_5 + \lambda'_3 &= \lambda'_1 + \lambda'_2 + \lambda'_3 \\ &= \lambda'_1 g + \lambda'_2 g + \lambda'_3 g \\ &= \lambda'_4 + \lambda'_2 g + \lambda'_3 \end{aligned}$$

implies $\lambda_2' g = \lambda_5'$. Further computation shows that

$$g\begin{bmatrix} x & x & 0 \\ -y & y & 0 \\ 0 & 0 & u \end{bmatrix}$$

where $u \in \mathcal{O}^*$, $2|x|^2 = 2|y|^2 = 1$. Then $\lambda_8' g$ is not a vertex of $D(\varphi')$. Therefore g must act as a permutation of S, and the proof is complete.

Proof of III.2.3. Note that the group G defined in the proof of III.2.2. is transitive on the set of forty-eight defining forms. It follows that we must examine the form whose domain is across one of the faces of $D(\varphi)$. For example, across the face determined by

$$\psi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} = {}^t\overline{M}\begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 2 & -1+i \\ 0 & -1-1 & 0 \end{bmatrix} M$$

lies the domain of the form $\varphi_1 = \varphi + \frac{1}{4}\psi$ given by the matrix

$$A_1 = \begin{bmatrix} 1 & 1/2 & (1+i)/2 \\ 1/2 & 1 & i/2 \\ (1-i)/2 & -i/2 & 1 \end{bmatrix}$$

(Using the diagonalization

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -i & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & i \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

it is easy to see that φ_1 is indeed perfect and has in common with φ the minimal vectors l_i for $i = 1, 3, 5, 6, 8, 10, 12, 13, 14$, and 15.)

Now let

$$g = \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & -i \\ -i & -i & -1 \end{bmatrix} \in \mathrm{SL}(3, \mathcal{O}).$$

Then $\varphi g^* = \varphi_1$, and, as in III.1.3., this suffices to complete the proof.

Chapter IV. The principal results of this chapter are Theorems IV.1.3 and IV.1.4 which give the homology groups $H_*(\mathrm{SL}(n, \mathcal{O}); \mathrm{St}(n) \otimes \Lambda)$ and $H_*(\mathrm{GL}(n, \mathcal{O}); \mathrm{St}(n) \otimes \Lambda)$ via Lemma IV.1.2. The method follows, in general, that of Lee-Szczarba [9] with addendum by Soulé [14]; however, we here require much more detail. In section one we recall results from [9] and [14] that we need and state the theorems. The tedious part of the proofs follows in sections two and three.

1. We work with the spaces

$$X_n = \mathrm{PH}(n)/\mathsf{R}_+^*,$$

$$X_n^* = \mathrm{PH}(n)^*/\mathsf{R}_+^*,$$

and

$$\partial X_n^* = X_n^* - X_n.$$

R_+^* denotes the positive real numbers and the action is the obvious family of dilatations. Give X_n^* the CW-topology defined by the cell structure on $\mathrm{PH}(n)^*$. As in the lemma of [14] we have

LEMMA IV.1.1. *For $n \geqslant 1$ the boundary ∂X_n^* of X_n^* has the homotopy type of the Tits building of $\mathsf{Q}(i)$ parabolic subgroups of $\mathrm{SL}(n, \mathsf{Q}(i))$ and X_n^* is contractible.*

We also restate Lemma 1.2. of [9]:

LEMMA IV.1.2.

$$H_q\Big((X_n^*, \partial X_n^*) \underset{G}{\times} EG; \Lambda\Big) \cong H_{q-n+1}(G; \mathrm{St}(n) \otimes \Lambda)$$

for $G = \mathrm{GL}(n, \Theta)$ or $G = \mathrm{SL}(n, \Theta)$. Here EG is the total space of the universal G-bundle.

From the cellular filtration on $(X_n^*, \partial X_n^*)$ we obtain a filtration on $(X_n^*, \partial X_n^*) \times_G EG$ and a spectral sequence $E^n_{*,*} \Rightarrow H_*((X_n^*, \partial X_n^*) \times_G EG; \Lambda)$. Let $(X_n^*)^p$ $(p \geqslant 0)$ denote the p-skeleton of X_n^* union the boundary ∂X_n^*. Let $(X_n^*)^{-1} = \partial X_n^*$. As usual

$$E^1_{p,q} = H_{p+q}\Big(\big[(X_n^*)^p, (X_n^*)^{p-1}\big] \underset{G}{\times} EG; \Lambda\Big)$$

$$\cong H_q\big(G; H_p((X_n^*)^p, (X_n^*)^{p-1}; \Lambda)\big),$$

by a spectral sequence argument;

$$\cong \bigoplus_{i \in I} H_q\Big(G; \Lambda G \underset{G_i}{\otimes} H_p(\sigma_i^p, \partial \sigma_i^p; \Lambda)\Big),$$

where the direct sum is over $i \in I$ indexing representative cells σ_i^p for each orbit of G in the set of p-cells and G_i denotes the automorphism group of σ_i^p in G;

$$\cong \bigoplus_{i \in I} H_q(G_i; H_p(\sigma_i^p, \partial \sigma_i^p; \Lambda))$$

by Shapiro's Lemma.

To interpret the components of the first differential d_1 let σ_0 be a p-cell and let $\sigma_1, \ldots, \sigma_r$ be the faces of σ_0 representing orbits of pairs (σ_0, face of σ_0) under the action of G_0 the group of σ_0. Let G_{0i} denote the group preserving the pair (σ_0, σ_i) and let G_i denote the group of σ_i. When we have the data to see the signs are correct it will be seen that the components of d_1 are the compositions

$$\begin{aligned} H_q(G_0; H_p(\sigma_0, \partial\sigma_0)) &\xrightarrow{\text{transfer}} H_q(G_{0i}; H_p(\sigma_0, \partial\sigma_0)) \\ &\xrightarrow{\cong} H_q\big(G_{0i}; H_{p-1}(\partial\sigma_0, \overline{\partial\sigma_0 - \sigma_i})\big) \\ &\xrightarrow{\cong} H_q\big(G_{0i}; H_{p-1}(\sigma_i, \partial\sigma_i)\big) \\ &\xrightarrow{\text{inclusion}} H_q\big(G_i; H_{p-1}(\sigma_i, \partial\sigma_i)\big). \end{aligned}$$

Using the spectral sequences and IV.1.2. we obtain the following theorems.

THEOREM IV.1.3.

$$H_n(\mathrm{SL}(2,\mathcal{O}); \mathrm{St}(2) \otimes \Lambda) \cong$$

$$H_n(\mathrm{GL}(2,\mathcal{O}); \mathrm{St}(2) \otimes \Lambda) = \begin{cases} 0, & n = 0,\ 1 \textit{ or } n \geqslant 3 \textit{ and } n \equiv 1, 0 \textit{ mod } 4. \\ \Lambda \oplus Z_3, & n = 2. \\ Z_3, & n \geqslant 5 \textit{ and } n \equiv 1, 2 \textit{ mod } 4. \end{cases}$$

THEOREM IV.1.4.

$$H_n(\mathrm{SL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \Lambda) \cong H_n(\mathrm{GL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \Lambda)$$

$$\begin{cases} 0, & n = 0,\ 1,\ 4, \textit{ or } n \geqslant 7, \equiv -1, 0 \textit{ mod } 4. \\ \Lambda \textit{ or } \Lambda \oplus Z_3, & n = 2. \\ \Lambda, & n = 3. \\ Z_3, & n = 5. \\ \Lambda \oplus Z_3^2, & n = 6. \\ Z_3^2, & n \geqslant 9 \textit{ and } n \equiv 1, 2 \textit{ mod } 4. \end{cases}$$

Theorem IV.1.3 is actually a separate calculation for each group. In both cases one arrives at an E^2-term whose nontrivial part extends upward.

6	0	0	0	0
5	0	0	Z_3	0
4	0	0	0	0
3	0	0	0	Z_3
2	0	0	0	0
1	0	0	Z_3	0
0	0	0	0	Λ
	0	1	2	3

Obviously the differentials vanish from this point on and there are no extension problems. In section two we outline how to get this E^2-term.

In Theorem IV.1.4 the isomorphism

$$H_n(\mathrm{SL}(3,\mathcal{O});\mathrm{St}(3)\otimes\Lambda)\xrightarrow{\cong}H_n(\mathrm{GL}(3,\mathcal{O});\mathrm{St}(3)\otimes\Lambda)$$

is immediate from the spectral sequence of the extension

$$1\to\mathrm{SL}(3,\mathcal{O})\to\mathrm{GL}(3,\mathcal{O})\to\mathbf{Z}_4\to 1$$

since $\mathbf{Z}_4$ acts trivially on $\mathrm{SL}(3,\mathcal{O})$ and on the coefficient module, and since the higher homology of $\mathbf{Z}_4$ with Λ-module coefficients vanishes. Now we show how to finish the calculation, given the E^2-term of the spectral sequence. In section three we will explain how to get this far.

9	0	0	0	$\mathbf{Z}_3$	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
7	0	0	0	0	$\mathbf{Z}_3$	$\mathbf{Z}_3$	0	0	$\mathbf{Z}_3$
6	0	0	0	0	0	0	0	0	0
5	0	0	0	$\mathbf{Z}_3$	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
3	0	0	0	0	$\mathbf{Z}_3$	$\mathbf{Z}_3$	0	0	$\mathbf{Z}_3$
2	0	0	0	0	0	0	0	0	0
1	0	0	0	$\mathbf{Z}_3$	0	0	0	0	0
0	0	0	0	0	Λ	Λ	0	0	Λ
	0	1	2	3	4	5	6	7	8

The ambiguity in the theorem is due to my inability to calculate the differential $d_2: E^2_{5,0}\to E^2_{3,1}$. Is it zero? However applying work of Brown [16] on the high-dimensional homology of groups of finite virtual dimension, we can resolve the extension problems.

His theorem states that when $i>6$,

$$H_i(\mathrm{SL}(3,\mathcal{O});\mathrm{St}(3)\otimes\Lambda)=\hat{H}^{5-i}(N_1;\Lambda)\oplus\hat{H}^{5-i}(N_2;\Lambda)$$

where $\hat{H}$ denotes Farrell-Tate cohomology and N_i denotes the normalizer of the group generated by h_i.

$$h_1=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&-1\end{bmatrix}\qquad\text{and}\qquad h_2=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

are representatives of the two conjugacy classes of elements of order three. (We remark on this below in section three.)

PROPOSITION IV.1.5. *N_1 and N_2 each map onto a group N with a 2-group for* *the kernels. N is a split extension*

$$1 \to \mathbf{Z} \oplus \mathbf{Z}_3 \to N \to \mathbf{Z}_2 \to 1,$$

with $\mathbf{Z}_2$ acting nontrivially on both generators of $\mathbf{Z} \oplus \mathbf{Z}_3$.

Then one computes

$$H^q(N_i : \Lambda) = \begin{cases} 0, & q \equiv 1,2 \pmod 4). \\ \mathbf{Z}_3, & q > 0, q \equiv -1, 0 \bmod 4. \end{cases}$$

and sees that the Farrell-Tate groups are periodic of period 4. See [16]. This proves all extension problems have the trivial solution except for the one along the line $p + q = 8$. For this case we calculate everything with $\mathbf{Z}_3$ coefficients. The universal coefficients theorem applied to $(X_3^*, \partial X_3^*) \times_G EG$ implies

$$\begin{aligned} \operatorname{Tor}(H_6(\mathrm{SL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \Lambda), \mathbf{Z}_3) &\cong H_7(\mathrm{SL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \mathbf{Z}_3) \\ &\cong H^{-2}(N_1; \mathbf{Z}_3) \oplus H^{-2}(N_2; \mathbf{Z}_3) \\ &= \mathbf{Z}_3^2 \end{aligned}$$

Proof of IV.1.5. (The author is grateful to K. Brown for this argument.) It will be useful to replace h_2 by its conjugate

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

which we still call h_2. Now

$$k = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

is an element of $\mathrm{SL}(3,\mathcal{O})$ conjugating h_i to h_i^{-1}, so it remains to find the centralizers.

Note that the integers $\mathcal{O}'$ in $\mathbf{Q}(\sqrt[12]{1}) = \mathbf{Q}(i)(\zeta)$, ζ a primitive cube root of unity, are $\mathcal{O} + \mathcal{O}\zeta$. Via h_i, $\mathcal{O}'$ embeds into $\mathrm{End}_{\mathcal{O}} \mathcal{O}^3$ and $P = \mathcal{O}^3/(\mathcal{O}'\text{-torsion})$ is then a projective rank 1 $\mathcal{O}'$-module isomorphic to $\mathcal{O}^2$ viewed also as an $\mathcal{O}'$-module. Therefore $\mathrm{End}_{\mathcal{O}'} M = \mathcal{O}'$ and $\mathrm{Aut}_{\mathcal{O}'} P = \mathcal{O}'^*$ may be interpreted as the subgroup of all elements of $\mathrm{Aut}_{\mathcal{O}} P$ commuting with the automorphism induced by h_i. $\mathcal{O}'^* = \mathbf{Z}_{12} \oplus \mathbf{Z}$, the generator of infinite order being $2 - \sqrt{3}$. It remains to lift this element back to an automorphism of $\mathcal{O}^3$ commuting with h_i. For h_1 the appropriate element is clearly

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & -2i \\ 0 & 2i & 2-i \end{bmatrix}.$$

For h_2 one can only lift the square of the generator $2-\sqrt{3}$ back; the appropriate element is

$$g_2 = \begin{bmatrix} 1 & -2-4i & -2+4i \\ 0 & 7+4i & -8i \\ 0 & 8i & 7-4i \end{bmatrix}.$$

It should be clear that k conjugates g_i to g_i^{-1} and that the kernel of $\mathrm{Aut}_{\mathcal{O}}\mathcal{O}^3 \to \mathrm{Aut}_{\mathcal{O}}P$ is two-primary.

2. Recall that the matrix $\lambda_i = l_i{}^t\bar{l}_i$. In this section and the next $c(i, j, \ldots, k)$ will denote the cell in X_n^* which is the projection of the span $\{r_i\lambda_i + r_j\lambda_j \cdots + r_k\lambda_k : r\text{'s} \geqslant 0, \text{ not all} = 0\}$ in $\mathrm{PH}(n)^*$. $\hat{c}(i, j, \ldots, k)$ denotes the form with matrix $\lambda_i + \lambda_j + \cdots + \lambda_k$, which we call the barycenter of $c(i, j, \ldots, k)$. Now we outline how to obtain the E^1-term of the spectral sequence for the $\mathrm{GL}(2, \mathcal{O})$ case, working from left to right.

$$E^1_{3,q} = \begin{cases} \Lambda, & q = 0; \\ \mathbf{Z}_3, & q \equiv 3 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

First it is necessary to recall from III.1.2 and III.1.3 that one cell c_3 is required here and that its group G_3 is an extension of S_4. Second we must see that G_3 acts on c_3 preserving the orientation. Starring c_3 at the barycenter decomposes c_3 into simplices and it is easy to check this assertion. This gives $E^1_{3,q}$ as claimed.

$$E^1_{2,q} = \begin{cases} \mathbf{Z}_3, & q = 1 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

In the Proof of III.1.3 we remarked that $c(2,3,6)$ and $c(1,3,6)$ represent the two orbits of $\mathrm{SG}(\varphi)$ in the set of faces of $G(\varphi)$. However

$$c(1,3,6)\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} = c(2,3,6)$$

so there is only one $\mathrm{GL}(2, \mathcal{O})$ orbit in the set of two-cells of X_2^*, and we take $c(2,3,6)$ as the representative. We leave it to the reader to verify that the stability group G_2 of $c(2,3,6)$ fits into an exact sequence $1 \to \mathbf{Z}_4 \to G_2 \to S_3 \to 1$ and contains an orientation-reversing element.

$E^1_{1,q} = 0$ and $E^1_{0,q} = 0$ are no trouble at all, and clearly all differentials vanish.

When setting up the E^1-term for the $\mathrm{SL}(2, \mathcal{O})$ case, we find

$$E^1_{3,q} = \begin{cases} \Lambda, & q = 0; \\ \mathbf{Z}_3, & q \equiv 1, 3 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

$$E^1_{2,q} = \begin{cases} \mathbf{Z}_3^2, & q \equiv 1 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

$$E^1_{1,q} = E^1_{0,q} = 0$$

That is, both $c(2,3,6)$ and $c(1,3,6)$ contribute to the E^1 term, and we will have to calculate a differential. As outlined in the beginning of the chapter, one may write

$$\partial[c_3] = ([c(1,3,6)] - [c(2,3,6)])(1 + h_1 + h_2 + h_3)$$

where $1, h_1, h_2$, and h_3 are coset representatives in G_3 for the cosets of the groups stabilizing the pairs $(c_3, c(2,3,6))$ and $(c_3, c(2,3,6))$. Using the formula for transfer in [4] on page 225 and the periodicity of the homology groups one shows $d_1 : E^1_{3,q} \to E^1_{2,q}$ induces the E^2 term as claimed.

3. Before we give the E^1-term of the spectral sequence which computes $H_*(\mathrm{SL}(3,\mathcal{O}); \mathrm{St}(3) \otimes \Lambda)$ we introduce more notation and make a few remarks. For a face $c = c(i, j, \ldots, k)$, $c[n]$ denotes the face obtained by omitting the vertex λ_n. We will inductively find representatives of the orbits of $\mathrm{SL}(3,\mathcal{O})$ in the codimension m cells by looking at the faces of the representative codimension $m-1$ cells and eliminating redundant cells according to the empirical principle that if $\det \hat{c} = \det \hat{c}'$ for the barycenters of c and c', then c and c' should be in the same $\mathrm{SL}(3,\mathcal{O})$ orbit. Existence or non-existence of symmetries of a cell may be inferred sometimes from the values of the determinants of barycenters of sets of subfaces of the given cell. A change of coordinates usually permits one to find a desired symmetry by inspection. One expects all this to be difficult in low codimensions where one must move around lots of vertices simultaneously. Therefore, we supply most of the data required to determine the spectral sequence in the right-most columns, leaving some verifications in the left-most columns to the diligent reader.

$$E^1_{8,q} = \begin{cases} \Lambda, & q = 0; \\ Z_3, & q \equiv 3 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

By III.2.3 there is one orbit among the top-dimensional cells and by III.2.2 we have the structure of the stability group G_8. Since $\mathrm{SL}(3,\mathbb{C})$ is connected G_8 preserves the orientation of the cell. Therefore, $E^1_{8,q}$ is as claimed.

$$E^1_{7,q} = 0.$$

We remarked in III.2.3 that G_8 is transitive on codimension one faces of c_8, so there is one orbit represented by $c_7 = c(1,2,4,5,6,7,8,10,12,14)$. Starring c_7 at its barycenter decomposes c_7 into eight simplices, and it is easily checked that orientation is reversed by

$$h_7 = \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & -i \\ -i & -i & -1 \end{bmatrix}.$$

We also need to note that no element of order three stabilizes c_7. (Elementary Galois theory shows that there are no elements of $\mathrm{SL}(3,\mathcal{O})$ of prime order $p \geqslant 5$.)

If there were such an element, it would have a fixed point set of dimension at least three in X_3^*, since it would have at least four orbits in the vertex set of c_7. But the fixed point sets of

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad h_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

are each two-dimensional and these two elements represent the distinct conjugacy classes of order three elements. This fact may be proved directly using a little elementary linear algebra or from the fact that elements of order three appearing in the stabilizer of a face of higher codimension are all conjugate to one of these.

$$E^1_{6,q} = \begin{cases} \Lambda^3, & q = 0; \\ \mathsf{Z}_3^2, & q \equiv 1 \bmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

This is clear, once we see that there are three orbits in the set of codimension two cells represented by

$$c_6^1 = c(2,5,6,8,10,12,14) \text{ with group } G_6^1 = \{1\},$$

$$c_6^2 = c(2,4,6,8,10,12,14) \text{ with group } G_6^2 = \mathsf{Z}_3, \text{ and}$$

$$c_6^3 = c(1,2,4,5,6,7,8,10,12) \text{ with group } G_6^3 = \mathsf{Z}_3.$$

Returning to the inequalities of chapter III, section 2, we can count the codimension two faces of $D(\varphi)$: There are $3 \cdot 2^6$ which are simplices and $3 \cdot 2^5$ which are not, like c_6^3. One can also verify that only the identity element of $\mathrm{SG}(\varphi) = G_8$ fixes c_6^1, c_6^2, or c_6^3, and that the order of G_8 is $3 \cdot 2^5$. Therefore the translates of the chosen cells fill out the set of codimension two faces of c_8. Then we observe that $\det \hat{c}_6^1 = 32 \neq 36 = \det \hat{c}_6^2$ so that these cells are in different $\mathrm{GL}(3, \mathcal{O})$ orbits. Therefore they form a complete, irredundant set of representatives.

To argue $G_6^1 = \{1\}$, start by computing determinants of the barycenters of the seven codimension one faces of c_6^1. $c_6^1[14]$ is the unique face such that $\det \hat{c}_6^1[14] = 16$, so it follows that G_6^1 stabilizes $c_6^1[14] = c_5^5$. However, we determine the group of this face below, and it is easily verified that only the trivial element fixes the vertex λ_{14}.

$G_6^2 = \mathsf{Z}_3$: Change corrdinates by

$$a_2 = \begin{bmatrix} 0 & 1 & -i \\ i & 1 & 0 \\ 1 & 1-i & 0 \end{bmatrix}.$$

Thus the vertices of $c_6^2 a_2$ are given by the following table:

TABLE IV.3.1

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}=\lambda_2 a_2 \qquad \begin{bmatrix}1&-i&0\\i&1&0\\0&0&0\end{bmatrix}=\lambda_8 a_2$$

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}=\lambda_{10} a_2 \qquad \begin{bmatrix}0&0&0\\0&1&-i\\0&i&1\end{bmatrix}=\lambda_{12} a_2$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\lambda_6 a_2 \qquad \begin{bmatrix}1&0&i\\0&0&0\\-i&0&1\end{bmatrix}=\lambda_4 a_2$$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}=\lambda_{14} a_2$$

Clearly $h_2 = a_2^{-1} g_2 a_2$ permutes the vertices and stabilizes the face. Calculating $\det \hat{c}_6^2[2] = \det \hat{c}_6^2[10] = \det \hat{c}_6^2[6] = 22$, $\det \hat{c}_6^2[8] = \det \hat{c}_6^2[12] = \det \hat{c}_6^2[4] = 20$, and $\det \hat{c}_6^2[14] = 18$ shows that any element of G_6^2 permutes $\{\lambda_2, \lambda_{10}, \lambda_6\}$ and is therefore conjugate by a_2 to an automorphism of the form given by the identity matrix. On the other hand it is easily checked from the table that only the powers of h_2 are conjugate to matrices stabilizing c_6^2.

$G_6^3 = \mathbf{Z}_3$: The argument is practically the same. Change coordinates by

$$a_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&-i&1\end{bmatrix}$$

From his own table the reader will easily recognize the symmetry $h_2 = a_3^{-1} g_3 a_3$ in the transformed vertices. On the other hand, observing that the only codimension one faces of c_6^3 which are not simplices are those faces omitting a vertex λ_8, λ_{10} or λ_{12} shows, as above, that $a_3^{-1} G_6^3 a_3$ is a subgroup of the automorphisms of the identity matrix. Again only powers of h_2 can belong to this subgroup.

$$E_{5,q}^1 = \begin{cases} \Lambda^4, & q = 0; \\ \mathbf{Z}_3^2, & q \equiv 1 \bmod 4; \\ \mathbf{Z}_3^3, & q \equiv 3 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

There are the following representative codimension three faces.

$$c_5^1 = c(1,2,4,5,6,7,8,12) \qquad c_5^2 = c(2,6,8,10,12,14)$$

$$c_5^3 = c(2,4,6,8,10,12) \qquad c_5^4 = c(1,4,7,8,10,12)$$

$$c_5^5 = c(2,5,6,8,10,12)$$

Their groups satisfy

$G_5^1 \supset Z_2$, is orientation-reversing, and has no 3-torsion;

$G_5^2 = \{1\}$; $G_5^3 = Z_3$; $G_5^4 = Z_3$ and

$1 \to (Z_2)^2 \to G_5^5 \to S_3 \to 1$ is orientation-preserving.

G_5^1: G_5^1 fails to contain three torsion for the same reason as G_7^1. One may check that G_5^1 contains

$$h_5 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and that this element reverses the orientation of c_5^1.

G_5^2: Calculation of the determinants of the barycenters of the faces of c_5^2 shows that G_5^2 permutes the set of vertices $\{\lambda_2, \lambda_6, \lambda_8, \lambda_{12}\}$. Thus G_5^2 is a subgroup of the group of $c(2,6,8,12)$ which is isomorphic to G_3^2. This group is described below and it is routine to see that no nontrivial element stabilizes c_5^2.

G_5^3 and G_5^4: Refer to the arguments and tables for G_6^2 and G_6^3.

G_5^5: The vertices of $c_5^5 a_3$ obviously admit a symmetry induced by $h_5 = a_3^{-1} h_3 a_3$. (Refer to one's own table of vertices of $c_6^3 a_3$.) h_3 and g_3 generate a subgroup of G_5^5 isomorphic to S_3. One checks that among pairs of opposite faces of c_5^5 having three vertices apiece the following set S satisfies $(\det \hat{c}(i,j,k), \det \hat{c}(l,m,n)) = (0,1)$. $S = \{(c(2,5,6), c(8,10,12)),\ (c(6,10,12), c(2,5,8)),\ (c(5,8,12), c(2,6,8)),\ (c(2,8,10), c(5,6,12))\}$. Therefore G_5^5 is a group of permutations of S. But G_5^5 acts transitively on S because it also contains the element

$$h_3' = a_3 \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} a_3^{-1}$$

It is now possible to prove that G_5^5 is generated by these orientation preserving elements and fits into the given exact sequence.

$$E_{4,q}^1 = \begin{cases} \Lambda, & q = 0; \\ Z_3, & q \equiv 3 \bmod 4; \\ 0, & \text{otherwise}. \end{cases}$$

Orbit representatives are:

$c_4^1 = c(1,2,4,5,6,7,8)$ $c_4^2 = c(5,6,8,10,12)$

$c_4^3 = c(4,6,7,10,12)$ $c_4^4 = c(2,6,8,12,14)$

We obtain the following information about the stability groups.

$1 \to \mathbf{Z}_4 \to G_4^1 \to S_4 \to 1,$ orientation-preserving.

$G_4^2 \supset (\mathbf{Z}_2 \times \mathbf{Z}_2) \times \mathbf{Z}_2,$ orientation-reversing.

$G_4^3 \supset \mathbf{Z}_2,$ orientation-reversing.

$G_4^4 \supset \mathbf{Z}_4,$ orientation-reversing.

G_4^1: c_4^1 is the cone on a copy of the domain constructed in the two-by-two case. Therefore, we have G_4^1 as described.

G_4^2, G_4^3, and G_4^4 contain no three torsion because no element of order three in $\mathbf{SL}(3,\mathcal{O})$ fixes two lines. Before giving generators, we remark that it is handy for the inductive step to know G_4^2 is transitive on a largest possible set of faces, so we give more than our orientation-reversing element.

$\mathbf{Z}_2^2 \subset G_4^2$ is generated by

$$h_4 = a_3\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} a_3^{-1} \quad \text{and} \quad h_4' = a_3\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} a_3^{-1}.$$

$$\mathbf{Z}_2 \text{ is generated by } k = a_3\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} a_3^{-1}.$$

$$\mathbf{Z}_2 \subset G_4^3 \text{ is generated by } h_4'' = a_2\begin{bmatrix} 1 & -i & i \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} a_2^{-1}.$$

$$\mathbf{Z}_4 \subset G_4^4 \text{ is generated by } h_4''' = a_2\begin{bmatrix} 1 & -i & 0 \\ -i & 0 & -i \\ -1 & 0 & 0 \end{bmatrix} a_2^{-1}.$$

$$E_{3,q}^1 = \begin{cases} \mathbf{Z}_3^3, & q \equiv 1 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

There are three representative cells in codimension five:

$$c_3^1 = c(2,5,6,8), \quad c_3^2 = c(5,6,8,10), \quad \text{and} \quad c_3^3 = c(2,6,8,14)$$

The groups all reverse orientation, G_3^1 is an extension of S_3,

$$1 \to \mathbf{Z}_4 \to G_3^1 \to S_3 \to 1,$$

$G_3^2 = S_4$, and $G_3^3 = S_3$.

It is easy to prove the assertions about the groups if one takes

$$b_1 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} i & i & i \\ 1 & 1-i & 0 \\ -1 & 0 & i \end{bmatrix}$$

and calculates the vertices of $c_3^1 a_3 b_1$, $c_3^2 a_3 b_2$, and $c_3^3 a_2 b_3$. However, the generator of $Z_2 \subset G_3^3$ may be a little hard to find. Its congugate by $a_2 b_3$ is

$$\begin{bmatrix} -1+i & -1-i & 1 \\ -1-i & 1 & -1+i \\ 1 & -1+i & -1-i \end{bmatrix}$$

$$E_{2,q}^1 = \begin{cases} Z_3^2, & q \equiv 1 \bmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

Representatives for the two classes of codimension six faces not in the boundary are $c_2^1 = c(6, 8, 10)$ and $c_2^2 = c(6, 8, 14)$. It is easy to make this inductive step and to show $G_2^1 \supset S_3$, $G_2^2 = S_3$ and both groups reverse orientations.

$E_{1,q}^1 = 0$ and $E_{0,q}^1 = 0$ are clear. Now we present the necessary facts about the differential d_1. $d_1 : E_{8,q}^1 \to E_{7,q}^1$ and $d_1 : E_{7,q}^1 \to E_{6,q}^1$ are clearly trivial. Keeping track of the orientations, we find the matrices for $d_1 : E_{6,q} \to E_{5,q}$ are

$$\begin{bmatrix} -2 & -3 & 3 \\ 4 & 1 & -3 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{if} \quad q = 0;$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{if} \quad q \equiv 1 \bmod 4;$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad q \equiv 3 \bmod 4;$$

or zero.

$d_1 : E_{5,q}^1 \to E_{4,q}^1$ is zero because the cell contributing to $E_{4,q}^1$ is not a face of any cell contributing to $E_{5,q}^1$.

$d_1 : E_{4,q}^1 \to E_{3,q}^1$ can have a component only in that part of $E_{3,q}^1$ coming from c_3^1 (consider conjugacy classes). Therefore, it is zero.

$$d_1 : E_{3,4j+1}^1 \to E_{2,4j+1}^1 \text{ has matrix } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a transfer calculation.

Now we comment on a procedure for eliminating redundant faces from the list of codimension m faces of a minimal set of representative codimension $m-1$ faces. As mentioned earlier we are trying to prove that if $\det \hat{c} = \det \hat{c}'$, and if c and c' have the same number of vertices, then c and c' represent the same orbit. One may try to find a sequence $g_1, \ldots, g_k$ of elements of $\mathrm{SL}(3, \mathcal{O})$ such that $cg_1 \cap c' < cg_1 g_2 \cap c' < \cdots < cg_1 g_2 \cdots g_k = c'$, writing "is a face of" as $<$. For example, suppose one knows a face c_1' of c' has a large stability group. Then one could try a calculation of determinants of barycenters of lower faces of c and c_1' to see which subface of c one should try to move onto c_1'. One could hope to finish moving c onto c' in one more step by finding g_2 inside the stability group of c_1'.

For example, we obtain most of the following data following this procedure.

Codimension two to codimension three. Faces of $c_6^3 = c(1,2,4,5,6,7,8,10,12)$:

$$c_5^1 = c_6^3[10] = c_6^3[8]\, g_3 = c_6^3[12](g_3)^2;$$

$$c_5^4 = c(1,4,7,8,10,12); \quad c_5^5 = c(2,5,6,8,10,12)$$

$$c(1,5,6,8,10,12) = c(2,4,6,8,10,12)g_3 = c(2,5,7,8,10,12)(g_3)^2$$

$$c(2,4,7,8,10,12) = c(1,5,7,8,10,12)g_3 = c(1,4,6,8,10,12)(g_3)^2$$

But

$$c(1,5,6,8,10,12)a_3 \begin{pmatrix} 1 & 0 & 0 \\ 1 & -i & 0 \\ 1 & 0 & i \end{pmatrix} a_2^{-1} = c_5^3;$$

$$c(2,4,7,8,10,12)a_3 \begin{pmatrix} -i & -i & -i \\ -i & 0 & 0 \\ -i & -1 & 0 \end{pmatrix} a_2^{-1} = c_5^2.$$

Faces of $c_6^2 = c(2,4,6,8,10,12,14)$:

$$\begin{aligned} c_5^3 &= c_6^2[14] \\ c_5^2 &= c_6^2[4] = c_6^2[12]\, g_2 = c_6^2[8](g_2)^2 \\ &\quad\; c_6^2[10] = c_6^2[2]\, g_2 = c_6^2[6](g_2)^2 \end{aligned}$$

But

$$c_6^2[10]a_2 \begin{pmatrix} -i & 1+i & -1 \\ -1 & -i & i \\ i & -1 & 1-i \end{pmatrix} a_3^{-1} = c_5^4$$

Faces of $c_6^1 = c(2,5,6,8,10,12,14)$:

$$c_6^1[14] = c_5^5$$

$$c_6^1[12]a_3\begin{bmatrix} i & 1 & -i \\ 0 & 1 & -i \\ 0 & 0 & -i \end{bmatrix}a_2^{-1} = c_5^3 \qquad c_6^1[10]a_3\begin{bmatrix} 0 & 0 & -i \\ 0 & 1 & -i \\ -i & 0 & -i \end{bmatrix}a_2^{-1} = c_5^2$$

$$c_6^1[8]a_3\begin{bmatrix} 1 & 0 & i \\ 1 & -i & i \\ 0 & 0 & i \end{bmatrix}a_2^{-1} = c_5^3 \qquad c_6^1[6]a_3\begin{bmatrix} -i & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -i & 0 \end{bmatrix}a_2^{-1} = c_5^3$$

$$c_6^1[5]a_3\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -i & 0 \end{bmatrix}a_2^{-1} = c_5^2 \qquad c_6^1[2]a_3\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{bmatrix}a_2^{-1} = c_5^3$$

Codimension three to codimension four. Faces of $c_5^5 = c(2,5,6,8,10,12)$:

$$c_4^2 = c_5^5[2] = c_5^5[5]\,g_3 = c_5^5[6](g_3)^2$$

$$c_5^5[12] = c_5^5[10]\,g_3 = c_5^5[8](g_3)^2$$

But

$$c_5^5[12]a_3\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}a_3^{-1} = c_4^2$$

Faces of $c_5^4 = c(1,4,7,8,10,12)$:

$$c_5^4[12] = c_5^4[10]\,g_3 = c_5^4[8](g_3)^2,$$

$$\text{but}\quad c_5^4[12]a_3\begin{bmatrix} 1 & 1-i & 1+i \\ 1 & 0 & i \\ 1 & -i & 0 \end{bmatrix}a_2^{-1} = c_4^3.$$

$$c_5^4[1] = c_5^4[4]\,g_3 = c_5^4[7](g_3)^2, \qquad \text{but}\quad c_5^4[1]a_3\begin{bmatrix} i & i & i \\ 0 & -1 & i \\ 0 & 0 & i \end{bmatrix}a_2^{-1} = c_4^4.$$

Faces of $c_5^3 = c(2,4,6,8,10,12)$:

$$c_4^3 = c_5^3[2] = c_5^3[6]\,g_2 = c_5^3[10](g_2)^2$$

$$c_5^5[5] = c_5^3[4] = c_5^3[12]\,g_2 = c_5^3[8](g_2)^2$$

Faces of $c_5^2 = c(2,6,8,10,12,14)$:

$$c_5^2[10] = c_4^4, \qquad c_5^2[14] = c_5^5[5]$$

$$c_5^2[2]a_3\begin{bmatrix} -1 & 0 & -1-i \\ 0 & 0 & i \\ 0 & i & 1 \end{bmatrix}a_2^{-1} = c_4^3, \qquad c_5^2[6]a_3\begin{bmatrix} 0 & i & -1 \\ 0 & 0 & i \\ -1 & 0 & 1-i \end{bmatrix}a_2^{-1} = c_4^3$$

$$c_5^2[8]a_3\begin{bmatrix} -1 & i & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}a_2^{-1} = c_4^3, \qquad c_5^2[12]a_3\begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & -i & -1 \end{bmatrix}a_2^{-1} = c_4^3$$

Faces of $c_5^1 = c(1,2,4,5,6,7,8,12)$:

$$c_4^1 = c_5^1[12] = c_5^1[8]h_5.$$

The other faces are simplices:

$$c_5^5[10] = c(2,5,6,8,12) = c(1,5,7,8,12)h_5$$

$$c_5^4[10] = c(1,4,7,8,12) = c(2,4,6,8,12)h_5$$

$$c(1,4,6,8,12) = c(2,4,7,8,12)h_5$$

$$c(1,5,6,8,12) = c(2,5,7,8,12)h_5$$

$$c(1,4,6,8,12)a_3\begin{bmatrix} i & 1 & 0 \\ i & 0 & -1 \\ i & 1 & -i \end{bmatrix}a_2^{-1} = c_4^3$$

$$c(1,5,6,8,12)a_3\begin{bmatrix} 0 & 0 & i \\ -i & 0 & i \\ 0 & 1 & i \end{bmatrix}a_3^{-1} = c_4^2$$

Codimension four to codimension five:

Since we are now dealing with cells having many automorphisms and few faces this is much easier. We can say that we need one face which is a simplex from c_4^1, the face c_3^1, by what we have done in the two-by-two case. The face which is not a simplex lies in ∂X_3^*. We have given enough elements of G_4^2 to see that it is transitive on faces containing λ_{12}. So from c_4^2 we consider one of these $c^2[10]$, and $c_3^2 = c_4^2[12]$. Similarly, from c_4^4 we need $c_4^4[14]$ and $c_3^3 = c_4^4[2]$. But we have to tabulate more cases for faces of c_4^3 since we only know $c_4^3[8]h_4'' = c_4^3[4]$.

We place the redundant faces as follows:

$$c_4^2[10]a_3\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 1 & 0 & 0\end{bmatrix}a_3^{-1} = c_3^1 \qquad c_4^4[10]a_2\begin{bmatrix}0 & -i & 0\\ -1 & 0 & 0\\ 0 & 0 & i\end{bmatrix}a_3^{-1} = c_3^2$$

$$c_4^3[12]a_2\begin{bmatrix}1 & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & i\end{bmatrix}a_3^{-1} = c_3^2 \qquad c_4^3[8]a_2\begin{bmatrix}-i & 0 & 0\\ i & 0 & -i\\ 1 & -1 & 0\end{bmatrix}a_2^{-1} = c_3^1$$

$$c_4^3[10]a_2\begin{bmatrix}-1+i & i & i\\ 1 & 0 & 1\\ -i & 0 & 0\end{bmatrix}a_2^{-1} = c_3^3 \qquad c_4^3[6]a_2\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}a_2^{-1} = c_4^3[10]$$

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