Number Theory and Random Matrix Theory

Michael Rubinstein

University of Waterloo

MSRI, Berkeley Feb 2011

Some background in random matrix theory:

In his *The Classical Groups*, Weyl worked out Haar measure for class functions on the classical compact groups: $U(N)$, and the orthogonal and symplectic groups.

Let $A \in U(N)$ be a unitary matrix, $AA^* = I$, with eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_N}$, $0 \le \theta_j < 2\pi$.

Let $f(A) = f(\theta_1, \dots, \theta_N)$ be a class function on $U(N)$, only depending on the conjugacy class that A belongs to, i.e. a symmetric function on the eigenangles θ_j.

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Haar measure for class functions on $U(N)$ is given in terms of its joint probability density function for eigenangles:

$$\langle f(A)\rangle_{U(N)} = \frac{1}{N!(2\pi)^N}\int_{[0,2\pi]^N} f(\theta_1,\ldots,\theta_N)\prod_{1\le j<k\le N}\left|e^{i\theta_k}-e^{i\theta_j}\right|^2 d\theta_1\ldots d\theta_N,$$

f integrable.

The statistics that we will consider:

- Eigenangle densities and correlations.
- Moments of characteristic polynomials.

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Another formula for this measure.

Define

$$S_N(\theta) = \sin(N\theta/2)/\sin(\theta/2),$$

and take $S_N(0) = N$. Then

$$\prod_{1\le j<k\le N} \left|\exp(i\theta_k) - \exp(i\theta_j)\right|^2 = \det_{N\times N}(S_N(\theta_k - \theta_j)).$$

Derive this formula by expressing the l.h.s. as a product of two Vandermonde determinants:

$$\det_{N\times N}(\exp(i(k-1)\theta_j)) \det_{N\times N}(\exp(-i(k-1)\theta_j)),$$

multiplying the two matrices, summing the geometric series, and simplifying.

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r-point density.

We would like to know, on average over $U(N)$, the number of eigenangles that lie in an interval $[a, b]$, and more generally, the density of r-tuples of eigenangles lying in a 'box'. Let r be a positive integer, and $f : [0, 2\pi]^r \to \mathbb{R}$ an integrable function. For $A \in U(N)$ with eigenangles $0 \leq \theta_1, \ldots, \theta_N < 2\pi$, we define the *$r$-point density*, weighted by f, to be the sum over all distinct r-tuples:

$$\sum_{1 \leq \substack{j_1, \ldots, j_r \\ \text{distinct}} \leq N} f(\theta_{j_1}, \ldots, \theta_{j_r}).$$

The sum is over $r!\binom{N}{r}$ ways to select our r-tuples of distinct θ's from the N eigenangles.

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The main result for $U(N)$, due to Gaudin and Mehta, is:

Theorem: Let $f : [0, 2\pi]^r \to \mathbb{R}$ be an integrable function. Then

$$\left\langle \sum_{\substack{1 \le j_1, \ldots, j_r \le N \\ \text{distinct}}} f(\theta_{j_1}, \ldots, \theta_{j_r}) \right\rangle_{U(N)}$$

equals the following r-dimensional integral:

$$\frac{1}{(2\pi)^r} \int_{[0,2\pi]^r} f(\theta_1, \ldots, \theta_r) \det_{r \times r}(S_N(\theta_k - \theta_j)) d\theta_1 \ldots d\theta_r.$$

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For $r = 1$ and integrable $f : [0, 2\pi] \to \mathbb{R}$, the theorem reads

$$\left\langle \sum_{j=1}^{N} f(\theta_j) \right\rangle_{U(N)} = \frac{N}{2\pi} \int_0^{2\pi} f(\theta) d\theta,$$

i.e. uniform density on $[0, 2\pi]$. Here we have used $S_N(0) = N$. However, if $r = 2$, then pairs of eigenangles are *not* uniformly dense in the box $[0, 2\pi]^2$. For integrable $f : [0, 2\pi]^2 \to \mathbb{R}$, we have

$$\left\langle \sum_{1 \le j_1 \ne j_2 \le N} f(\theta_1, \theta_2) \right\rangle_{U(N)} = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} f(\theta_1, \theta_2)(N^2 - S_N(\theta_2 - \theta_1)^2) d\theta_1 d\theta_2.$$

The integrand is small when θ_2 is close to θ_1. The non-uniformity is reflected in the fact that unitary eigenvalues tend to repel away from one another.

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Outline of proof. The r-point density is a symmetric function of the eigenangles. Hence we can find its average by integrating against the joint probability density function for unitary eigenangles:

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However, the measure above is a symmetric function with respect to the θ's (easiest to see from the Vandermonde squared), so each term in the sum contributes the same amount, and we get:

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Two useful properties:

$$\int_0^{2\pi} S_N(\theta_j - \theta) S_N(\theta - \theta_k) d\theta = 2\pi S_N(\theta_j - \theta_k),$$

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These two properties allow us (Gaudin's Lemma) to integrate out w.r.t. $\theta_{r+1}, \ldots \theta_N$ and rewrite the r-point density as:

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Scaling Limit

Let $f \in L^1(\mathbb{R}^r)$, and normalize the eigenangles

$$\tilde{\theta}_i = \theta_i N/(2\pi)$$

to account for the fact that the eigenvalues are getting more dense on the unit circle. Then, as $N \to \infty$,

$$\left\langle \sum_{\substack{1 \le j_1, \ldots, j_r \le N \\ \text{distinct}}} f(\tilde{\theta}_{j_1}, \ldots, \tilde{\theta}_{j_r}) \right\rangle_{U(N)}$$

$$\to \int_{[0,\infty]^r} f(x_1, \ldots, x_r) \det_{r \times r}(S(x_k - x_j)) dx_1 \ldots dx_r,$$

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$$S(x) = \sin(\pi x)/(\pi x).$$

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Pair correlation

Let $f \in L^1(\mathbb{R})$. Applying the two point density to the average pair correlation gives:

$$\left\langle \frac{1}{N} \sum_{1 \le j \ne k \le N} f(\tilde{\theta}_k - \tilde{\theta}_j). \right\rangle_{U(N)}$$

$$= \frac{1}{N} \int_0^N \int_0^N f(x_2 - x_1) \det_{2\times 2}(S_N((x_k - x_j)2\pi/N)/N) dx_1 dx_2.$$

(we have changed variables $x_j = \theta_j N/(2\pi)$). One can show that, as $N \to \infty$ this tends to

$$= \int_{-\infty}^{\infty} f(t) \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt.$$

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r-point correlations can similarly be defined and evaluated.

Let $f \in L^1(\mathbb{R}^{r-1})$. Then, as $N \to \infty$,

$$\left\langle \frac{1}{N} \sum_{\substack{1 \leq j_1, \dots, j_r \leq N \\ \text{distinct}}} f(\tilde{\theta}_{j_r} - \tilde{\theta}_{j_1}, \dots, \tilde{\theta}_{j_2} - \tilde{\theta}_{j_1}) \right\rangle_{U(N)}$$

$$\to \int_{\mathbb{R}^{r-1}} f(t_1, \dots, t_{r-1}) \det_{r \times r}(S(t_{k-1} - t_{j-1})) dt_1 \dots dt_{r-1}.$$

In the determinant we use the convention that $t_0 = 0$.

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For example, the three-point correlation reads as:

$$\lim_{N\to\infty} \left\langle \frac{1}{N} \sum_{1 \le \substack{j_1, j_2, j_3 \\ \text{distinct}} \le N} f(\tilde{\theta}_{j_3} - \tilde{\theta}_{j_1}, \tilde{\theta}_{j_2} - \tilde{\theta}_{j_1}) \right\rangle_{U(N)}$$

$$= \int_{\mathbb{R}^2} f(t_1, t_2) \begin{vmatrix} 1 & S(t_1) & S(t_2) \\ S(t_1) & 1 & S(t_2 - t_1) \\ S(t_2) & S(t_2 - t_1) & 1 \end{vmatrix} dt_1 \dots dt_2.$$

We have cleaned up the entries of the determinant slightly using $S(-x) = S(x)$.

Zeros of L-functions

Why might the Riemann Hypothesis be true?

- Hilbert and Polya: the Riemann Hypothesis is true for spectral reasons- the zeros of the zeta function are associated to the eigenvalues of some Hermitian or unitary operator acting on some Hilbert space.
- Katz and Sarnak studied families of function field zeta functions (for example, associated to the number of solutions over finite fields of plane algebraic curves). They were the first to suggest that the statistics of all the classical compact groups should be relevant for L-functions over number fields, such as the Riemann zeta function.

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Montgomery achieved the first result connecting zeros of zeta with eigenvalues of unitary operators.

Write a typical non-trivial zero of ζ as

$$1/2 + i\gamma.$$

Assume RH for now, so that the γ's are real. The zeros come in conjugate pairs, so focus on those lying above the real axis and order them

$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \cdots$$

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Montgomery's Conjecture

Let $0 \le \alpha < \beta$. Then

$$\frac{1}{M}|\{1 \le i < j \le M : \tilde{\gamma}_j - \tilde{\gamma}_i \in [\alpha, \beta)\}| \sim \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt.$$

as $M \to \infty$.

Notice that the integrand is small when t is near 0. Zeros of zeta tend to repel away from one another.

Montgomery was able to prove that

$$\frac{1}{M}\sum_{1\le i<j\le M} f(\tilde\gamma_j-\tilde\gamma_i)\to \int_0^\infty f(t)\left(1-\left(\frac{\sin\pi t}{\pi t}\right)^2\right)dt$$

as $M\to\infty$, for smooth and rapidly decaying functions f satisfying the stringent restriction that $\hat f$ be supported in $(-1,1)$.

Rudnick and Sarnak generalized this to any primitive L-function (assuming a weak form of the Ramanujan conjectures in the case of higher degree L-functions). They also gave a smoothed version of the above theorem in the case that RH is false.

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Pair correlation from data, bins of size .01, versus $1 - \sin(\pi t)^2/(\pi t)^2$.

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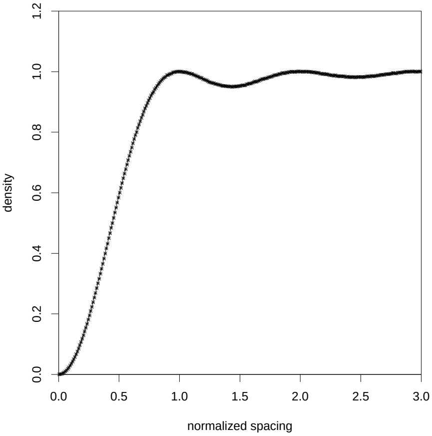

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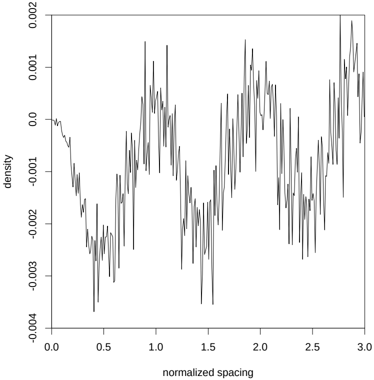

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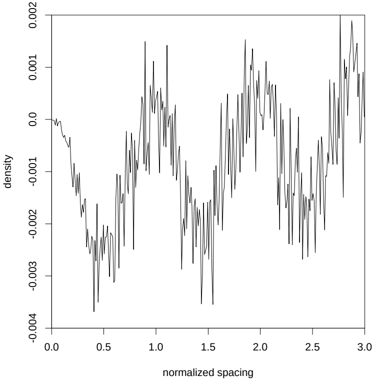

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How Montgomery and Rudnick-Sarnak's theorems are proven: Use Weil's explicit formula to relate sums over zeros of zeta to sums over primes:

Let $\epsilon > 0$ and $\phi(z)$ analytic in $-1/2 - \epsilon \leq \Im(z) \leq 1/2 + \epsilon$ and satisfy $\phi(z) = O(|z|^{-1-\epsilon})$ in that strip. Assume further that $\hat{\phi}(u) = O(\exp(-(\pi + \epsilon)u))$ as $u \to \infty$. Then

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$\Lambda(n) = \log(p)$ if $n = p^k$, 0 otherwise. The sum on the l.h.s. is over the non-trivial zeros $1/2 + i\gamma$ of $\zeta(s)$ each term counted with multiplicity of the zero. The Riemann Hypothesis (i.e. $\gamma \in \mathbb{R}$) is *not* assumed.

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$$R_2(T,f,h) = \sum_{j \neq k} h_1(\gamma_j/T) h_2(\gamma_k/T) f\left((\gamma_j - \gamma_k)\frac{\log T}{2\pi}\right).$$

Think of h as pulling out the zeros roughly up to height T.
Theorem (Montgomery , Rudnick-Sarnak version which does not assume RH).

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$$f\left((\gamma_j-\gamma_k)\frac{\log T}{2\pi}\right)=\int_{-\infty}^{\infty}\hat{f}(u)e^{iu(\gamma_j-\gamma_k)\log T}du.$$

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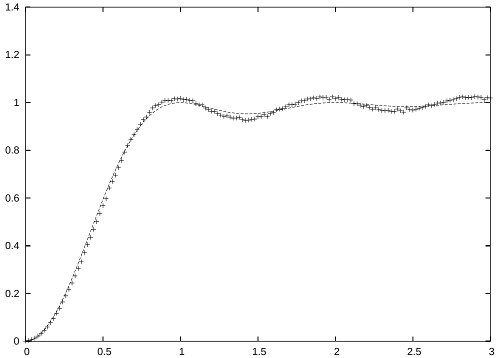
Apply the explicit formula, multiply out all the terms. In a nutshell: the support condition, $|u|<1$ restricts us, on the prime side, to the region where only the diagonal sum contributes.

R. data

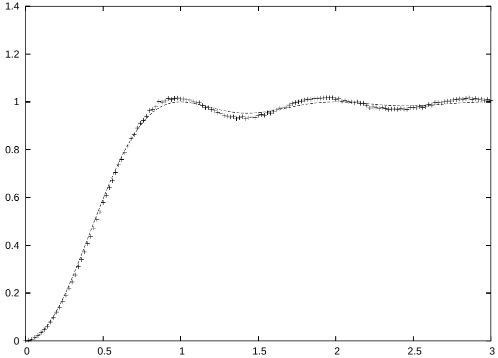
Pair correlation for five million zeros of $L(s, \chi)$, $q = 3$.



Normalization: $\tilde{\gamma} = \gamma \log(\gamma q/(2\pi e))/(2\pi)$.

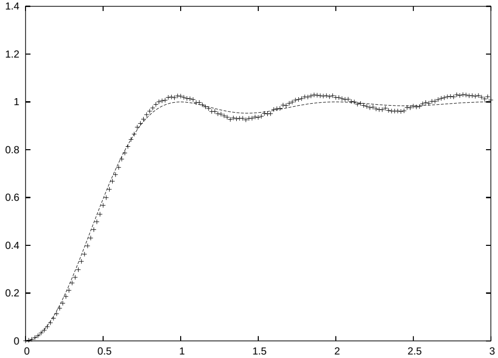
R. data

Pair correlation for five million zeros of $L(s, \chi)$, $q = 4$.



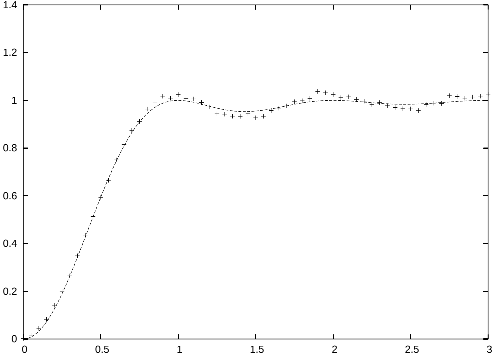
R. data

$L(s,\chi)$, $q=5$, 4 graphs averaged, 2 million zeros each.



R. data

300,000 zeros of the Ramanujan tau L-function.



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Katz-Sarnak

Let A be a matrix in one of the classical compact groups:

- Unitary: $AA^* = I$. Eigenvalues on unit circle.
- Orthogonal: $AA^t = I$, real entries. Eigenvalues come in conjugate pairs. Distinguish SO($2N$), vs SO($2N+1$). Latter always has an eigenvalue at $z = 1$.
- Unitary Symplectic: $A \in U(2N)$, $A^t JA = J$, $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ Eigenvalues come in conjugate pairs.

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$$\lim_{N\to\infty} \left\langle \sum_{\substack{1\le j_1,\ldots,j_r \le N \\ \text{distinct}}} f(\tilde{\theta}_{j_1},\ldots,\tilde{\theta}_{j_r}) \right\rangle_{G(N)}$$
$$= \int_0^\infty \cdots \int_0^\infty f(x) W_G^{(r)}(x) dx$$

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Main point: Gives a specific test that can be used to detect the different classical compact groups.

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Density of zeros for quadratic Dirichlet L-functions

Let

$$D(X) = \{d \text{ a fundamental discriminant} : |d| \leq X\}$$

and let $\chi_d(n) = \left(\frac{d}{n}\right)$ be Kronecker's symbol. We consider the zeros of $L(s, \chi_d)$, quadratic Dirichlet L-functions. Write the non-trivial zeros above the real axis of $L(s, \chi_d)$ as

$$1/2 + i\gamma_j^{(d)}, \qquad j = 1, 2, 3 \ldots$$

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r-point density for $L(s, \chi_d)$, R.

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \sum_{\substack{j_i\geq 1\\ \text{distinct}}} f\left(\tilde{\gamma}_{j_1}^{(d)}, \tilde{\gamma}_{j_2}^{(d)}, \ldots, \tilde{\gamma}_{j_r}^{(d)}\right)$$
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Assumes f smooth and rapidly decreasing with $\hat{f}$ supported in $\sum |u_i| < 1$. Does not assume GRH.

This generalized the $r = 1$ case that had been achieved by Özlük and Snyder (and also Katz and Sarnak). $W_{\mathrm{USp}}^{(1)}(x)$ equals

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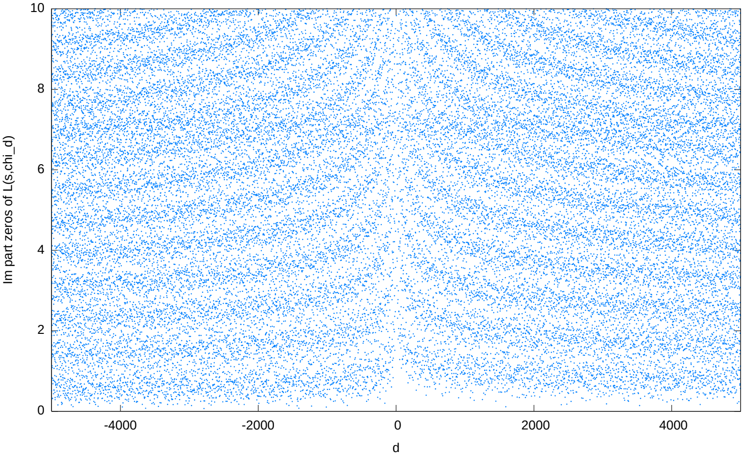


Zeros of $L(s, \chi_d)$ for $-5,000 < d < 5,000$.

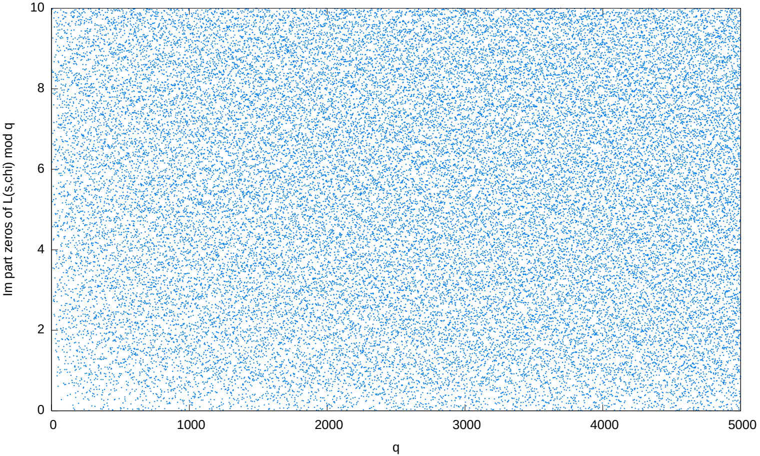
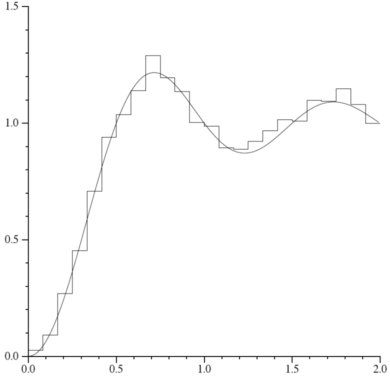
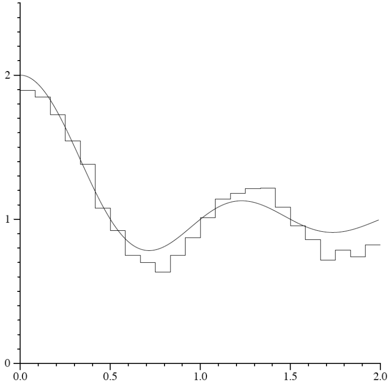


Figure: For comparison: Zeros of $L(s,\chi)$ for a generic complex primitive χ mod q, $q \leq 5,000$. 1-point density is uniform.



1-point density of zeros of $L(s, \chi_d)$ for 7,000 values of $|d| \approx 10^{12}$. Compared against the random matrix theory prediction, $1 - \sin(2\pi x)/(2\pi x)$.



One-level density and distribution of the lowest zero of even quadratic twists of the Ramanujan τ L-function, $L_\tau(s, \chi_d)$, for 11,000 values of $d \approx 500,000$ vs prediction (for large even orthogonal matrices), $1 + \sin(2\pi x)/(2\pi x)$.

Moments of the zeta function.

Obtain the asymptotics, as $T \to \infty$, of

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt.$$

$k = 1$: Hardy and Littlewood, Ingham

$k = 2$: Ingham, Heath-Brown

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Hardy and Littlewood, main term for $k = 1$

$$\begin{aligned} & \int_0^T |\zeta(1/2 + it)|^2 dt \\ \sim \quad & T \log(T) \end{aligned}$$

Ingham, full asymptotics

$$\begin{aligned}&\int_0^T |\zeta(1/2+it)|^2 dt \\ = \quad & T\log(T/(2\pi)) + T(2\gamma-1) + O(T^{1/2}\log(T))\end{aligned}$$

Balsubramanian

$$\int_0^T |\zeta(1/2+it)|^2 dt = T\log(T/(2\pi)) + T(2\gamma-1) + O(T^{1/3+\epsilon})$$

Ivic

$$
\begin{aligned}
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= \quad & T\log(T/(2\pi)) + T(2\gamma-1) + O(T^{35/108+\epsilon})
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Ingham, main asymptotics for $k = 2$

$$\int_0^T |\zeta(1/2 + it)|^4 dt$$

$$\sim \frac{T \log(T)^4}{2\pi^2}$$

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Heath-Brown, full asymptotics:

$$\int_0^T |\zeta(1/2+it)|^4 dt$$

$$= T\sum_{r=0}^{4} c_r \log(T)^{4-r} + O(T^{7/8+\epsilon})$$

$$\begin{aligned} c_0 &= 1/(2\pi^2) \\ c_1 &= 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)/\pi^2)/\pi^2 \end{aligned}$$

with c_2, c_3, c_4 implicitly given but not worked out explicitly by Heath-Brown.

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$$a_k = \prod_p \left(1-p^{-1}\right)^{k^2} \sum_{n=0}^{\infty} \binom{n+k-1}{n}^2 p^{-n}.$$

The inner sum is ${}_2F_1(k,k;1;1/p)$.

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Three heuristic approaches to studying the moments:

- Keating and Snaith, based on the analogous result in rmt.
- CFKRS, based on approximate functional equation, guided by rmt.
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Let $A \in U(N)$ with eigenvalues $\exp(i\theta_1), \ldots, \exp(i\theta_N)$.

Characteristic polynomial, evaluated on unit circle:

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Problem with this model: how does the a_k arise? Keating and Snaith method is ad hoc. And, why compare density of zeros? Seems to produce the right answer, but that is just good (great!) luck.

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Using number theoretic heuristics, and guided by techniques and results from random matrix theory, Conrey, Farmer, Keating, R., and Snaith conjectured:

For positive integer k, and any $\epsilon > 0$,

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt = \int_0^T P_k \left(\log \tfrac{t}{2\pi}\right) dt + O(T^{1/2+\epsilon}),$$

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$$P_k(x) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \qquad \oint \cdots \oint \frac{F(z_1, \ldots, z_{2k}) \Delta^2(z_1, \ldots, z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}}$$

$$\times e^{\frac{x}{2} \sum_{i=1}^{k} z_i - z_{i+k}} \, dz_1 \ldots dz_{2k},$$

with the path of integration over small circles about $z_i = 0$.

$$F(z_1,\dots,z_{2k}) = A_k(z_1,\dots,z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k} \zeta(1+z_i-z_{j+k}),$$

and A_k is the product over primes:

$$\begin{aligned} &A_k(z_1,\dots,z_{2k}) \\ &= \prod_p \prod_{i,j=1}^{k} (1-p^{-1-z_i+z_{k+j}}) \\ &\times \int_0^1 \prod_{j=1}^{k} \left(1-\frac{e(\theta)}{p^{\frac{1}{2}+z_j}}\right)^{-1} \times \left(1-\frac{e(-\theta)}{p^{\frac{1}{2}-z_{k+j}}}\right)^{-1} d\theta. \end{aligned}$$

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Example, $k = 1$

In this case, $A_1(z_1, z_2) = 1$

$$\begin{aligned} P_1(x) &= -\frac{1}{(2\pi i)^2} \oint \cdots \oint \frac{\zeta(1+z_1-z_2)(z_2-z_1)^2}{z_1^2 z_2^2} e^{\frac{x}{2}(z_1-z_2)}\, dz_1 dz_2 \\ &= x + 2\gamma \end{aligned}$$

by extracting the coefficient of $z_1 z_2$ of the numerator.

So, the full asymptotics of the second moment is given by:

$$\int_0^T (\log(t/(2\pi)) + 2\gamma) dt = T \log(T/(2\pi)) + T(2\gamma - 1)$$

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In this case, $A_2(z_1, z_2, z_3, z_4) = \zeta(2 + z_1 + z_2 - z_3 - z_4)^{-1}$, and computing the residue gives:

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P_2(x) \quad = \quad & \frac{1}{2\pi^2} x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right) x^3 \\
& + \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right) x^2 \\
& + \frac{12}{\pi^8}\Big(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 \\
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& \qquad + 3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 \\
& \qquad - 864\gamma_1\zeta'(2)^2\pi^4 + 5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 \\
& \qquad - 15552\zeta''(2)\zeta'(2)^2\pi^2 - 96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4\Big),
\end{aligned}
$$

consistent with Heath-Brown.

Example, $k = 2$

In this case, $A_2(z_1, z_2, z_3, z_4) = \zeta(2 + z_1 + z_2 - z_3 - z_4)^{-1}$, and computing the residue gives:

$$
\begin{aligned}
P_2(x) \quad = \quad & \frac{1}{2\pi^2}x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right)x^3 \\
& + \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right)x^2 \\
& + \frac{12}{\pi^8}\Big(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 \\
& \qquad + 288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4\Big)x \\
& + \frac{4}{\pi^{10}}\Big(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4 \\
& \qquad + 3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 \\
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\end{aligned}
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consistent with Heath-Brown.

We developed formulas and algorithms to compute the coefficients of $P_k(x)$ and found, for example,

$$\begin{aligned}
P_3(x) = \quad & 0.000005708527034652788398376841445252313\, x^9 \\
+ \quad & 0.00040502133088411440331215332025984\, x^8 \\
+ \quad & 0.011072455215246998350410400826667\, x^7 \\
+ \quad & 0.14840073080150272680851401518774\, x^6 \\
+ \quad & 1.0459251779054883439385323798059\, x^5 \\
+ \quad & 3.9843850948235347247479640734 29\, x^4 \\
+ \quad & 8.6073191457812067561483476362 9\, x^3 \\
+ \quad & 10.274330830703446134183009522\, x^2 \\
+ \quad & 6.5939130206497581046571339 2\, x \\
+ \quad & 0.9165155076378930590178543.
\end{aligned}$$

In our paper we got up to $k = 7$. With my Master's student, Shuntaro Yamagishi, we extended our tables to $k = 13$.

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$$\begin{aligned}
P_3(x) = \quad & 0.00000570852703465278839837684144525231 3\, x^9 \\
+ \quad & 0.000405021330884114403312153320259 84\, x^8 \\
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+ \quad & 3.984385094823534724747964073429\, x^4 \\
+ \quad & 8.607319145781206756148347636 29\, x^3 \\
+ \quad & 10.274330830703446134183009522\, x^2 \\
+ \quad & 6.593913020649758104657133 92\, x \\
+ \quad & 0.916515507637893059017854 3.
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As k grows, the leading coefficients become very small. Because we are evaluating this as a polynomial in $\log t/(2\pi)$, which increases slowly, the lower terms are very relevant for checking the conjecture.

Hiary-R. have worked out the uniform asymptotics of these coefficients, in the case of rmt, and partially here. Yamagishi is considering the same problem for orthogonal and unitary symplectic moment polynomials.

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For example, expand the Keating Snatih $U(N)$ moment polynomial:

$$\prod_{j=0}^{k-1}\left(\frac{j!}{(j+k)!}\prod_{i=0}^{k-1}(N+i+j+1)\right)=\sum_{r=0}^{k^2}c_r(k)N^{k^2-r},$$

and let

$$\mu := \sum_{j=1}^{k}\frac{j}{j+1}+\sum_{j=k+1}^{2k}\frac{2k-j}{j+1}=k\log 4-\log(k/2)+1/2-\gamma+O(1/k$$

Then, Hiary-R. prove that there exists $\rho > 0$ such that, for all k sufficiently large, a maximal $c_r(k)$ occurs for some

$$r \in \left[k^2-\mu-\rho\log(k)^2/k, k^2-\mu+1+\rho\log(k)^2/k\right], \qquad (1)$$

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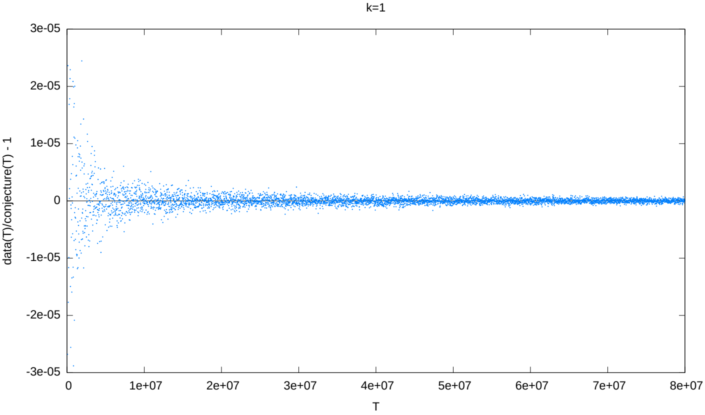
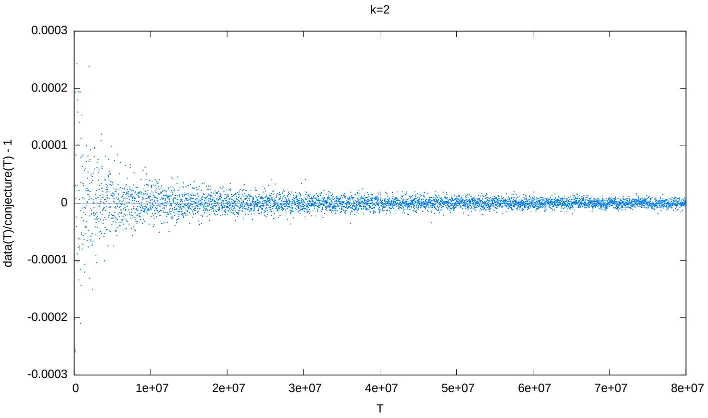
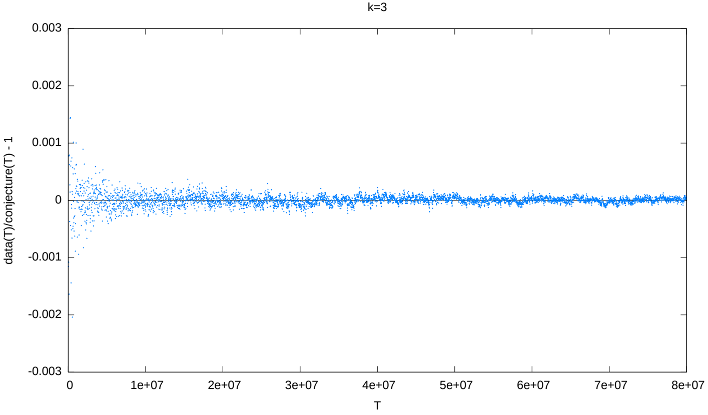


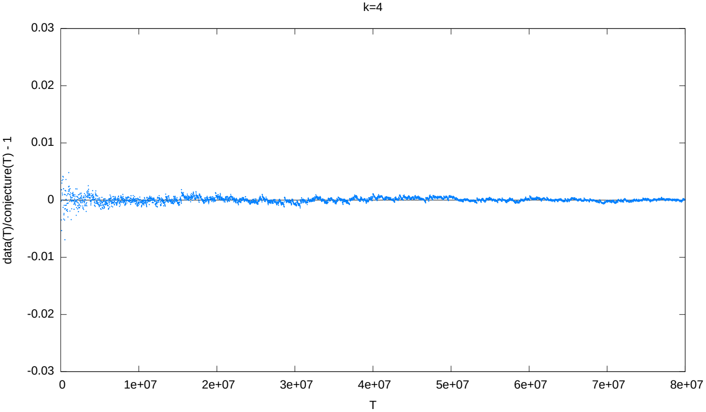
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^2 dt}{\int_0^T P_1(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 7 decimal places out of 9. Joint with Shuntaro Yamagishi (Master's thesis).



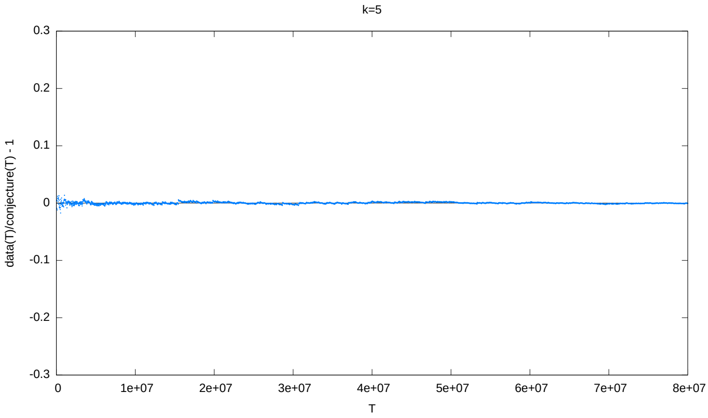
Graph of: $\frac{\int_0^T |\zeta(1/2+it)|^4 dt}{\int_0^T P_2(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 5-6 decimal places out of 12.



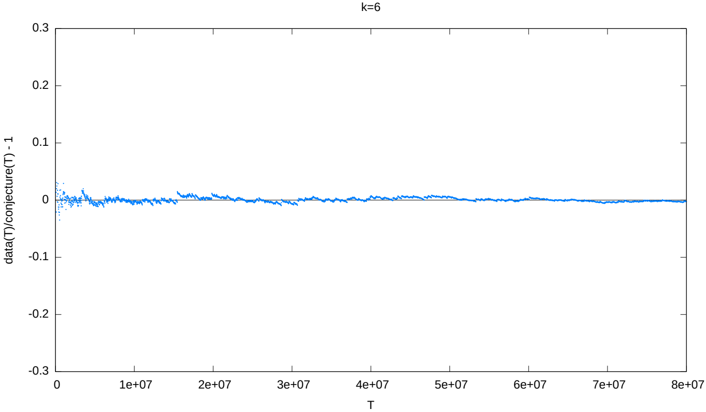
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^6 dt}{\int_0^T P_3(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 4-5 decimal places out of 15.



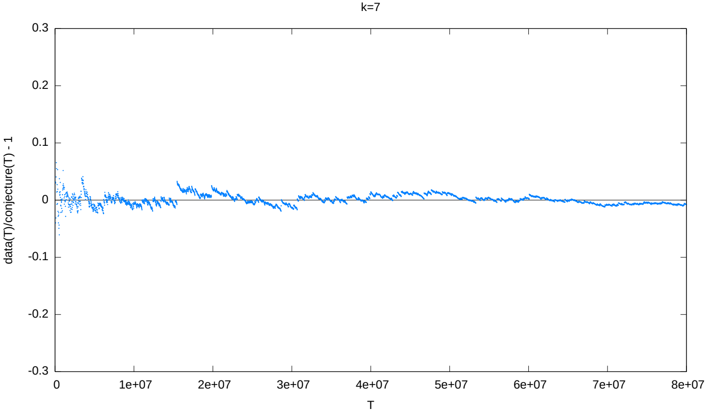
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^8 dt}{\int_0^T P_4(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 4 decimal places out of 18.



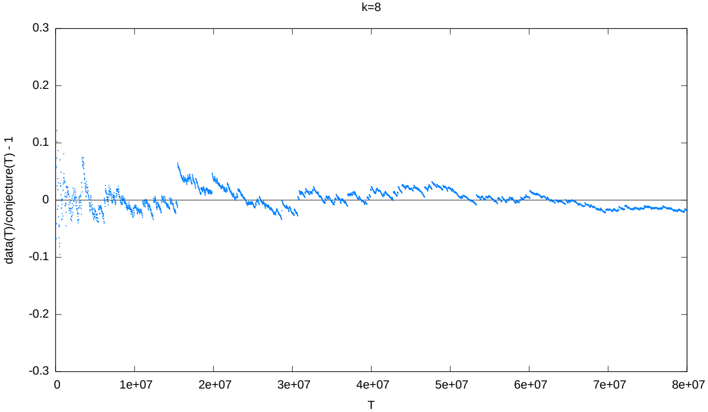
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^{10} dt}{\int_0^T P_5(\log(t)/(2\pi)) dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 3 decimal places out of 21.



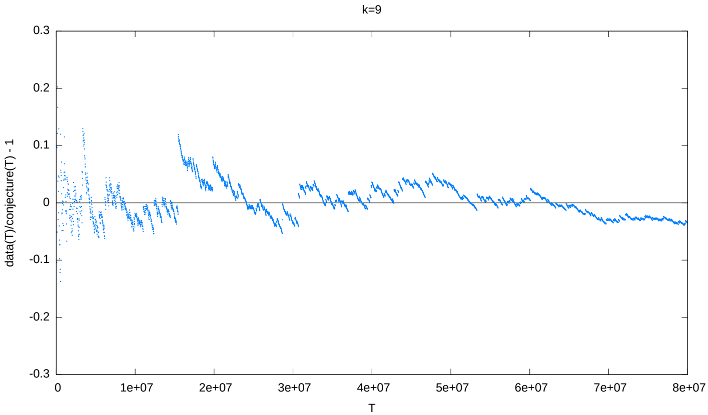
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^{12} dt}{\int_0^T P_6(\log(t)/(2\pi)) dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 2-3 decimal places out of 25.



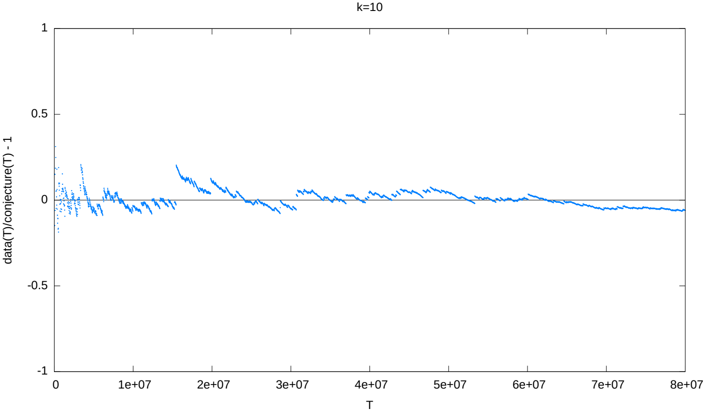
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^{14} dt}{\int_0^T P_7(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 2 decimal places out of 28.



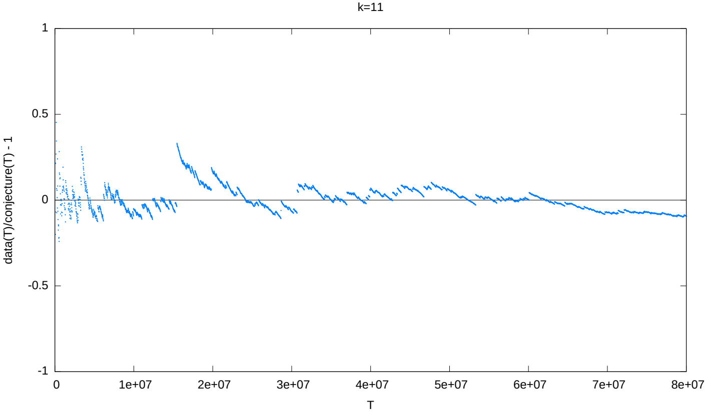
Graph of: $\frac{\int_0^T |\zeta(1/2+it)|^{16} dt}{\int_0^T P_8(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 1-2 decimal places out of 32.



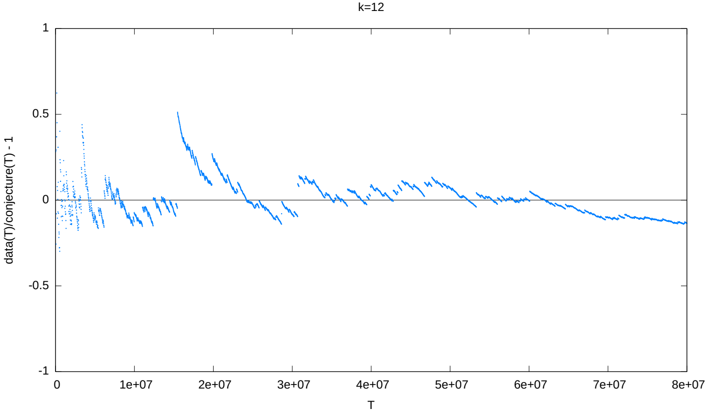
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^{18} dt}{\int_0^T P_9(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 1-2 decimal places out of 36.



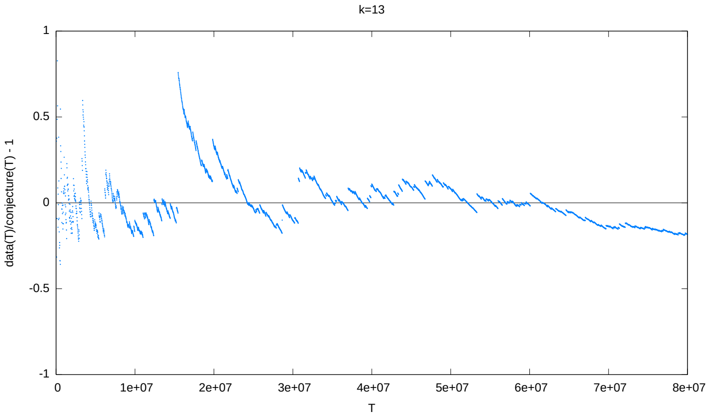
Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^{20}dt}{\int_0^T P_{10}(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 1 decimal place out of 39.



Graph of: $\frac{\int_0^T |\zeta(1/2+it)|^{22} dt}{\int_0^T P_{11}(\log(t)/(2\pi))dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 1 decimal place out of 43.



Graph of: $\dfrac{\int_0^T |\zeta(1/2+it)|^{24} dt}{\int_0^T P_{12}(\log(t)/(2\pi)) dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 1 decimal place out of 47.



Graph of: $\frac{\int_0^T |\zeta(1/2+it)|^{26} dt}{\int_0^T P_{13}(\log(t)/(2\pi)) dt} - 1$, for $0 < T < 8 \times 10^7$.
Agreement is to about 1 decimal place out of 51.

Moments of $L(1/2, \chi_d)$
Conjecture (Keating-Snaith):

$$\frac{1}{|D(X)|} \sum_{d \in D(X)} L(\tfrac{1}{2}, \chi_d)^k \sim a_k \prod_{j=1}^{k} \frac{j!}{(2j)!} \log(X)^{k(k+1)/2}$$

where

$$a_k = \prod_p \frac{(1-\frac{1}{p})^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}} \left(\frac{(1-\frac{1}{\sqrt{p}})^{-k} + (1+\frac{1}{\sqrt{p}})^{-k}}{2} + \frac{1}{p} \right).$$

Proved by Jutila for $k = 1, 2$ and Soundararajan for $k = 3$.

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Let $A \in USp(2N)$ with eigenvalues $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$.

Characteristic polynomial, evaluated at $z = 1$

$$\prod_{1}^{N} |1 - \exp(i\theta_j)|^2.$$

kth moment, over $USp(2N)$, is asymptotically, for $k \in \mathbb{Z}$:

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Density $2N/(2\pi)$ versus $\log |d|/(2\pi)$.

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Let $D_{\pm}(X)$ denote the sets of positive (respectively negative) fundamental discriminants with $|d| \le X$. Conjecture (CFKRS):

$$\sum_{d \in D_{\pm}(X)} L(\tfrac{1}{2}, \chi_d)^k = \frac{3}{\pi^2} \int_0^X Q_{\pm}(k, \log|t|) dt + o(X)$$

To define $Q_{\pm}$, let $a = 0$ if $d > 0$ and $a = 1$ if $d < 0$, and

$$X(s,a) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1+a-s}{2})}{\Gamma(\frac{s+a}{2})},$$

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$$G(z_1, \ldots, z_k) = A_k(z_1, \ldots, z_k) \prod_{j=1}^{k} X(\tfrac{1}{2}+z_j, a)^{-\frac{1}{2}} \prod_{1 \le i \le j \le k} \zeta(1+z_i+z_j),$$

Let $D_\pm(X)$ denote the sets of positive (respectively negative) fundamental discriminants with $|d| \le X$. Conjecture (CFKRS):

$$\sum_{d \in D_\pm(X)} L(\tfrac{1}{2}, \chi_d)^k = \frac{3}{\pi^2} \int_0^X Q_\pm(k, \log|t|) dt + o(X)$$

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and A_k is the Euler product, absolutely convergent for $|\Re z_j| < \frac{1}{2}$, defined by

$$\begin{aligned} A_k(z_1, \ldots, z_k) &= \prod_p \prod_{1 \le i \le j \le k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \\ &\times \left(\frac{1}{2}\left(\prod_{j=1}^{k}\left(1 - \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^{k}\left(1 + \frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right) \right. \\ &\times \left(1 + \frac{1}{p}\right)^{-1}. \end{aligned}$$

$Q_\pm(k, x)$ is the polynomial of degree $k(k+1)/2$ given by the k-fold residue

$$\frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)^2}{\prod_{j=1}^{k} z_j^{2k-1}} e^{\frac{x}{2}\sum_{j=1}^{k} z_j} dz_1 \ldots dz_k,$$

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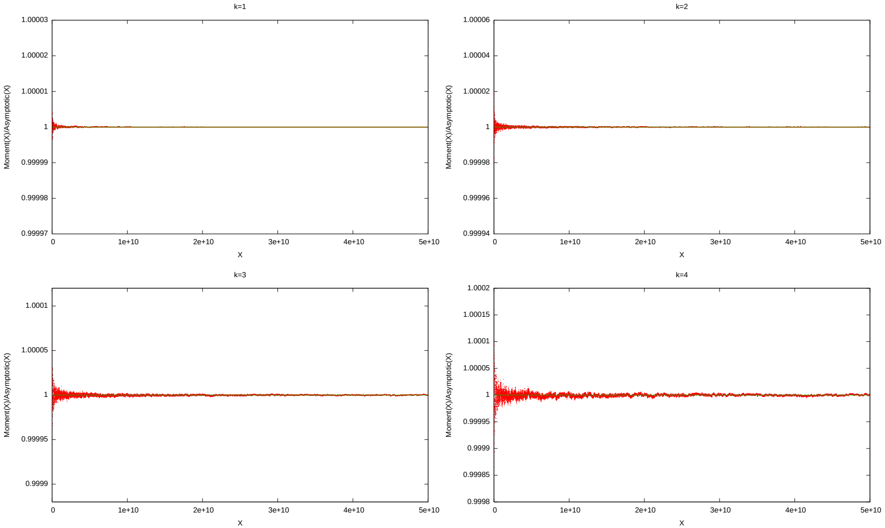
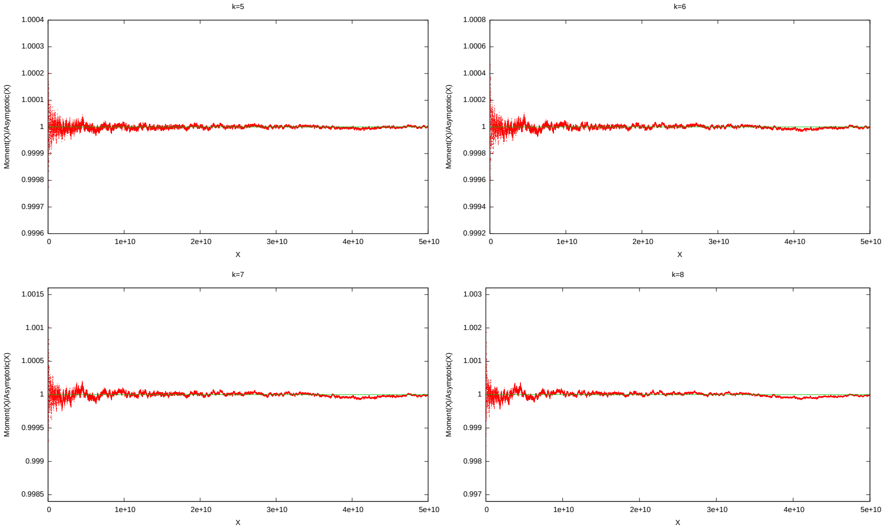
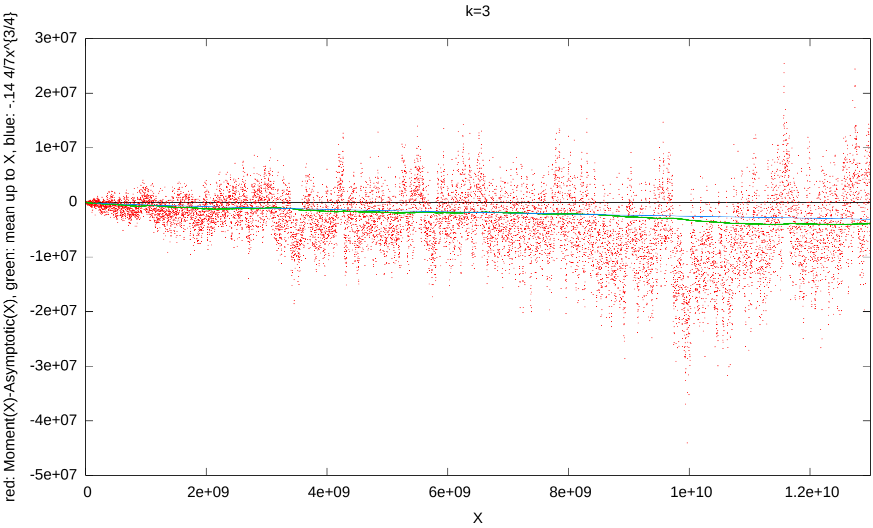


Ratio of: data vs asymptotic, $0 < d < 5 \times 10^{10}$, $k = 1, 2, 3, 4$. With Master's student Matthew Alderson.

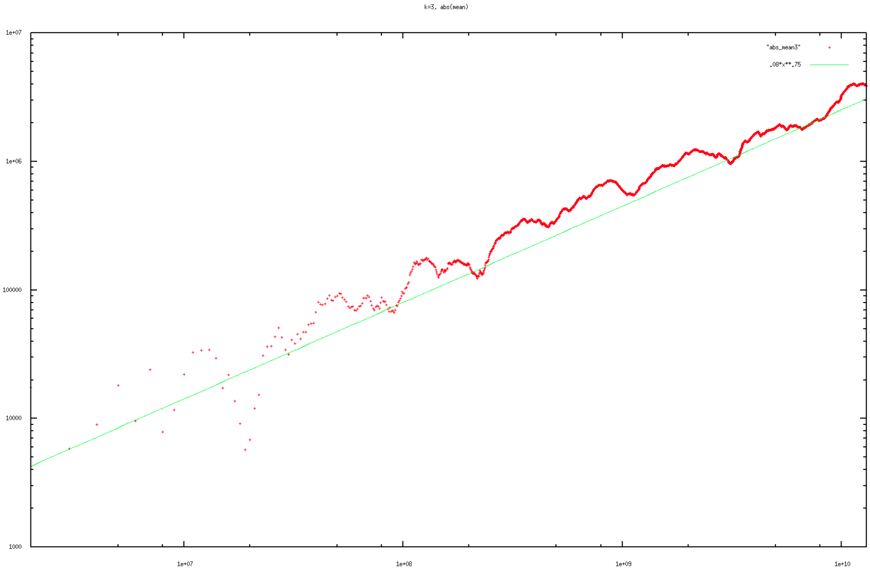


Ratio of: data vs asymptotic, $0 < d < 5 \times 10^{10}$, $k = 5, 6, 7, 8$..

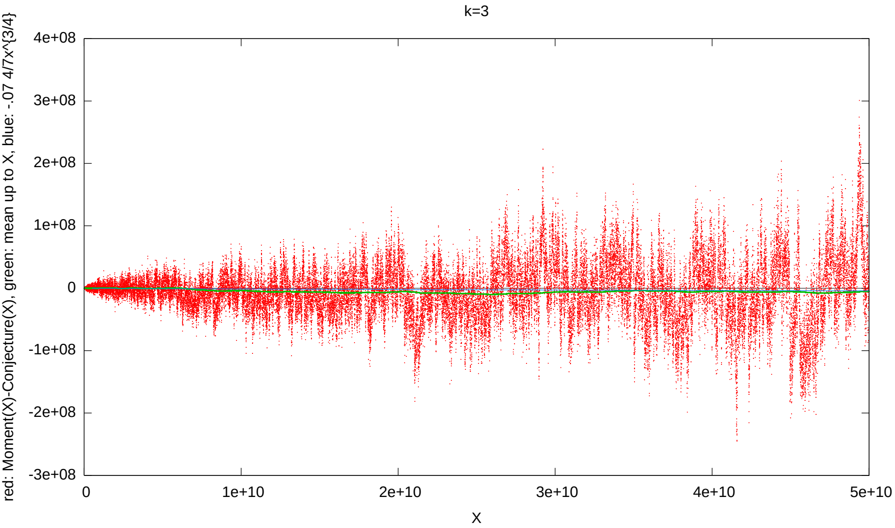
Diaconu, Goldfeld, and Hoffstein conjectured that further lower order terms exists for $k \in \mathbb{Z}$, $k \geq 3$. For $k = 3$ they conjecture an additional term of the form $bx^{3/4}$. Qiao Zhang computed $b = -.07$ for $d > 0$, and $b = -.14$ for $d < 0$.



Red: moment data − CFKRS asymptotics. Green: Running average of Red. Blue: Zhang. $d > 0$.



$d > 0$. log log plot of abs(average of the remainder)



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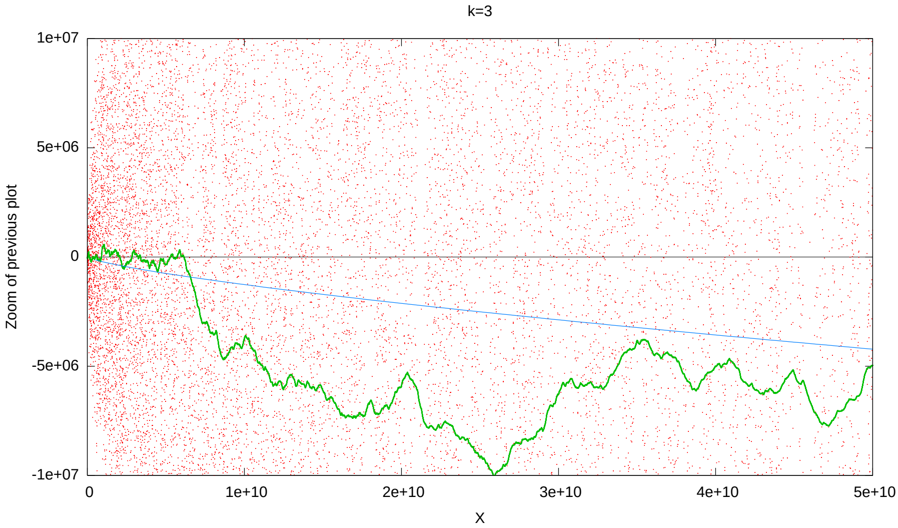

Red: moment data − CFKRS asymptotics. Green: Running average of Red. Blue: Zhang. $d < 0$. Zoom.

$d < 0$. log log plot of abs(average of the remainder)

Lower terms for the moments of elliptic curve L-functions.

Let E be an elliptic curve over $\mathbb{Q}$ and let it's L-function be

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p|Q} \left(1 - a_p p^{-s}\right)^{-1} \prod_{p\nmid Q} \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1}$$
$$= \prod_{p} \mathcal{L}_p(1/p^s), \qquad \Re(s) > 3/2.$$

Q is the conductor of E, and $a_p = p + 1 - \#E(\mathbb{F}_p)$.

$L_E(s)$ has analytic continuation to $\mathbb{C}$ and satisfies a functional equation of the form

$$\left(\frac{2\pi}{\sqrt{Q}}\right)^{-s}\Gamma(s)L_E(s) = w_E\left(\frac{2\pi}{\sqrt{Q}}\right)^{s-2}\Gamma(2-s)L_E(2-s),$$

with $w_E = \pm 1$.

Let

$$L_E(s,\chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s}$$

be the L-function of the elliptic curve E_d, the quadratic twist of E by the fundamental discriminant d. If $(d,Q)=1$, then $L_E(s,\chi_d)$ satisfies the functional equation

$$\begin{aligned}\left(\frac{2\pi}{\sqrt{Q}|d|}\right)^{-s} &\Gamma(s) L_E(s,\chi_d) \\ &= \chi_d(-Q) w_E \left(\frac{2\pi}{\sqrt{Q}|d|}\right)^{s-2} \Gamma(2-s) L_E(2-s,\chi_d).\end{aligned} \tag{2}$$

Focus on even functional equation: $\chi_d(-Q)w_E = 1$.

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Let

$$S(X) = \{|d| \leq X; \chi_d(-Q)w_E = 1\}.$$

For a fixed prime $q \nmid Q$, let

$$R_q(X) = \frac{\sum_{\substack{d \in S(X) \\ L_E(1,\chi_d)=0 \\ \chi_d(q)=1}} 1}{\sum_{\substack{d \in S(X) \\ L_E(1,\chi_d)=0 \\ \chi_d(q)=-1}} 1}$$

be the ratio of the number of vanishings of $L_E(1, \chi_d)$ sorted according to whether $\chi_d(q) = 1$ or -1.

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$$R_q = \left(\frac{q+1-a_q}{q+1+a_q} \right)^{1/2}.$$

A conjecture (ckrs 2000) asserts that, for $q \nmid Q$,

$$\lim_{X\to\infty} R_q(X) = R_q.$$

The power $1/2$ comes from the pole at $k = -1/2$ in the moments, as predicted by the moments in SO($2N$). We can also restrict to subsets such as $d < 0$ or $d > 0$ (the arithmetic factor is the same for these two families).

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Using the full asymptotics for the moments in both families, we can derive (conjecturally) more terms for $R_q(X)$:

$$R_q(X) = \left(\frac{q+1-a_q}{q+1+a_q}\right)^{1/2}\left(1-\frac{\alpha_q}{\log X}+O(\log(X)^{-2})\right)$$

where

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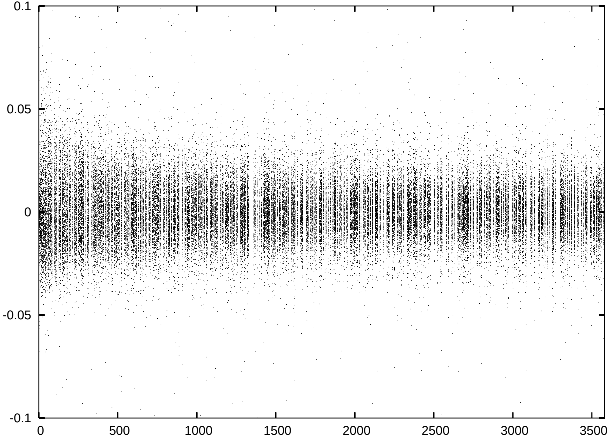
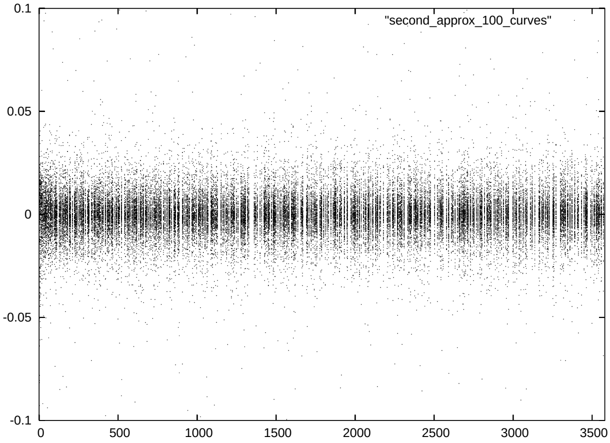
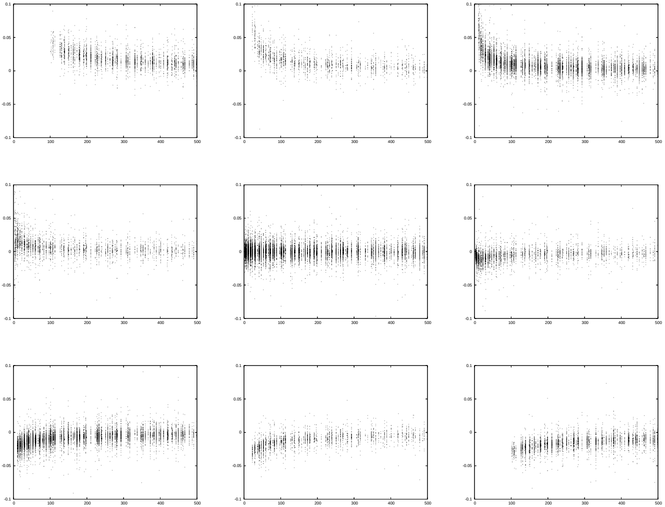


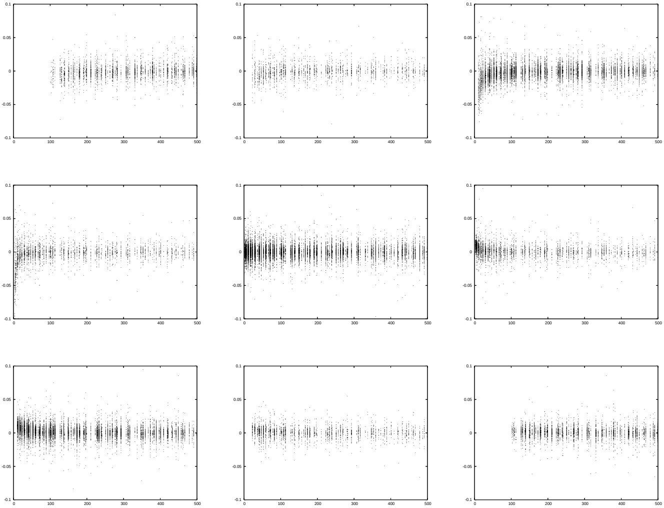
A plot for one hundred data sets. q, horizontal, versus $R_q(10^8) - R_q$, vertical.



Taking into account the next term in the asymptotics.



Left to right, top to bottom: $n = -20, -9, -6, -3, -1, 3, 6, 9, 20$. Values of $R_q(10^8) - R_q$, $2 \leq q < 500$, for the subset of our elliptic curves satisfying $a_q = n$. The blank white areas on the left reflect Hasse's theorem.



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