Dynamical Systems in Sage

Benjamin Hutz

Department of Mathematics and Statistics
Saint Louis University

November 17, 2019
Sage-days 104

This workshop is supported by NSF grant DMS-1906266

A Quick Review of Dynamical Systems

Definition

Given a set A and a map $f : A \to A$ we can iterate the map f on the set A to create a dynamical system. We denote the n-th iterate of f as

$$f^n = f \circ f^{n-1}.$$

Definition

1. We say a point x is periodic if there exists an $n \in \mathbb{N}$ such that $f^n(x) = x$.
2. We say a point x is preperiodic if there exists an $m \in \mathbb{N}$ such that $f^m(x)$ is periodic.

Examples

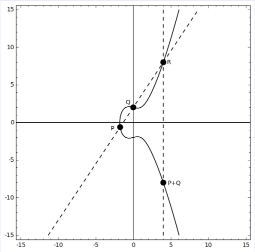
Example (Newton's Method)

The classic example of a dynamical system is Newton's Method for a differentiable rational function F. We define

$$f(x) = x - \frac{F(x)}{F'(x)}.$$

Then the fixed points of f are the zeros of F.

Example (EC Point Doubling)

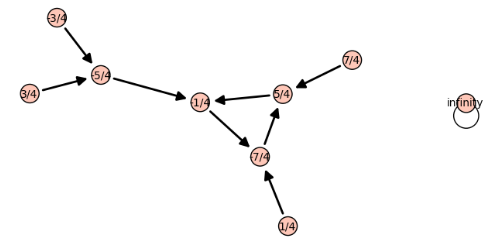


Examples: Periodic Points

Example

Consider $A = \mathbb{Q}$

1. $f(x) = x^2 - \frac{29}{16} \quad : \quad \frac{3}{4} \to -\frac{5}{4} \to \left[-\frac{1}{4} \to -\frac{7}{4} \to \frac{5}{4} \to -\frac{1}{4}\right]$



Dynamical Systems and Sage

Sage-dynamics project

1. Started at ICERM in 2012
2. Sage-days 55 in 2013
3. NSF grant DMS-1415294, 2014-2017 (PI: Hutz)
4. IMA coding sprints 2017
5. ICERM REU 2019

Resources

- Project Page: `http://wiki.sagemath.org/dynamics/ArithmeticAndComplex`
- Reference card: (`https://wiki.sagemath.org/quickref/`)
- Google group: sage-dynamics

Where is the code?

- Sage has a schemes framework: Affine, Projective, Toric, etc.
- Sage has a homomorphism framework that specifies maps between objects (homset)
- Sage has a DynamicalSystem object (hom with domain and range the same)

Functionality Overview

1. Points and functions ($\mathbb{A}^n$, $\mathbb{P}^n$, $\mathbb{P}^n \times \mathbb{P}^m$)
2. Iteration, orbits, and preimages
3. Heights, local heights, and canonical heights
4. Periodic points and dynatomic polynomials
5. Critical points, critical height, and post-critically finite maps
6. Conjugation, invariants, and minimal models
7. Automorphism groups and conjugating sets
8. Rational maps, indeterminacy, dynamical degrees

Basic Examples: Iteration

```
sage: A.<x> = AffineSpace(QQ, 1)
sage: f = DynamicalSystem_affine([x^2-1])
sage: P = A(2)
sage: f(P), f(f(P)), f(f(f(P))), f.nth_iterate(P,4)
((3), (8), (63), (3968))
sage: f.orbit(P,[0,3])
[(2), (3), (8), (63)]
sage: f.nth_iterate_map(2)
Dynamical System of Affine Space of dimension 1 over
    Rational Field
  Defn: Defined on coordinates by sending (x) to
        (x^4 - 2*x^2)
```

Basic Examples: Varieties

```
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem([(x-2*y)^2,(x-2*z)^2,x^2])
sage: X = P.subscheme(y-z)
sage: for Y in f.orbit(X,3):
sage:     Y.defining_polynomials()
(y - z,), (x - y,), (x - z,), (y - z,)
```

Height definitions

Given a point $P = (P_0 : \cdots : P_N)$

- The (absolute) global height is defined as

$$h(P) = \frac{1}{[K:\mathbb{Q}]} \log(\prod_v \max_i(|P_i|_v))$$

- The local height at a place v is defined as

$$\lambda_v(P) = \log(\max(|P_i|_v, 1))$$

- The canonical height with respect to a morphism f is defined as

$$\hat{h}(P) = \lim_{n\to\infty} \frac{h(f^n(P))}{\deg(f)^n}.$$

- The local canonical height (Green's function) at a place v.

Basic Examples: Heights

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([4*x^2-3*y^2, 4*y^2])
sage: Q = P(3,5)
sage: Q.local_height(2), Q.local_height(5)
(0.000000000000000, 1.60943791243410)
sage: Q.global_height()
1.60943791243410
sage: f.canonical_height(Q, error_bound=0.001)
2.3030927691516823627114122790
sage: sum([f.green_function(Q,t) for t in [0,2,5]])
2.3025850929940456840179914547
```

Application of Heights to Preperiodic points

Theorem

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over a number field.

1. $\hat{h}_f(Q) = 0$ *if and only if* Q *is preperiodic*
2. *There exists a constant* C_1 *depending only on* f *such that*
$$|h(f(Q)) - dh(Q)| \leq C_1.$$
3. *There exists a constant* C_2 *depending only on* f *such that*
$$\left|h(Q) - \hat{h}_f(Q)\right| \leq C_2.$$

Application of Heights to Preperiodic points

Combining these we get a test for preperiodic points

Preperiodic point check

1. If $\hat{h}(P) > C_2$, then not preperiodic.
2. Compute forward images
   1. If we encounter a cycle, return preperiodic
   2. If the height becomes $> C_2$, return not preperiodic.

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([4*x^2-3*y^2, 4*y^2])
sage: Q = P(3,2)
sage: Q.is_preperiodic(f, return_period=True)
(0,1)
```

All preperiodic points from heights

Algorithm

1. Find the height difference bound via Nullstellensatz
2. Check all rational point up to bound for preperiodic

```
sage: %time
sage: B = f.height_difference_bound();print B
sage: L = []
sage: for Q in P.points_of_bounded_height(bound=exp(B)):
sage:     if Q.is_preperiodic(f):
sage:         L.append(Q)
sage: L
2.48490664978800
[(1/2 : 1), (-1/2 : 1), (3/2 : 1), (-3/2 : 1), (1 : 0)]
CPU time: 2.58 s,  Wall time: 2.70 s
```

Solving the equations

The periodic points of specified period form a zero-dimensional scheme.

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([4*x^2-3*y^2, 4*y^2])
sage: X = f.periodic_points(1,return_scheme=True);X
Closed subscheme of Projective Space of dimension 1
    over Rational Field
defined by:
  4*x^2*y - 4*x*y^2 - 3*y^3
sage: X.rational_points()
[(-1/2 : 1), (1 : 0), (3/2 : 1)]
```

Definitions: Dynatomic Polynomial

Let $f \in K[z]$ be a single variable polynomial.

Definition

The *n*-th dynatomic polynomial for f is defined as

$$\Phi_n^*(f) = \prod_{d|n} (f^d(z) - z)^{\mu(n/d)},$$

where μ is the Moebius function.

Definition

We can also define a generalized dynatomic polynomial for preperiodic points as

$$\Phi_{m,n}^*(f) = \frac{\Phi_n^*(f^m)}{\Phi_n^*(f^{m-1})}.$$

More general definitions can be made using intersection theory.

Basic Example: Dynatomic Polynomial

We can find preperiodic points by finding the roots of the dynatomic polynomials.

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem([4*x^2-3*y^2, 4*y^2])
sage: f.dynatomic_polynomial(1).factor()
y * (2*x - 3*y) * (2*x + y)
sage: f.dynatomic_polynomial(2).factor()
(4) * (2*x + y)^2
sage: f.dynatomic_polynomial([1,1]).factor()
(16) * y * (2*x - y) * (2*x + 3*y)
```

Definitions

Theorem (Northcott)

The set of rational preperiodic points for a morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ is a set of bounded height.

Conjecture (Morton-Silverman)

Given a morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree d, defined over a number field of degree D, then there exists a constant $C(d, D, N)$ such that

$$\#\operatorname{PrePer}(f) \leq C(d, D, N).$$

Reducing mod p

Definition

Let $f(x) = \frac{p(x)}{q(x)}$ for polynomials p, q. Define the resultant as

$$\operatorname{Res}(f) = \operatorname{Res}(p(x), q(x)).$$

Proposition

Let $f(x) = \frac{p(x)}{q(x)}$. The following are equivalent:

1. $\deg f = \deg \overline{f}$,
2. *$p(x)$ and $q(x)$ have no common zeros modulo p,*
3. $\operatorname{Res}(f) \not\equiv 0 \pmod{p}$.

Reducing mod p

Definition

Let $f(x)$ be a rational function. We say that a prime p is a prime of good reduction if any condition of the Proposition is satisfied. Otherwise we say p is a prime of bad reduction.

For a prime of good reduction iteration commutes with reduction mod p:

$$\overline{f^n(x)} \equiv \overline{f}^n(\overline{x}).$$

Multiplier

Definition

Let $f(x)$ be a rational function and z a periodic point of minimal period n. Then, the multiplier of z is

$$\lambda_z = (f^n)'(z).$$

Example

For $f(x) = x^2 - 2$, we have $z = 2$ is a fixed point with multiplier 4.

A precise description of n

Theorem (Morton-Silverman 1994, Zieve)

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{Q}$ with $\deg(f) \geq 2$. Assume that f has good reduction at p with a rational periodic point P. Define

$n =$ *minimal period of* P.

$m =$ *minimal period of* P *modulo* p.

$r =$ *the multiplicative order of* $(f^m)'(P) \mod p$

Then

$$n = m \quad or \quad n = mrp^e$$

for some explicitly bounded integer $e \geq 0$.

See [Hut09] for higher dimensions.

Algorithm for finding all rational preperiodic points [Hut15]

1. For each prime p in PrimeSet with good reduction find the list of possible global periods:
   1. Find all of the periodic cycles modulo p
   2. Compute m, mrp^e for each cycle.
2. Intersect the lists of possible periods for all primes in PrimeSet.
3. For each n in PossiblePeriods
   1. Find all rational solutions to $f^n(x) = x$.
4. Let PrePeriodicPoints = PeriodicPoints.
5. Repeat until PrePeriodicPoints is constant
   1. Add the first rational preimage of each point in PrePeriodicPoints to PrePeriodicPoints.

Use Weil restriction of scalars for number fields (polynomials in dimension 1).

Examples

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem([x^2-29/16*y^2,y^2])
sage: f.rational_preperiodic_graph()
Looped digraph on 9 vertices
```

```
sage: R.<t> = PolynomialRing(QQ)
sage: K.<v> = NumberField(t^3 + 16*t^2 - 10496*t + 94208)
sage: PS.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem([x^2-29/16*y^2,y^2]) #Hutz
sage: f.rational_preperiodic_graph().show()  #10s
```

Examples

```
sage: K.<w> = QuadraticField(33)
sage: PS.<x,y> = ProjectiveSpace(K,1)
sage: f = DynamicalSystem([x^2-71/48*y^2,y^2]) #Stoll
sage: len(f.rational_preperiodic_points())
13
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ,2)
sage: f = DynamicalSystem([2*x^3 - 50*x*z^2 + 24*z^3,5
    *y^3 - 53*y*z^2 + 24*z^3,24*z^3])
sage: f.rational_preperiodic_graph().show()
```

Definitions

Definition

We say that P is a critical point for f if the jacobian of f does not have maximal rank at P.

Definition

We say that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is post-critically finite if all of the critical points are preperiodic.

Definition

The critical height of f is defined as

$$\sum_{c \in crit} \hat{h}(c).$$

Algorithm: is_postcritically_finite()

1. Compute the critical points of f over $\overline{\mathbb{Q}}$
2. Determine if each is preperiodic.

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem([x^2 + 12*y^2, 7*x*y])
sage: f.critical_points(R=QQbar)
[(-3.464101615137755? : 1), (3.464101615137755? : 1)]
sage: f.is_postcritically_finite()
False
sage: f.critical_height(error_bound=0.001)
2.8717614996729500069637701410
```

Example: Lattès maps

```
sage: P.<x,y> = ProjectiveSpace(QQ,1)
sage: E = EllipticCurve([0,0,0,0,2]);E
Elliptic Curve defined by y^2 = x^3 + 2 over Rational
    Field
sage: f = P.Lattes_map(E,2)
sage: f.is_postcritically_finite()
True
sage: f.critical_point_portrait()
Looped digraph on 10 vertices
sage: f.critical_height(error_bound=0.001)
8.2900752025070323779826707582e-17
```

Conjugation

Definition

Given a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ we can conjugate by an element $\alpha \in \mathrm{PGL}_2$

$$f^\alpha = \alpha \circ f \circ \alpha^{-1}.$$

This preserves the dynamical properties of f.

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \alpha & & \downarrow \alpha \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

Example

Every degree 2 polynomial is conjugate to a polynomial of the form

$$f_c(z) = z^2 + c$$

Example

Consider $f(z) = z^2 - 2z + 1$. Let $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2$ then $\alpha : z \mapsto z + 1$.

$$\begin{aligned} f^{\alpha}(z) &= \alpha^{-1}(f(\alpha(z))) = f(\alpha(z)) + 1 \\ &= f(z+1) - 1 = z^2 - 1. \end{aligned}$$

Sage Example

```
sage: P.<x,y> = ProjectiveSpace(QQbar,1)
sage: f = DynamicalSystem([x^2 -y^2,y^2])
sage: g = DynamicalSystem([x^2 - 2*x*y + y^2,y^2])
sage: f.is_conjugate(g)
True
sage: f.conjugating_set(g)
[[ 1 -1]
[ 0  1]]
```

Polynomial Forms

Definition

A polynomial map on $\mathbb{P}^1$ is a map with a totally ramified fixed point. A polynomial is in monic centered form if it is of the form

$$x^d + a_{d-2}x^{d-2} + \cdots + a_1 x + a_0.$$

```
sage: P.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([4*x^3 + 7*x^2*y + 5*x*y^2 +
      y^3, -3*x^3 - 4*x^2*y - 2*x*y^2])
sage: f.is_polynomial()
True
sage: f.normal_form()
Scheme endomorphism of Projective Space of dimension 1
      over Rational Field
  Defn: Defined on coordinates by sending (x : y) to
        (x^3 + 2*x*y^2 + y^3 : y^3)
```

Moduli Space

Definition

We define Hom_d to be the space of degree d morphisms on $\mathbb{P}^1$.

This conjugation action gives rise to a moduli space

$$M_d = \mathrm{Hom}_d \,/\, \mathrm{PGL}_2 \,.$$

It is known that

1. $M_2 \cong \mathbb{A}^2$, Milnor (1993) (as schemes over $\mathbb{Z}$)
2. The quotient is a geometric quotient, Petsche-Szpiro-Tepper (2009)
3. M_d is a rational variety for all d, Levy (2011)

Invariant Functions

Theorem (Milnor ($\mathbb{C}$), Silverman ($\mathbb{Z}$))

There are two PGL_2 *invariant functions* σ_1, σ_2 *such that*

$$(\sigma_1, \sigma_2) : M_2 \cong \mathbb{A}_2.$$

Theorem (Milnor ($\mathbb{C}$), Silverman ($\mathbb{Z}$))

Every invariant function of M_2 *is a polynomial in* $\mathbb{Z}[\sigma_1, \sigma_2]$.

Multiplier Spectra, Sigma invariants

Definition

Given a point z of period n for $f : \mathbb{P}^1 \to \mathbb{P}^1$ we define the multiplier at z to be

$$\lambda_z = (f^n)'(z)$$

Definition

We define the *n*-th multiplier spectrum of f, Λ_n, to be the set of multipliers of n periodic points (with multiplicity)

$$\Lambda_n(f) = \{\lambda_z : z \in \mathrm{Per}_n(f)\}$$

Definition

We define $\sigma_{n,i}$ to be the i-th symmetric function on Λ_n. We denote $\boldsymbol{\sigma}_n = (\sigma_{n,1}, \ldots, \sigma_{n,d^n+1})$.

Quadratic Polynomials

Recall that every quadratic polynomial is conjugate to exactly one polynomial of the form $f_c(z) = z^2 + c$. We can compute

$$\mathrm{Per}_1(f) = \left\{ \frac{1 \pm \sqrt{1-4c}}{2}, \infty \right\}$$
$$\Lambda_1(f) = \{1 \pm \sqrt{1-4c}, 0\}$$
$$\boldsymbol{\sigma}_1(f) = \{2, 4c, 0\}$$

In particular, the family of quadratic polynomials represents the line $\sigma_1 = 2$ in M_2.

```
sage: R.<c> = QQ[]
sage: P.<x,y> = ProjectiveSpace(R,1)
sage: f = DynamicalSystem([x^2+c*y^2,y^2])
sage: f.sigma_invariants(1)
[2, 4*c, 0]
```

We say a representation f of $[f] \in M_d$ is minimal if

$$\mathrm{Res}(f) \leq \mathrm{Res}(f^{\alpha}) \text{ for all } \alpha \in \mathrm{PGL}_2 .$$

```
sage: PS.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem([6*x^2+12*x*y+7*y^2, 12*x*y]
    )
sage: f.is_PGL_minimal()
False
sage: f.resultant()
6048
```

Bruin-Molnar algorithm [BM12]

```
sage: g=f.minimal_model();g
Scheme endomorphism of Projective Space of dimension 1
     over Rational
Field
  Defn: Defined on coordinates by sending (x : y) to
        (x^2 + 12*x*y + 42*y^2 : 2*x*y)
sage: g.resultant()
168
```

Reduced Model

Having found the minimal resultant, we can now conjugate by any element of $\mathrm{SL}_2(\mathbb{Z})$ without changing the resultant. We would like the model with smallest coefficients.

Apply Cremona-Stoll [CS03] to the fixed point binary form. This gives *almost* the minimal model

```
sage: PS.<x,y> = ProjectiveSpace(QQ, 1)
sage: f = DynamicalSystem([x^3 + x*y^2, y^3])
sage: m = matrix(QQ, 2, 2, [-221, -1, 1, 0])
sage: f = f.conjugate(m);f
...Defn: Defined on coordinates by sending (x : y) to
(x^3 : 10793861*x^3 + 146524*x^2*y + 663*x*y^2 + y^3)
sage: f.reduced_form()[0]
...Defn: Defined on coordinates by sending (x : y) to
(x^3 + x*y^2 : y^3)
```

Automorphism groups

Definition

The automorphism group of f is the group

$$\mathrm{Aut}(f) = \{\alpha \in \mathrm{PGL}_{N+1} : f^{\alpha} = f\}.$$

Determining Automorphism Groups [FMV14]

```
sage: R.<x,y> = ProjectiveSpace(QQ,1)
sage: f = DynamicalSystem([x^2-2*x*y-2*y^2,-2*x^2-2*x*
    y+y^2])
sage: f.automorphism_group(return_functions=True)
[x, 2/(2*x), -x - 1, -2*x/(2*x + 2), (-x - 1)/x, -1/(x
     + 1)]
sage: f.conjugate(matrix([[-1,-1],[1,0]])) == f
True
```

Indeterminacy

Definition

For maps $f = (f_0, \ldots, f_n) : \mathbb{P}^n \to \mathbb{P}^n$ a point of indeterminacy is where

$$f_i(P) = 0 \quad \forall i.$$

Definition

Define the dynamical degree of f to be

$$\lim_{n \to \infty} \deg(f^n)^{1/n}.$$

```
sage: P2.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: f = DynamicalSystem([x*y, y*z, z^2])
sage: f.indeterminacy_points()
[(0 : 1 : 0), (1 : 0 : 0)]
sage: f.dynamical_degree(N=50)
1.08181102597739
sage: ``degrees of iterates''
2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11 , 12 ,
```

References

Nils Bruin and Alexander Molnar, *Minimal models for rational functions in a dynamical setting*, LMS J. Comput. Math **15** (2012), 400–417.

John Cremona and Michael Stoll, *On the reduction theory of binary forms*, J. Reine Angew. Math. **565** (2003), 79–99.

Xander Faber, Michelle Manes, and Bianca Viray, *Computing conjugating sets and automorphism groups of rational functions*, Journal of Algebra **423** (2014), 1161–1190.

Benjamin Hutz, *Good reduction of periodic points*, Illinois J. Math. **53** (2009), no. 4, 1109–1126.

______, *Determination of all rational preperiodic points for morphisms of PN*, Mathematics of Computation **84** (2015), no. 291, 289–308.