Symbolic Computation assists Algebraic Cryptanalysis: SCrypt

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Algebraic cryptanalysis

- represent cryptographic problems as systems of polynomial equations
- solve these systems

Cryptosystems	$\rightarrow$	systems of equations	$\rightarrow$	solutions
$f_k : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$		variables: key,		XL, XSL
$f_k(p) = c$		plaintext, ciphertext,		F_4, F_5
		internal state		

Questions

- "suitable" representation
- solution methods

Algebraic cryptanalysis

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Problem

Only a few examples available.

- Write down equations for specific components by hand, use a computer algebra system to put them together.
  - Tedious.
  - Resulting systems are big, hard to manipulate and work with.
- This process can be automated!

Goal

Given the specification of a cryptosystem, allow plugging in different representations of components to generate systems of polynomial equations.

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SCrypt: Symbolic computation & cryptography

- Easy structural description of cryptosystems.
- Compute intermediate states as symbolic expressions.
- Provide different ways of converting symbolic expressions to algebraic models.

```
self.SBox0 = function('SBox0',4,2)
self.SBox1 = function('SBox1',4,2)

Emat = matrix(ZZ, 8, 4, [0,0,0,1,
                         1,0,0,0,
                         0,1,0,0,
                         0,0,1,0,
                         0,1,0,0,
                         0,0,1,0,
                         0,0,0,1,
                         1,0,0,0])
ExpPermuation = MatrixOp('ExpPerm', Emat)

lpmat = matrix(ZZ, 4, 4, [0,1,0,0,
                          0,0,0,1,
                          0,0,1,0,
                          1,0,0,0])
LastPerm = MatrixOp('LastPerm',lpmat)
```

```
F = OpChain(4)
F.chain_op(ExpPermuation)
F.chain_op(KeyAddition)
F.chain_op(ParallelOp('S', [self
    .SBox0, self.SBox1]))
F.chain_op(LastPerm)

FeistelCipher.__init__(self, F,
    8, 10, 1, 0)
```

SCrypt: Symbolic computation & cryptography

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```
sage: sd.states[2]
regs: [p4 + rr_0_5, p5 + rr_0_7, p6 + rr_0_6, p7 + rr_0_4, p0 + rr_0_1, p1 +
    rr_0_3, p2 + rr_0_2, p3 + rr_0_0]
varpref: rr_0_
pref_inds: {'p': 8, 't': 0, 'rr_0_': 8, 'k': 10}
rels: [(SBox0(k0 + p7, k6 + p4, k8 + p5, k3 + p6), [rr_0_0, rr_0_1]), (SBox1(k7 +
    p5, k2 + p6, k9 + p7, k5 + p4), [rr_0_2, rr_0_3]), (SBox0(p3 + rr_0_0, p0 +
    rr_0_1, p1 + rr_0_3, p2 + rr_0_2), [rr_0_4, rr_0_5]), (SBox1(p1 + rr_0_3, p2
    + rr_0_2, p3 + rr_0_0, p0 + rr_0_1), [rr_0_6, rr_0_7])]
```

SCrypt: Symbolic computation & cryptography

- Easy structural description of cryptosystems.
- Compute intermediate states as symbolic expressions.
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```
sage: from scrypt.chain import State
sage: from scrypt.eqgen import EqGen
sage: st = State([])
sage: t0 = st.new_var()
sage: eq = EqGen(st, 4, base_field = GF(2))
sage: eq.sringel_to_polys(t0)
[t0_0, t0_1, t0_2, t0_3]
sage: from scrypt.op import Not
sage: eq.sringel_to_polys(Not(t0))
[t0_0 + 1, t0_1 + 1, t0_2 + 1, t0_3 + 1]
sage: eq = EqGen(st, 4, base_field = GF(4,'a'))
sage: eq.sringel_to_polys(t0)
[t0_0, t0_1]
sage: eq.sringel_to_polys(Not(t0))
[t0_0 + (a + 1), t0_1 + (a + 1)]
```

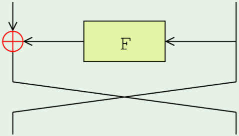
Describing structure

SCrypt provides

- Constructs for common cipher components
  (linear diffusion, rotation, modulo sums, field multiplication, etc.)
- Cryptosystem design patterns
  (block ciphers, Feistel networks, stream ciphers, etc.)
- Framework to connect components together similar to circuit diagram patterns commonly used in design specifications

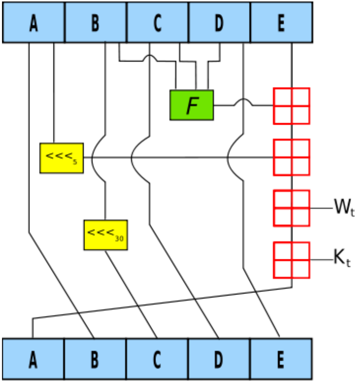
Example

```
ApplyF = BinaryOp('ApplyF',2,0,1,0,func2=F)
HalfRound = OpChain(2, adjust_blocks=True)
HalfRound.chain_op(ApplyF)
HalfRound.chain_op(Swap)
```



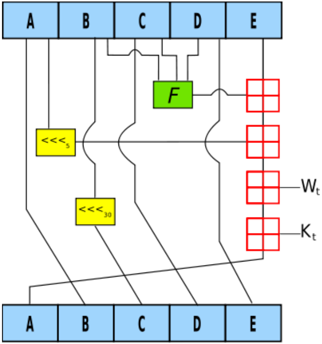
SHA1 compression function

```
#(b and c) or ((not b) and d)
F1 = function('F1', 3, 1)
#b xor c xor d
F2 = function('F2', 3, 1)
#(b and c) or (b and d) or (c and d)
F3 = function('F3', 3, 1)

RotL5 = RotL(5)
F = ParamFunc('F')

@fn2Op(cache_result=False)
def AFunc(state, args, kwds):
    return ModSum(F(*(state[1:4]), **kwds),
        state[4], RotL5(state[0]),
        args[0], args[1])

RotR2 = RotR(2)
CFunc = OpChain(5)
CFunc.chain_op(Proj('P2',2))
CFunc.chain_op(RotR2)

SHA1C = ParallelOp('SHA1C',
        [AFunc , Proj('P1',1), CFunc, Proj('P3', 3), Proj('P4', 4)],
        input_all=True, args_all=True, cache_result=False)
```



SHA1 compression function

```
sage: from scrypt.symb import var
sage: from scrypt.chain import State
sage: s = [var('x' + str(i))
        for i in range(5)]
sage: w0, w1 = var('w0'), var('w1')
sage: res = SHA1C(w0, w1,
        state=State(s), F=F1)
sage: res
regs: [ModSum(x4, w0, w1, RotL5(x0),
    F1(x1, x2, x3)), x1, RotR2(x2), x3, x4]
varpref: t
pref_inds: {'t': 0}
rels: []
sage: w2, w3 = var('w2'), var('w3')
sage: SHA1C(w2, w3, state=res, F=F2)
regs: [ModSum(x4, w2, w3, RotL5(ModSum(x4, w0, w1,
     RotL5(x0), F1(x1, x2, x3))), F2(x1, RotR2(x2)
    , x3)), x1, RotR2(RotR2(x2)), x3, x4]
varpref: t
pref_inds: {'t': 0}
rels: []
```



Symbolic expressions

- Symbolic variables in a ring of characteristic 2
  ($+ \rightarrow$ xor, $\times \rightarrow$ and)
- Symbolic expressions used to denote other constructs
- Implementation based on PolyBoRi in Sage

Example

```
sage: from scrypt.symb import var, function
sage: x, y = var('x'), var('y')
sage: from scrypt.op import FMul, ModSum, Not
sage: FMul16 = FMul(GF(16,'a'))
sage: x + ModSum(FMul16(x, y), Not(x))
x + ModSum(Not(x), FMul16(x, y))
sage: SBox = function('SBox', 2, 2)
sage: SBox(x, y)
SBox(x, y)
```

Symbolic relations

Multiple outputs

If a function has multiple outputs (i.e., an SBox),

- new symbolic variables are created to represent the outputs
- a relation is recorded in the relevant data structure.

Example

```
sage: import ctc
sage: c = ctc.CTC(1, 1)
sage: c.calc_states()
sage: c.states[1]
regs: [k0 + p0, k1 + p1, k2 + p2]
...
sage: c.states[2]
regs: [rr_0_0 + rr_0_1, rr_0_1 + rr_0_2, rr_0_0]
...
rels: [(SBox(k0 + p0, k1 + p1, k2 + p2),
        [rr_0_0, rr_0_1, rr_0_2])]
```

Symbolic expressions to equations

- Specify how many bits in a block and base field,
- SCrypt processes the symbolic expressions to create systems of polynomial equations

Example

```
sage: from scrypt.chain import State
sage: from scrypt.eqgen import EqGen
sage: st = State([])
sage: t0 = st.new_var()
sage: eq = EqGen(st, 4, base_field = GF(2))
sage: eq.sringel_to_polys(t0)
[t0_0, t0_1, t0_2, t0_3]
sage: from scrypt.op import Not
sage: eq.sringel_to_polys(Not(t0))
[t0_0 + 1, t0_1 + 1, t0_2 + 1, t0_3 + 1]
sage: eq = EqGen(st, 4, base_field = GF(4,'a'))
sage: eq.sringel_to_polys(t0)
[t0_0, t0_1]
sage: eq.sringel_to_polys(Not(t0))
[t0_0 + (a + 1), t0_1 + (a + 1)]
```

Example

```
sage: F = GF(16, 'a')
sage: a = F.gen()
sage: from scrypt.symb import constant
sage: from scrypt.op import FMul
sage: FMul16 = FMul(F)
sage: FElem3 = constant('FElem3', a+1)
sage: t1 = st.new_var()
sage: eq = EqGen(st, 4, base_field = GF(2))
sage: eq.sringel_to_polys(FMul16(t0,FElem3))
[t0_0 + t0_3, t0_0 + t0_1 + t0_3, t0_1 + t0_2, t0_2 + t0_3]
sage: eq.sringel_to_polys(t0 + FMul16(t0,Not(t1)))
[t0_0*t1_0 + t0_3*t1_1 + t0_2*t1_2 + t0_1*t1_3 + t0_1 + t0_2 + t0_3, t0_1*t1_0 +
    t0_0*t1_1 + t0_3*t1_1 + t0_2*t1_2 + t0_3*t1_2 + t0_1*t1_3 + t0_2*t1_3 + t0_0
    + t0_1, t0_2*t1_0 + t0_1*t1_1 + t0_0*t1_2 + t0_3*t1_2 + t0_2*t1_3 + t0_3*t1_3
     + t0_0 + t0_1 + t0_2, t0_3*t1_0 + t0_2*t1_1 + t0_1*t1_2 + t0_0*t1_3 + t0_3*
    t1_3 + t0_0 + t0_1 + t0_2 + t0_3]
```