SECRETS OF...

... Singular and PolyBoRi

The Systems

* Singular: general commutative algebra
* PolyBoRi: commutative algebra in Boolean Rings

Singular - Terms

- A term in a polynomial is a struct containing:
  - coefficient
  - exponent vector
  - pointer to the next term

- A polynomial is identified with a pointer to its leading term

Singular - monomial orderings

* A monomial ordering can be represented by a matrix A
* $x^\alpha > x^\beta \Leftrightarrow A \cdot \alpha >_{lex} A \cdot \beta$
* these products are also stored in the term structure to speed up comparison of terms
* The terms are ordered by monomial ordering
* So the leading term is always the first term

Singular - polynomial structure

* highly manipulateable
  * coef
  * exp
  * next pointer
* very compact in rings up to a medium number of variables
* very fast ordered iteration of polynomials
* arbitrary monomial ordering
* sparse

code example: cancel every multiple of „monom“

```
poly prev=c->S->m[i];
poly tail=c->S->m[i]->next;
 while((tail!=NULL)&& (pLmCmp(tail, monom)>=0))
 {
   if (p_LmDivisibleBy(monom,tail,c->r))
   {
     prev->next=tail->next;
     tail->next=NULL;
     p_Delete(& tail,c->r);
     tail=prev;
   }
   prev=tail;
   tail=tail->next;
 }
```

Consequences of this Style

* avoid a lot of copying/ allocation
* very direct manipulation of polynomials possible
* „intuitive“ mainstream imperative style
* very fast polynomial arithemetic

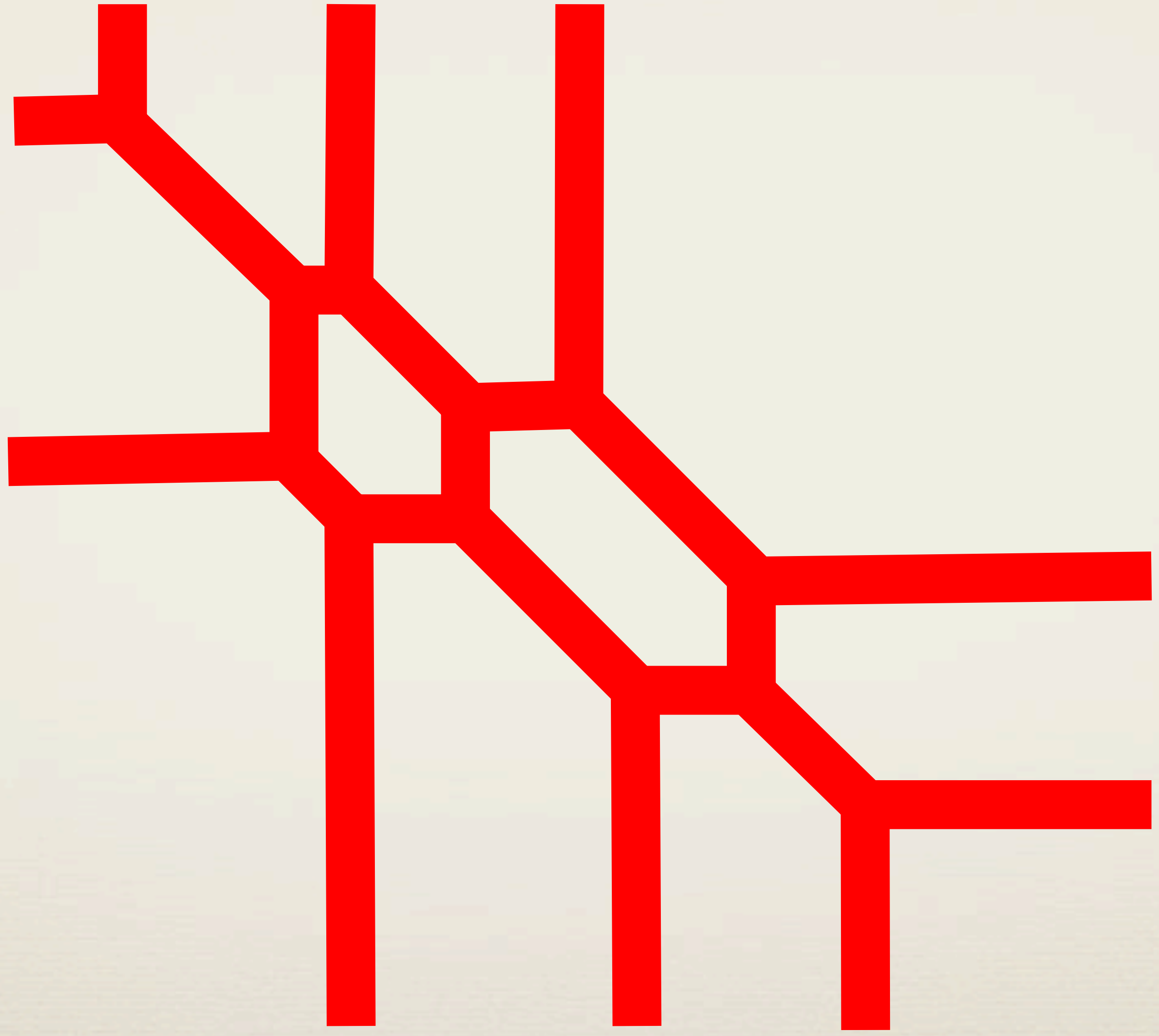
* can introduce funny bugs and memory holes
* mutability of objects makes caching hard

Singular 3-1-0 - new rings

* Polynomial rings over
  * the integers
  * Z/m
* Implemented for these rings:
  * arithmetic
  * Gröbner bases/normal forms
* Implemented by Oliver Wienand

Singular goes tropical...

My curves come to a point!



Tropical Geometry

* tropical.lib
  * Anders Jensen
  * Hannah Markwig
  * Thomas Markwig
* tropical lifting (calling gfan)
* visualization
* j-invariants
* weierstrass form
* polymake.lib: Thomas Markwig

Noncommutative News

discretize.lib	finite difference schemes
dmodapp.lib	applications d-modules
jacobson.lib	Smith/Jacobson Form
nchomolog.lib	noncommutative homological algebra
freegb.lib	two sided Gröbner bases in free algebras

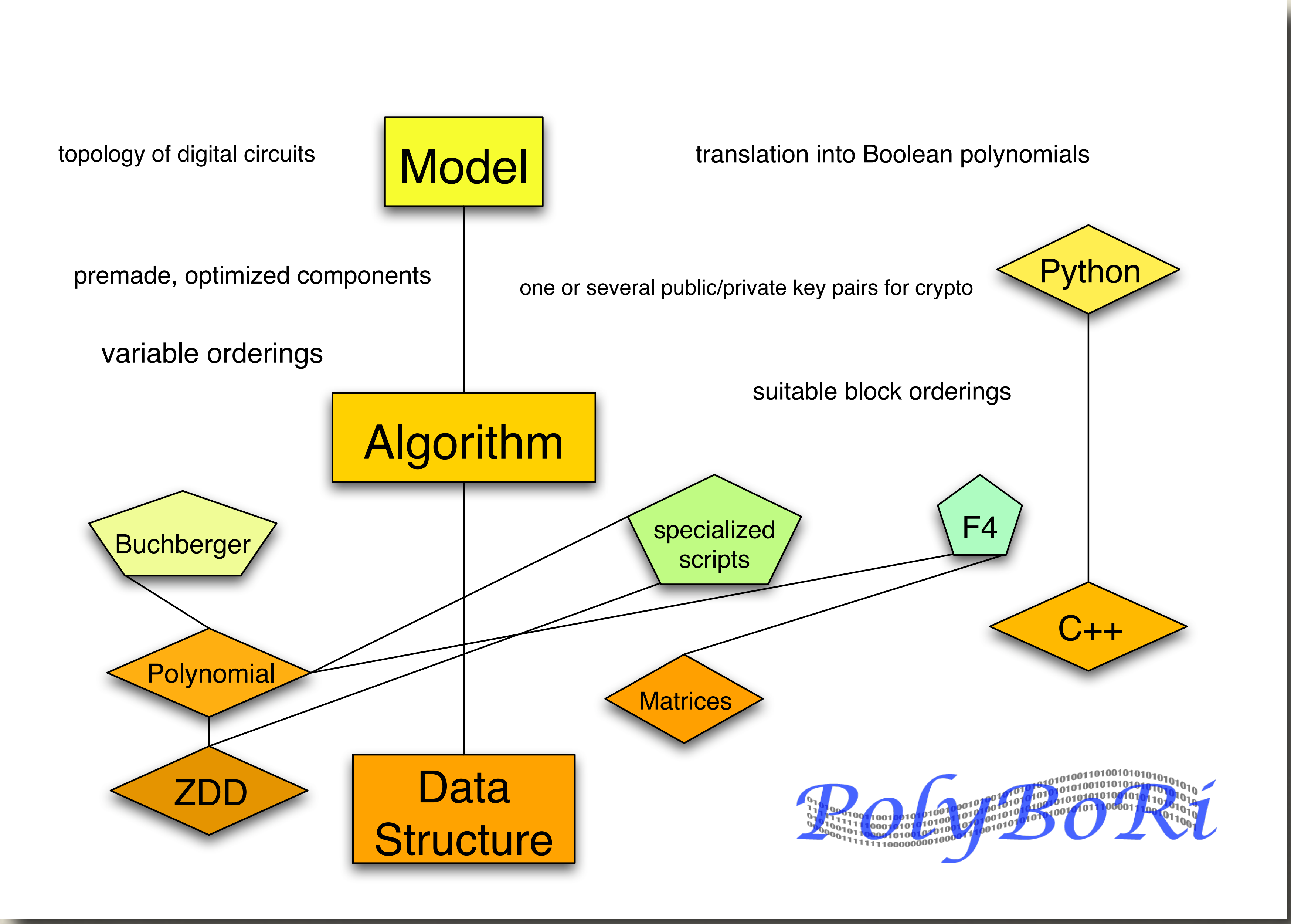
Singular 3-0-1 further libraries

- redcgslib (Reduced Comprehensive Gröbner Systems)
- bfct.lib (Bernstein-Sato polynomial)
- decodegb.lib (Coding theory)

Secret pre-release version

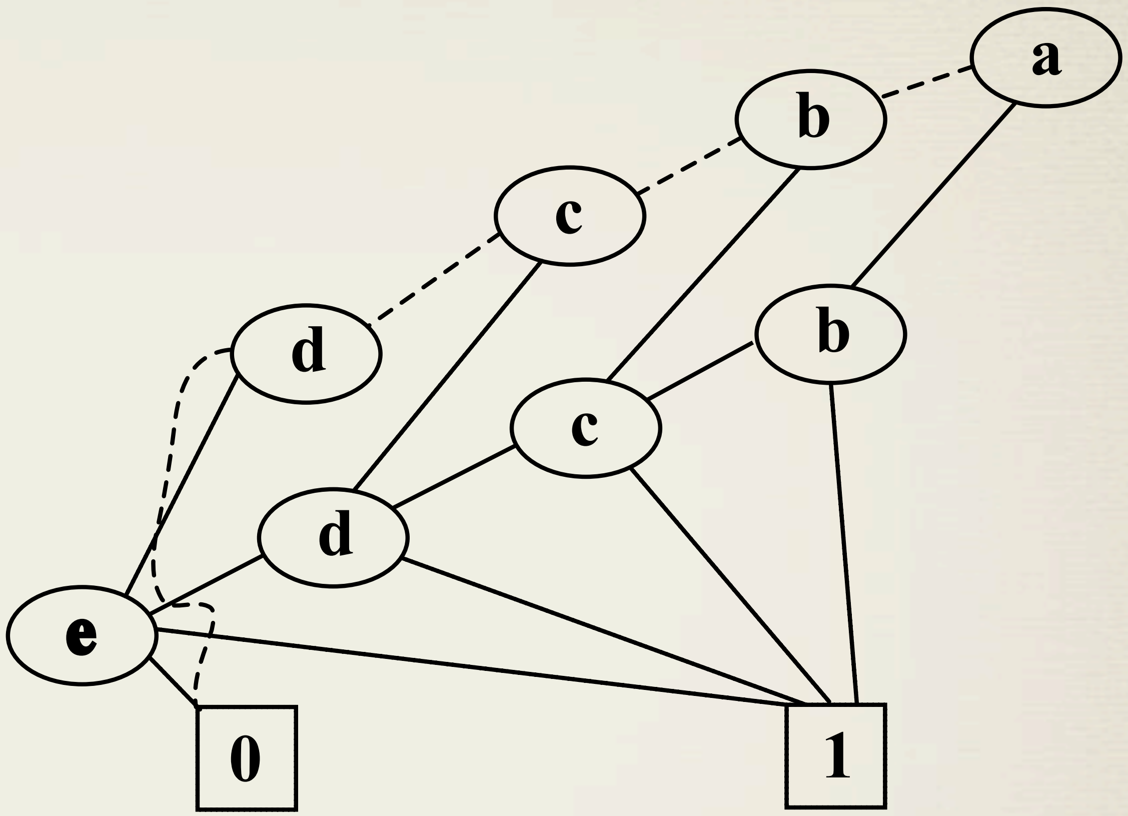
http://www.mathematik.uni-kl.de/ftp/pub/Math/Singular/devel/pre-3-1/

PolyBoRi


topology of digital circuits
Model
translation into Boolean polynomials
premade, optimized components
one or several public/private key pairs for crypto
Python
variable orderings
suitable block orderings
Algorithm
Buchberger
specialized scripts
F4
C++
Polynomial
Matrices
ZDD
Data Structure
PolyBoRi

Decision diagrams

* diagram decides if a term occurs in the polynomial
* term occurs if it exists as path leading to one
* Example: all Boolean terms of degree two



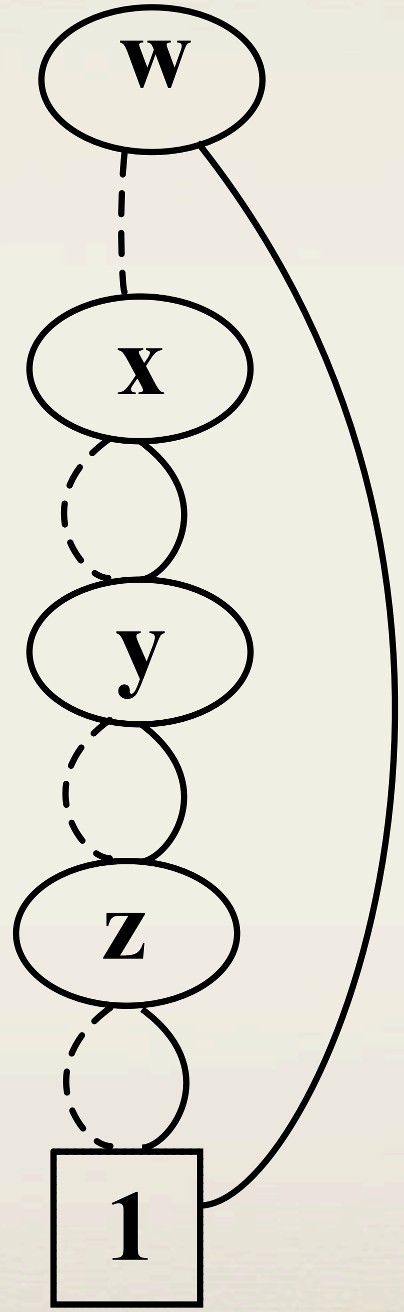
a*b+a*c+a*d+a*e+b*c+b*d
+b*e+c*d+c*e+d*e

Monomial orderings

- Given an monomial ordering > and a polynomial in ZDD form, it is a priori unclear, how to
  - calculate the leading term
  - iterate over the terms (following the monomial ordering)
- Implemented that for:
  - lexicographical orderings (easy)
  - Degree (reverse) lexicographical ordering
  - Block orderings
- For every ordering you have to find a special trick
- no general matrix orderings are supported

Lexicographical Ordering

- Leading Term:
  - always go right
- ordered iteration:
  - begin with lead
  - jump back and go left (repeatedly)



w

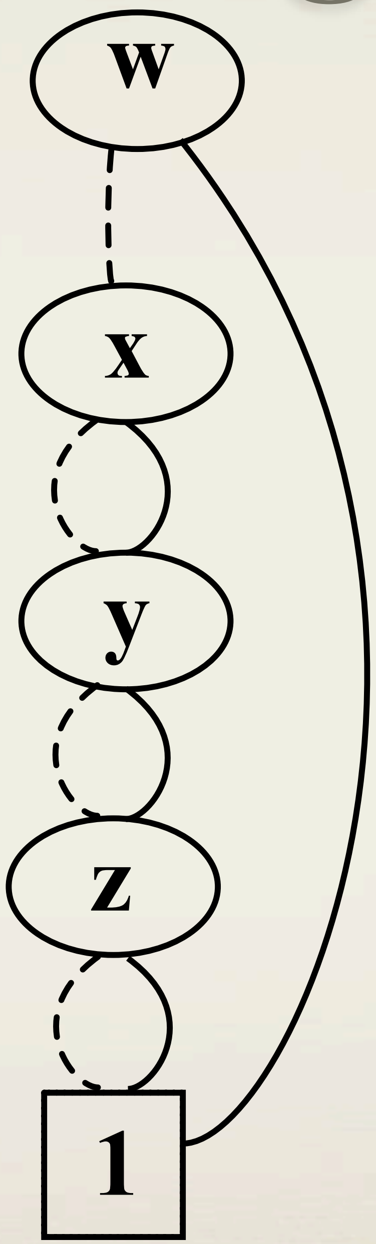
+x*y*z+x*y+x*z+x

+y*z+y

+z+1

Degree Lexicographical Ordering

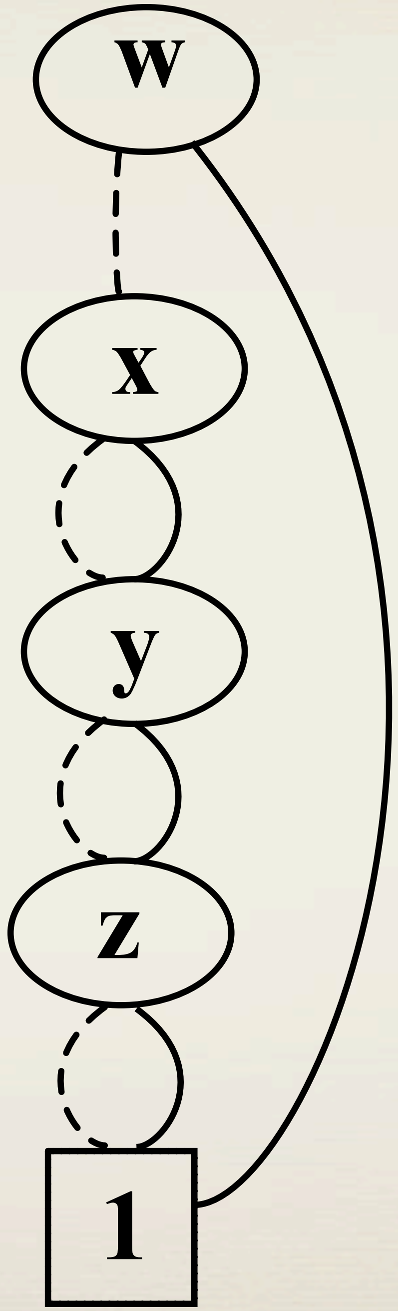
* Leading Term:
    * always go right, if exists max. degree term in this branch
* ordered iteration:
    * iterate lexicographical and jump over terms of „wrong“ degree



x*y*z
+x*y+x*z+y*z
+w+x+y+z
+1

Further Orderings

- Degree Reverse Lexicographical (w<x<y<z)
- Block Orderings
  - dlex blocks
  - degrevlex blocks
  - Example:
    - derevlex: (w),(x,y,z)



x*y*z
+y*z+x*z+x*y
+z+y+x+w
+1

w
+x*y*z
+y*z+x*z+x*y
+z+y+x
+1

Functional Style

* Every manipulation is forbidden/immutable objects
* Pro
  * Operations on polynomials can be cached on the level of diagram nodes
* Contra
  * Always creating new nodes can generate a lot of overhead (every node is garantied to be unique)

Example: canceling every multiple of monom

* p=p.set()
* p=p.diff(p.multiplesOf(monom))
* p=Polynomial(p)

Recursive Implementation

```
template <class CacheType,
  class NaviType, class SetType>
SetType
dd_first_multiples_of(
      const CacheType& cache_mgr,
      NaviType navi, NaviType rhsNavi,
      SetType init){
    typedef typename SetType::dd_type dd_type;

  if(rhsNavi.isConstant())
    if(rhsNavi.terminalValue())
      return cache_mgr.generate(navi);
    else
      return cache_mgr.generate(rhsNavi);

  if (navi.isConstant() || (*navi > *rhsNavi))
    return cache_mgr.zero();

  if (*navi == *rhsNavi)
    return dd_first_multiples_of(
        cache_mgr, navi.thenBranch(),
          rhsNavi.thenBranch(), init).change(*navi);

 // Look up old result - if any
  NaviType result = cache_mgr.find(navi, rhsNavi);

  if (result.isValid())
    return cache_mgr.generate(result);

  // Compute new result
  init = dd_type(*navi,
                 dd_first_multiples_of(
                   cache_mgr, navi.thenBranch(),
                        rhsNavi, init).diagram(),
                 dd_first_multiples_of(cache_mgr,
                   navi.elseBranch(),
                                  rhsNavi, init).diagram() );

  // Insert new result in cache
  cache_mgr.insert(navi, rhsNavi, init.navigation());

  return init;
}
```

Solutions for overhead problem

- Replace many small operations by a few bigger ones
- Accept the overhead, understand the style decision diagram operations should be implemented and win in total by caching
- For a few operations, use alternative data structures (e.g. vectors of integers for Exponents of Boolean monomials)

The structure most different from a ZDD is ... ... a dense matrix

* libm4ri
  * Gregory Bard
  * Martin Albrecht
  * William Hart
  * ...
* a good team:
  * dense matrices for calculation with dense, random like systems
  * ZDDs for structured/sparse polynomials