Decomposable Objects and Combinatorial Species

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Decomposable Objects

What are decomposable objects?

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Decomposable objects include trees, graphs, functions, relations, permutations, sets, subsets, cycles, lists, and much more...

Decomposable Objects

What do we want to do with decomposable objects?

- Count them.
- Generate them.
- Generate random ones.
- ...

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(in both the labeled and unlabeled cases)

Existing Software

- combstruct in Maple (Project Algo, Mishna, Murray, and Zimmermann)
- CS in MuPAD (Project Algo, Corteel, Denis, Dutour, Sarron, and Zimmerman)
- decomposableObject in MuPAD-Combinat (Cellier, Hivert, and Thiéry)
- Aldor-Combinat in Aldor/FriCAS (Hemmecke and Rubey)

Aldor-Combinat

- Started in 2006 by Ralf Hemmecke and Martin Rubey.
- Written as a fully literate program in the language of Aldor that tries to stay as close as possible to the theory of species as outlined in "Combinatorial Species and Tree-like Structures" by Bergeron, Labelle, and Leroux.
- Can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/aldor/combinat/

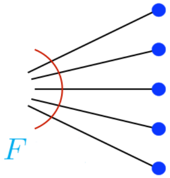
What are species?

Let $\mathbb{B}$ be the category of finite sets with bijections. A *species* is simply a functor

$$F : \mathbb{B} \longrightarrow \mathbb{B}.$$

What are species?

- For every finite set A, we get a finite set $F[A]$ whose elements are said to be the *structures* of F on the underlying set A.

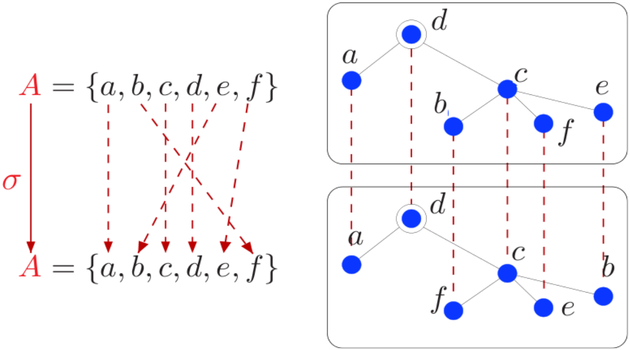


What are species?

- For each bijection $\sigma : A \to B$, we have a bijection

$$F[\sigma] : F[A] \to F[B]$$

which is called the transport of F-structures along σ.



What are species?

- F is *functorial*, which means that
  1. $F[\mathrm{Id}_A] = \mathrm{Id}_{F[A]}$
  2. $F[\psi\sigma] = F[\psi]F[\sigma]$.

Example: Partition Species

We define the species of partitions P by letting $P[A]$ be all set partitions of A.

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$$\begin{aligned} P[\{1,2,3\}] &= [\{\{1,2,3\}\}, \{\{1,3\},\{2\}\}, \{\{1,2\},\{3\}\}, \\ &\quad \{\{2,3\},\{1\}\}, \{\{1\},\{2\},\{3\}\}]. \end{aligned}$$

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Let $\sigma : \{1,2,3\} \to \{1,2,3\}$ be the bijection which sends 2 to 3 and 3 to 2. Then,

$$P[\sigma](\{\{1,3\},\{2\}\}) = \{\{1,2\},\{3\}\}$$

Example: Partition Species

```
sage: P = species.PartitionSpecies()
sage: P.structures([1,2,3]).list()
[{{1, 2, 3}},
 {{1, 3}, {2}},
 {{1, 2}, {3}},
 {{2, 3}, {1}},
 {{1}, {2}, {3}}]
```

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[{{1, 2, 3}},
 {{1, 3}, {2}},
 {{1, 2}, {3}},
 {{2, 3}, {1}},
 {{1}, {2}, {3}}]
```

```
sage: a = _[1]; a
{{1, 3}, {2}}
sage: a.transport(PermutationGroupElement((2,3)))
{{1, 2}, {3}}
```

Building Blocks

- Partitions
- Permutations
- Cycles
- Sets
- Subsets
- Linear orders (sequences)
- Singleton and empty set species

Addition

$$(F + G)[A] = F[A] + G[A]$$

The sum on the right side corresponds to a disjoint union.

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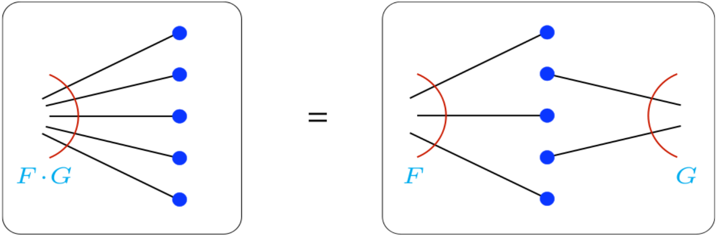
Example:

```
sage: P = species.PartitionSpecies()
sage: P.structures([1,2]).list()
[{{1, 2}}, {{1}, {2}}]

sage: F = P+P
sage: F.structures([1,2]).list()
[{{1, 2}}, {{1}, {2}}, {{1, 2}}, {{1}, {2}}]
```

Multiplication

$$(F \cdot G)[A] = \sum_{B+C=A} F[B] \times G[C]$$



Multiplication

Example:

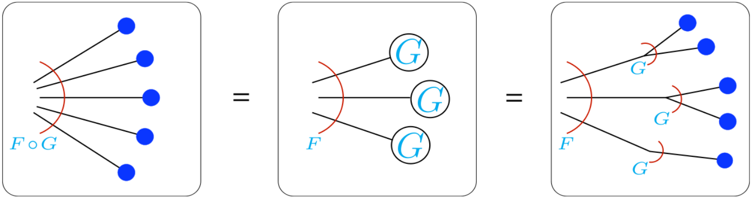
```
sage: P = species.PartitionSpecies()
sage: F = P*P
sage: F.structures([1,2]).list()

[{}*{{1, 2}},
 {}*{{1}, {2}},
 {{1}}*{{2}},
 {{2}}*{{1}},
 {{1, 2}}*{},
 {{1}, {2}}*{}]
```

Substitution

When $G[\emptyset] = \emptyset$,

$$(F \circ G)[A] = \sum_{\pi \in P[A]} F[\pi] \times \prod_{B \in \pi} G[B]$$



Substitution

Example:

```
sage: E = species.SetSpecies()
sage: Eplus = species.SetSpecies(min=1)
sage: F = E(Eplus)
sage: F.structures([1,2,3]).list()

[F-structure: {{1, 2, 3}}; G-structures: [{1, 2, 3}],
 F-structure: {{1, 3}, {2}}; G-structures: [{1, 3}, {2}],
 F-structure: {{1, 2}, {3}}; G-structures: [{1, 2}, {3}],
 F-structure: {{2, 3}, {1}}; G-structures: [{2, 3}, {1}],
 F-structure: {{1}, {2}, {3}}; G-structures: [{1}, {2}, {3}]]
```

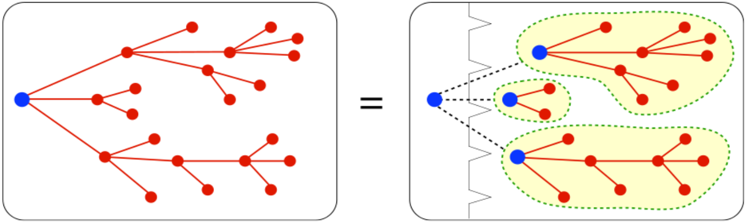
Other Operations

- Functorial composition
- Derivative
- Pointing
- ...

Recursive definition / Implicit Equations

"A rooted tree is a root which is attached to a set of rooted trees."

$$A = X \cdot E(A)$$



Recursive definition / Implicit Equations

"A binary tree is either a leaf or a pair of binary trees."

$$B = X + B * B$$

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Example:

```
sage: B = species.CombinatorialSpecies()
sage: X = species.SingletonSpecies()
sage: B.define(X+B*B)
sage: B.structures([1,2,3]).list()
[1*(2*3),
 1*(3*2),
 ...
 (2*3)*1,
 (3*2)*1]
```

Generating Series

The primary tools used in the theory of species are *generating series*. With each species, we can associate three different generating series:

1. (Exponential) Generating Series
2. Isomorphism Type Generating Series
3. Cycle Index Series

(Exponential) Generating Series

The *(exponential) generating series* of a species F is given by

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

where f_n is the number of elements of $F[A]$ for any A with n elements.

(Exponential) Generating Series

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$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

where f_n is the number of elements of $F[A]$ for any A with n elements.

Example:

The generating series for the species of partitions is given by

$$P(x) = \sum_{n \geq 0} B_n \frac{x^n}{n!}$$

where B_n are the Bell numbers.

(Exponential) Generating Series

Example:

```
sage: P = species.PartitionSpecies()
sage: gs = P.generating_series()
sage: gs.coefficients(5)
[1, 1, 1, 5/6, 5/8]
sage: gs
1 + x + x^2 + 5/6*x^3 + 5/8*x^4 + O(x^5)

sage: gs.counts(5)
[1, 1, 2, 5, 15]
```

Isomorphic Structures

Two structures $a \in F[A]$ and $b \in F[B]$ are said to be isomorphic if there exists a bijection $\sigma : A \to B$ such that

$$F[\sigma](a) = b.$$

Example:

```
sage: a
{{1, 3}, {2}}
sage: b
{{1, 2}, {3}}
sage: a.transport(PermutationGroupElement((2,3)))
{{1, 2}, {3}}
sage: a.is_isomorphic(b)
True
```

Isomorphic Structures

```
sage: P.isotypes([1,2,3,4]).list()
[{{1, 2, 3, 4}},
 {{1, 2, 3}, {4}},
 {{1, 2}, {3, 4}},
 {{1, 2}, {3}, {4}},
 {{1}, {2}, {3}, {4}}]
```

Isomorphic Structures

```
sage: P.isotypes([1,2,3,4]).list()
[{{1, 2, 3, 4}},
 {{1, 2, 3}, {4}},
 {{1, 2}, {3, 4}},
 {{1, 2}, {3}, {4}},
 {{1}, {2}, {3}, {4}}]

sage: B.isotypes([1,2,3,4]).list()
[1*(2*(3*4)),
 1*((2*3)*4),
 (1*2)*(3*4),
 (1*(2*3))*4,
 ((1*2)*3)*4]
```

Isomorphism Type Generating Series

The isomorphism type generating series of F is defined to be

$$\tilde{F}(x) = \sum_{n \geq 0} \tilde{f}_n x^n$$

where $\tilde{f}_n$ is the number of non-isomorphic elements of $F[A]$ for any A with n elements.

Isomorphism Type Generating Series

Example:

The isomorphism type generating series for the species of partitions is given by

$$\tilde{P}(x) = \sum_{n \geq 0} p_n x^n$$

where p_n is the number of integer partitions of n.

Isomorphism Type Generating Series

Example:

```
sage: P = species.PartitionSpecies()
sage: itgs = P.isotype_generating_series()
sage: itgs.coefficients(5)
[1, 1, 2, 3, 5]
sage: itgs
1 + x + 2*x^2 + 3*x^3 + 5*x^4 + O(x^5)
```

Generating Series

The generating series play nicely with the operations on species:

$$(F+G)(x) = F(x) + G(x), \widetilde{(F+G)}(x) = \widetilde{F}(x) + \widetilde{G}(x)$$

$$(F \cdot G)(x) = F(x) \cdot G(x), \widetilde{(F \cdot G)}(x) = \widetilde{F}(x) \cdot \widetilde{G}(x)$$

$$(F \circ G)(x) = F(G(x))$$

Putting It Together

Rooted Trees

```
sage: E = species.SetSpecies()
sage: X = species.SingletonSpecies()
sage: A = species.CombinatorialSpecies()
sage: A.define(X*E(A))
sage: A.isotype_generating_series().coefficients(10)
[0, 1, 1, 2, 4, 9, 20, 48, 115, 286]
sage: sloane_find(_)[0][1]
Searching Sloane's online database...

'Number of rooted trees with n nodes
(or connected functions with a fixed point).'
```

Weighted Species: Ordered Trees

```
sage: q = QQ['q'].gen()
sage: leaf = species.SingletonSpecies()
sage: internal_node = species.SingletonSpecies(weight=q)
sage: L = species.LinearOrderSpecies(min=1)
sage: T = species.CombinatorialSpecies()

sage: T.define(leaf + internal_node*L(T))
sage: T.isotype_generating_series().coefficient(4)
q^3 + 3*q^2 + q
```

Future Work (This Week!)

- Automatically recognizing recurrence relations
- More efficient random generation
- Multisort species
- Plugging in data structures into the generation routines

Thanks!