Shifted combinatorial Hopf algebras from K-theory

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Outline and setup

- Goal: introduce some interesting bases of (quasi)symmetric functions from the perspective of **combinatorial Hopf algebras** and ***K*-theory**.

- Will start with some classical objects (already implemented in Sage), then discuss some semi-classical things (partially implemented), finally talk about new constructions (not yet implemented).

  Results joint w/ **Yu-Cheng Chiu**, **Joel Lewis**, **Brendan Pawlowski**.

- Conventions: all maps f are **linear**, meaning $\mathbb{Z}$-linear with

$$f\left(\sum_{i\in I} a_i\right) = \sum_{i\in I} f(a_i)$$

  even for infinite sums. Choice of scalar ring $\mathbb{Z}$ is mostly arbitrary, could be replaced by any integral domain.

Algebras, coalgebras, and bialgebras

- Two commutative algebras: polynomials $\mathbb{Z}[x]$ and power series $\mathbb{Z}[\![x]\!]$. We have a natural nondegenerate bilinear form $\mathbb{Z}[x] \times \mathbb{Z}[\![x]\!] \to \mathbb{Z}$:

$$\langle f, g \rangle := g(\tfrac{d}{dx}) f(x)\big|_{x=0} \quad \Rightarrow \quad \langle x^m, x^n \rangle = \tfrac{d^n}{dx^n} x^m\big|_{x=0} = n! \cdot \delta_{mn}.$$

- Define $\Delta : \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes \mathbb{Z}[x]$ and $\Delta : \mathbb{Z}[\![x]\!] \to \mathbb{Z}[\![x]\!] \hat{\otimes} \mathbb{Z}[\![x]\!]$ by

$$\langle \Delta(f), g_1 \otimes g_2 \rangle = \langle f, g_1 g_2 \rangle \quad \text{and} \quad \langle f_1 \otimes f_2, \Delta(g) \rangle = \langle f_1 f_2, g \rangle$$

  Here we evaluate $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle := \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle$.

- Both Δ's are linear and co-associative: $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$. Small miracle: **both maps Δ are algebra morphisms**. Can compute

$$\Delta(x^n) = \sum_{i+j=n} \binom{n}{i} x^i \otimes x^j.$$

Conclusion: $\mathbb{Z}[x]$ and $\mathbb{Z}[\![x]\!]$ are **dual bialgebras via the form $\langle \cdot, \cdot \rangle$.**

Antipodes and Hopf algebras

- Suppose H is a bialgebra with product $\nabla : f \otimes g \mapsto fg$, coproduct Δ. The set $\mathrm{End}(H)$ of linear maps $f : H \to H$ is an algebra with product

$$f_1 * f_2 := \nabla \circ (f_1 \otimes f_2) \circ \Delta.$$

The unit of this **convolution algebra** is **not** the identity map id_H. Instead, it is the composition $\iota \circ \epsilon$ of the **unit** and **counit** of H.

- In all examples today, H will be a subset of formal power series, and the composition $\iota \circ \epsilon$ is just the map setting all variables to zero.
- If id_H has 2-sided inverse $\mathbf{S} : H \to H$ for $*$ then H is a **Hopf algebra** with **antipode S**. If $\mathbf{S}$ exists then it is unique, and $\mathbf{S}(ab) = \mathbf{S}(b)\mathbf{S}(a)$.
- Both $\mathbb{Z}[x]$ and $\mathbb{Z}[\![x]\!]$ are Hopf algebras with $\mathbf{S}(x) = -x$ as

$$(\mathbf{S} * \mathrm{id})(x^n) = (\mathrm{id} * \mathbf{S})(x^n) = \sum_{i+j=n} \binom{n}{i}(-x)^i x^j = (x-x)^n = x^n\big|_{x=0}.$$

Malvenuto-Reutenauer algebra and symmetric functions

- A **packed word** $w = w_1 w_2 \cdots w_p$ has $\{w_1, \ldots, w_p\} = \{1, \ldots, n\}$ for some $n \leq p$. If v, w are packed words with $\max(v) = m$ then let
$$v \sqcup\!\sqcup\, w = \sum (\text{shuffles of } v \text{ and } (w_1 + m)(w_2 + m) \cdots (w_p + m)).$$
Example: $21 \sqcup\!\sqcup\, 12 = 3421 + 3241 + 3214 + 2341 + 2314 + 2134 + 2134.$
- $\widehat{\mathbf{Perm}}$ = infinite linear comb's of **permutations** $w \in \bigsqcup_{n\geq 0} S_n$,
Perm = finite linear comb's of permutations $\rightsquigarrow$ both algebras for $\sqcup\!\sqcup$.
- $\widehat{\mathbf{Sym}}$ = symmetric power series in $\mathbb{Z}[\![x_1, x_2, \ldots]\!]$,
Sym = power series in $\widehat{\mathbf{Sym}}$ of bounded degree.
- Define $\langle \cdot, \cdot \rangle : \mathbf{Sym} \times \widehat{\mathbf{Sym}} \to \mathbb{Z}$ by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ for **Schur functions**.
Define $\langle \cdot, \cdot \rangle : \mathbf{Perm} \times \widehat{\mathbf{Perm}} \to \mathbb{Z}$ by $\langle v, w \rangle = \delta_{v^{-1}w}$ for $v, w \in \bigsqcup_{n\geq 0} S_n$.

Theorem

Sym *and* $\widehat{\mathbf{Sym}}$ *(resp.* **Perm** *and* $\widehat{\mathbf{Perm}}$*) are dual Hopf algebras via* $\langle \cdot, \cdot \rangle$*.*

Quasisymmetric functions

For compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ let $M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$.
Define $\widehat{\mathbf{QSym}}$ = infinite linear combinations of M_α's. This is an algebra.

Proposition

$\widehat{\mathbf{QSym}}$ is a Hopf algebra for $\Delta(M_\alpha) := \sum_{i=0}^k M_{(\alpha_1,\ldots,\alpha_i)} \otimes M_{(\alpha_{i+1},\ldots,\alpha_k)}$.

- A **combinatorial Hopf algebra** is a Hopf algebra H with an algebra morphism $\zeta : H \to \mathbb{Z}[\![t]\!]$ satisfying counit condition $\zeta(h)\big|_{t=0} = \epsilon(h)$.
- Call ζ the **character** of H. We view $\widehat{\mathbf{QSym}}$ as a combinatorial Hopf algebra for the character ζ_Q that sets $x_1 = t$ and $x_2 = x_3 = \cdots = 0$.

Theorem (Aguiar-Bergeron-Sottile, 2006)

For each combinatorial Hopf algebra (H, ζ) there is a unique Hopf algebra morphism $\Psi : H \to \widehat{\mathbf{QSym}}$ such that $\zeta = \zeta_Q \circ \Psi$.

Fundamental quasisymmetric functions

For an n-letter word w let $\alpha(w)$ be composition of n giving lengths of maximal increasing subwords. For example $\alpha(\underline{1346}\overline{279}\underline{58}) = (4, 3, 2)$.

Define $\zeta_<(w) = t^n$ if $w = 123 \cdots n$ and $\zeta_<(w) = 0$ if w not increasing.

Proposition

There is a unique Hopf alg. morph. $\Psi : \widehat{\mathbf{Perm}} \to \widehat{\mathbf{QSym}}$ *with* $\zeta_< = \zeta_{\mathrm{Q}} \circ \Psi$. *This map has* $\Psi(v) = \Psi(w)$ *for permutations* v, w *iff* $\alpha(v) = \alpha(w)$.

Define $\boxed{L_\alpha = \Psi(w)}$ for w with $\alpha = \alpha(w)$ and $\boxed{R_\alpha = \sum_{\alpha(w)=\alpha} w \in \mathbf{Perm}}$.

$\{L_\alpha\}$ is basis for $\widehat{\mathbf{QSym}}$, $\{R_\alpha\}$ is basis for a subalgebra $\mathbf{NSym} \subset \mathbf{Perm}$.

Theorem

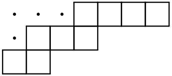
$\mathbf{NSym}$ *and* $\widehat{\mathbf{QSym}}$ *are dual Hopf algebras via form with* $\langle R_\alpha, L_\gamma \rangle = \delta_{\alpha\gamma}$.

A diagram of Hopf algebras

These objects fit into the classical diagram:

$$\begin{array}{ccccc} \mathbf{Sym} & \twoheadleftarrow & \mathbf{NSym} & \hookrightarrow & \mathbf{Perm} \\ | & & | & & | \\ \widehat{\mathbf{Sym}} & \hookrightarrow & \widehat{\mathbf{Q}}\mathbf{Sym} & \twoheadleftarrow & \widehat{\mathbf{P}}\mathbf{erm} \end{array}$$

- The vertical lines indicate dualities via the three forms $\langle \cdot, \cdot \rangle$.
- Each $f : A_1 \hookrightarrow A_2$ is an inclusion and each $g : B_2 \twoheadrightarrow B_1$ is surjective. These maps come in adjoint pairs satisfying $\langle f(a_1), b_2 \rangle = \langle a_1, g(b_2) \rangle$.
- The bottom right map sends $w \mapsto L_{\alpha(w)}$.
- Top left map sends $R_\alpha \mapsto s_{\lambda/\mu}$ where λ/μ is the **ribbon of type α.** For example if $\alpha = (2, 3, 4)$ then $\lambda = (7, 4, 2)$ and $\mu = (3, 1)$ so that

$\lambda/\mu =$ 

Sage implementations

$$
\begin{array}{ccccc}
\mathbf{Sym} & \twoheadleftarrow & \mathbf{NSym} & \hookrightarrow & \mathbf{Perm} \\
| & & | & & | \\
\widehat{\mathbf{Sym}} & \hookrightarrow & \widehat{\mathbf{QSym}} & \twoheadleftarrow & \widehat{\mathbf{Perm}}
\end{array}
$$

- All of the objects here (at least in top row) are in Sage
  See documentation for "Combinatorial Hopf algebras"
- **Perm** is called *FQSym* "free quasisymmetric functions"
- **NSym** is called *NCSF* "non-commutative symmetric functions"
- L_α's are the *QSym.F* basis while R_α's are the *NCSF.ribbon* basis.

Polynomials from K-theory

- Let $\mathbf{Mat}_{n\times n}$ be the set of $n \times n$ matrices over $\mathbb{C}$.
  Let B be the group upper-triangular invertible $n \times n$ matrices over $\mathbb{C}$.
- B acts on $\mathbf{Mat}_{n\times n}$ on left and right, orbits are indexed by (partial) permutation matrices w. Let X_w be the closure of orbit of $w \in S_n$.
- T-equivariant K-theory class of X_w is an inhomogeneous polynomial

$$\mathfrak{G}_w \in \mathbb{Z}[x_1, x_2, \dots, x_n] = K_T(\mathbf{Mat}_{n\times n}).$$

These **Grothendieck polynomials** have a few special properties:

- $\mathfrak{G}_w = \mathfrak{S}_w +$ (higher degree) where $\mathfrak{S}_w$ is **Schubert polynomial**.
- If $w \times 1^k = \left[\begin{array}{c|c} w & 0 \\ \hline 0 & I_k \end{array}\right]$ then $\mathfrak{G}_w = \mathfrak{G}_{w\times 1^k}$.
- If $1^k \times w = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & w \end{array}\right]$ then $G_w := \lim_{k\to\infty} \mathfrak{G}_{1^k\times w} \in \widehat{\mathbf{Sym}} - \mathbf{Sym}$.

Stable Grothendieck polynomials

- If w_λ is the **dominant permutation** of shape λ then $\mathfrak{G}_{w_\lambda} = x^\lambda$.
  Example: if $\lambda = (2,1,1)$ then $w_\lambda = \begin{bmatrix} \square & \square & 1 & \\ \square & 1 & & \\ \square & & & 1 \\ 1 & & & \end{bmatrix}$ and $\mathfrak{G}_{w_\lambda} = x_1^2 x_2 x_3$.
- Define **stable Grothendieck polynomial** $G_\lambda := G_{w_\lambda}$ for partitions λ.
  One has $G_\lambda = s_\lambda +$ (higher degree terms) where s_λ is **Schur function**.
  $\Rightarrow \{G_\lambda\}$ is a basis for $\widehat{\mathbf{Sym}}$, but each G_λ has unbounded degree.
- Define $\{g_\lambda\}$ to be unique symmetric functions with $\langle g_\lambda, G_\mu \rangle = \delta_{\lambda\mu}$.

Theorem (Buch, 2002)

Each G_w and $G_\lambda G_\mu$ is in (finite) $\mathbb{N}$-span$\{G_\nu\} \Rightarrow \{G_\nu\}$ generates a ring.

Later: G_λ and g_λ have certain explicit weight generating functions.

Some authors work with equivalent defn. $G_\lambda^{(\beta)} := \frac{1}{\beta^{|\lambda|}} G_\lambda(\beta x_1, \beta x_2, \dots)$.

K-theoretic Hopf algebras of multipermutations

Define $\sim$ on packed words by $w = w_1 \cdots w_i \cdots w_m \sim w_1 \cdots w_i w_i \cdots w_m$. This means that $121 \sim 1121 \sim 1221 \sim 1211 \sim 11221 \sim \cdots$ and so forth.

- Define $\mathfrak{m}\widehat{\mathbf{Perm}} =$ (infinite linear span of elements $[\![w]\!] := \sum_{v \sim w} v$).
  Define $\mathfrak{M}\mathbf{Perm} =$ (linear span of packed words) $/ \langle v - w : v \sim w \rangle$.[1]
  These spaces are algebras for shifted shuffle product $ш$.
- Bases given by $\{[\![w]\!]\}$ and $\{w\}$ as w ranges over **multipermutations**: packed words like 13243212 with no adjacent repeated letters.
  Let $\langle \cdot, \cdot \rangle : \mathfrak{M}\mathbf{Perm} \times \mathfrak{m}\widehat{\mathbf{Perm}} \to \mathbb{Z}$ be the form with $\langle v, [\![w]\!] \rangle = \delta_{vw}$ for multipermutations v, w.

Theorem (Lam-Pylyavskyy, 2007)

The algebras $\mathfrak{M}\mathbf{Perm}$ and $\mathfrak{m}\widehat{\mathbf{Perm}}$ are dual Hopf algebras via $\langle \cdot, \cdot \rangle$.

[1] This linear span of packed words should be given a different algebra structure dual to the shuffle product ш. The correct definition is a little too involved to include here.

Multifundamental quasisymmetric functions

Recall: $\zeta_<(w) = t^n$ if $w = 123\cdots n$ and $\zeta_<(w) = 0$ for all other w.

Can evaluate $\zeta_<$ on sums $[\![w]\!]$, sends $[\![w]\!] \mapsto t^{\ell(w)}$ if w strictly increasing.

Proposition (Lam-Pylyavskyy, 2007)

$\exists$! Hopf algebra morphism $\Psi_< : \mathfrak{m}\widehat{\mathbf{Perm}} \to \widehat{\mathbf{Q}}\mathbf{Sym}$ with $\zeta_< = \zeta_\mathsf{Q} \circ \Psi_<$. One has $\Psi_<([\![v]\!]) = \Psi_<([\![w]\!])$ for multipermutations v, w iff $\alpha(v) = \alpha(w)$.

Define $\boxed{\tilde{L}_\alpha := \Psi_<([\![w]\!]) \in \widehat{\mathbf{Q}}\mathbf{Sym}}$ for any multiperm w with $\alpha = \alpha(w)$.

Define $\boxed{\tilde{R}_\alpha := \sum_{\alpha(w)=\alpha} [\![w]\!] \in \mathfrak{M}\mathbf{Perm}}$ (sum over multipermutations w).

$\{\tilde{L}_\alpha\}$ is basis for $\widehat{\mathbf{Q}}\mathbf{Sym}$, $\{\tilde{R}_\alpha\}$ is basis for subalgebra $\mathfrak{M}\mathbf{NSym} \subset \mathfrak{M}\mathbf{Perm}$.

Theorem (Lam-Pylyavskyy, 2007)

$\mathfrak{M}\mathbf{NSym}$ and $\widehat{\mathbf{Q}}\mathbf{Sym}$ are dual Hopf algebras via form with $\langle \tilde{R}_\alpha, \tilde{L}_\gamma \rangle = \delta_{\alpha\gamma}$.

A diagram K-theoretic Hopf algebras

These objects fit into the following modified diagram:

$$\begin{array}{ccccc}
\mathbf{Sym} & \twoheadleftarrow & \mathfrak{M}\mathbf{NSym} & \hookrightarrow & \mathfrak{M}\mathbf{Perm} \\
\Big| & & \Big| & & \Big| \\
\widehat{\mathbf{Sym}} & \hookrightarrow & \widehat{\mathbf{QSym}} & \twoheadleftarrow & \mathfrak{m}\widehat{\mathbf{Perm}}
\end{array}$$

- Vertical lines are again dualities via the three forms $\langle \cdot, \cdot \rangle$.
- The $\hookrightarrow$ maps are inclusions, while $\twoheadrightarrow$ maps are adjoint surjections.
- The bottom right map sends $[\![w]\!] \mapsto \tilde{L}_{\alpha(w)}$.
- Top left map sends $\tilde{R}_\alpha \mapsto g_{\lambda/\mu}$ where λ/μ is the **ribbon of type α.**

Theorem (Lam-Pylyavskyy, 2007)

Each G_λ expands as (potentially infinite) $\mathbb{N}$-linear combination of $\tilde{L}_\alpha$'s.

Generating functions for stable Grothendieck polynomials

Recall $s_\lambda = \sum_{T\in\mathsf{SSYT}(\lambda)} x^T$ is sum over **semistandard Young tableaux**.

A **set-valued tableau** of shape λ is a **weakly increasing** filling of λ by **finite nonempty sets** of positive integers, no repetitions in a column:

$$T = \begin{array}{|c|c|c|c|}\hline 12 & 256 & 6 & 6 \\ \hline 34 & 7 \\ \cline{1-2}\end{array} \in \mathsf{SVT}(\lambda) \quad \text{and} \quad x^T = x_1x_2^2x_3x_4x_5x_6^3x_7.$$

A **reverse plane partition (RPP)** of shape λ is a weakly increasing filling T of λ by positive integers. The **weight** of T is $\mathrm{wt}(T) = (a_1, a_2, \dots)$ where a_i is number of **distinct columns** containing i:

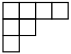
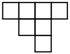
$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 1 & 3 \\ \cline{1-2}\end{array} \in \mathsf{RPP}(\lambda) \quad \text{and} \quad x^{\mathrm{wt}(T)} = x_1x_2^3x_3.$$

Theorem (Buch, 2002; Lam-Pylyavskyy, 2007)

One has $G_\lambda = \sum_{T\in\mathsf{SVT}(\lambda)} x^T$ *and* $g_\lambda = \sum_{T\in\mathsf{RPP}(\lambda)} (-1)^{|\lambda|}(-x)^{\mathrm{wt}(T)}$.

Shifted set-valued tableaux

Fix a **strict partition** $\lambda = (\lambda_1 > \cdots > \lambda_k > 0)$ with all distinct parts.
The **shifted diagram** of λ is formed by shifting row i to the right by $i - 1$:

$\lambda = (4, 2, 1) =$  $\rightsquigarrow$ 

A **shifted set-valued tableau** of shape λ is a weakly increasing filling of shifted diagram by **finite nonempty subsets** of $\{1' < 1 < 2' < 2 < \ldots\}$, no primed (resp., unprimed) numbers repeated in a row (resp., column):

$$T = \begin{array}{|c|c|c|c|} \hline 1'1 & 1 & 23' & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 3'3 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \in \mathrm{ShSVT}(\lambda) \quad \text{and} \quad x^T = x_1^3 x_2^2 x_3^4 x_4.$$

Define $\boxed{GQ_\lambda := \sum_{T \in \mathrm{ShSVT}(\lambda)} x^T}$ and $\boxed{GP_\lambda := \sum_{\substack{T \in \mathrm{ShSVT}(\lambda) \\ \text{no primes on diagonal}}} x^T}$

Shifted stable Grothendieck polynomials

For strict λ: $GQ_\lambda = Q_\lambda +$ (higher order) and $GP_\lambda = P_\lambda +$ (higher order).

Theorem (Ikeda-Naruse, 2013)

Both $\{GQ_\lambda\}$ and $\{GP_\lambda\}$ are linearly independent subsets of $\widehat{\mathbf{Sym}} - \mathbf{Sym}$.

Let $\widehat{\mathbf{G}}\mathbf{Q}$ and $\widehat{\mathbf{G}}\mathbf{P}$ be infinite linear spans of $\{GQ_\lambda\}$ and $\{GP_\lambda\}$.

Theorem (Ikeda-Naruse, Clifford-Thomas-Yong, 2014; Lewis-M., 2022+)

Both $\widehat{\mathbf{G}}\mathbf{Q}$ and $\widehat{\mathbf{G}}\mathbf{P}$ are subalgebras of $\widehat{\mathbf{Sym}}$. More strongly, the sets $\{GQ_\lambda\}$ and $\{GP_\lambda\}$, λ ranging over strict partitions, each generate a ring.

Theorem (M.-Pawlowski, 2020)

*GQ_λ and GP_λ are stable limits of equivariant K-theory representatives of B-orbit closures in varieties of **symmetric** and **skew-symmetric matrices**.*

One has $Q_\lambda = 2^{\ell(\lambda)} P_\lambda$. Likewise GQ_λ is a finite $\mathbb{Z}$-linear comb of GP_μ's.

Shifted reverse plane partitions

Continue to let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0)$ be a strict partition.

A **shifted RPP** of shape λ is a weakly increasing filling T of shifted diagram by numbers in $\{1' < 1 < 2' < 2 < \dots\}$. The **weight** of T is

$$\mathrm{wt}(T) = (a_1 + b_1, a_2 + b_2, \dots)$$

where a_i (resp., b_i) counts **columns** (resp., **rows**) containing i (resp., i'):

$$T = \begin{array}{|c|c|c|c|} \hline 1' & 1' & 3 & 3 \\ \hline \end{array}\begin{array}{c} \\ \end{array}$$

$$T = \begin{matrix} 1' & 1' & 3 & 3 \\ & 2 & 3 & \\ & & 3 & \end{matrix} \in \mathrm{ShRPP}(\lambda) \quad \text{and} \quad x^{\mathrm{wt}(T)} = x_1 x_2 x_3^2.$$

Define

$$gq_\lambda := \sum_{T \in \mathrm{ShRPP}(\lambda)} (-1)^{|\lambda|} (-x)^{\mathrm{wt}(T)}$$

$$gp_\lambda := \sum_{\substack{T \in \mathrm{ShRPP}(\lambda) \\ \text{all diagonal entries primed}}} (-1)^{|\lambda|} (-x)^{\mathrm{wt}(T)}.$$

Shifted dual stable Grothendieck polynomials

For strict λ: $gq_\lambda = Q_\lambda +$ (lower order) and $gp_\lambda = P_\lambda +$ (lower order).

Theorem (Lewis-M., 2022+)

Both $\{gq_\lambda\}$ and $\{gp_\lambda\}$ are linearly independent subsets of **Sym**.

Theorem (Lewis-M., 2022+)

Both $\mathbf{gq} := \mathbb{Z}$*-span*$\{gq_\lambda\}$ *and* $\mathbf{gp} := \mathbb{Z}$*-span*$\{gp_\lambda\}$ *are algebras.*

Define bilinear forms $[\cdot,\cdot] : \mathbf{gq} \times \widehat{\mathbf{GP}} \to \mathbb{Z}$ and $[\cdot,\cdot] : \mathbf{gp} \times \widehat{\mathbf{GQ}} \to \mathbb{Z}$ by

$$[gq_\lambda, GP_\mu] = [gp_\lambda, GQ_\mu] = \delta_{\lambda\mu}. \quad \text{This form is not } \langle\cdot,\cdot\rangle \neq [\cdot,\cdot].$$

Theorem (Lewis-M., 2022+)

$\mathbf{gq}$ *and* $\widehat{\mathbf{GP}}$ *(resp.,* $\mathbf{gp}$ *and* $\widehat{\mathbf{GQ}}$*) are dual Hopf algebras via* $[\cdot,\cdot]$.

These results were conjectured by Nakagawa and Naruse (2018).

A diagram of shifted K-theoretic Hopf algebras

These objects fit into larger diagram of Hopf algebras (not all yet defined):

$$\begin{array}{ccccccccc}
\mathbf{gp} & \twoheadleftarrow & \mathfrak{M}\mathbf{Peak}_P & \hookrightarrow & \mathfrak{M}\mathbf{Perm} & \hookleftarrow & \mathfrak{M}\mathbf{Peak}_Q & \twoheadrightarrow & \mathbf{gq} \\
| & & | & & | & & | & & | \\
\widehat{\mathbf{GQ}} & \hookrightarrow & \widehat{\Pi}\mathbf{Sym}_Q & \twoheadleftarrow & \mathfrak{m}\widehat{\mathbf{Perm}} & \twoheadrightarrow & \widehat{\Pi}\mathbf{Sym}_P & \hookleftarrow & \widehat{\mathbf{GP}}
\end{array}$$

- Hopf algebras $\widehat{\Pi}\mathbf{Sym}_Q$ and $\widehat{\Pi}\mathbf{Sym}_P$ have bases $\{K_\alpha\}$ and $\{\bar{K}_\alpha\}$ indexed by **peak compositions** α with $\alpha_i \geq 2$ for $i < \ell(\alpha)$.

Theorem (Lewis-M., 2019)

GP_λ expands positively into $\bar{K}_\alpha$'s and GQ_λ expands positively into K_α's.

- Hopf algebras $\mathfrak{M}\mathbf{Peak}_Q$ and $\mathfrak{M}\mathbf{Peak}_P$ are free as algebras, with generators $\{\pi q_n\}$ and $\{\pi p_n\}$. The top left and right maps are surjections sending $\pi q_n \mapsto gq_\lambda$ and $\pi p_n \mapsto gp_\lambda$ for $\lambda = (n)$.
- If scalars are $\mathbb{Q}$ not $\mathbb{Z}$ then P- and Q-versions of each object coincide.

Multi-peak quasisymmetric functions

For an n-letter word w let $\alpha_{\text{peak}}(w)$ be composition of n giving lengths of maximal "$\vee$" subwords. For example $\alpha_{\text{peak}}(\underline{321234}\overline{32123}\underline{2}) = (6, 5, 1)$.

- Recall $\zeta_<([\![w]\!]) = t^n$ if $w = 123\cdots n$ and $\zeta_<([\![w]\!]) = 0$ otherwise.
- Define $\zeta_>([\![w]\!]) = t^n$ if $w = n\cdots 321$ and $\zeta_>([\![w]\!]) = 0$ otherwise.
- Also $\exists$ unique morphism $\Psi_> : \mathfrak{m}\widehat{\mathbf{Perm}} \to \hat{\mathbf{Q}}\mathbf{Sym}$ with $\zeta_> = \zeta_{\mathrm{Q}} \circ \Psi_>$. But $\left\{\Psi_>([\![w]\!]) : \text{multiperms } w\right\} = \left\{\Psi_<([\![w]\!]) : \text{multiperms } w\right\}$.

To get something new, let $\zeta_{>|<} := \nabla \circ (\zeta_> \otimes \zeta_<) \circ \Delta : \mathfrak{m}\widehat{\mathbf{Perm}} \to \mathbb{Z}[\![t]\!]$.

Proposition (Lewis-M., 2019)

$\exists$! Hopf algebra morphism $\Psi : \mathfrak{m}\widehat{\mathbf{Perm}} \to \hat{\mathbf{Q}}\mathbf{Sym}$ with $\zeta_{>|<} = \zeta_{\mathrm{Q}} \circ \Psi$. One has $\Psi([\![v]\!]) = \Psi([\![w]\!])$ for multiperms v, w iff $\alpha_{\text{peak}}(v) = \alpha_{\text{peak}}(w)$.

Define $\boxed{K_\alpha := \Psi([\![w]\!]) \in \hat{\mathbf{Q}}\mathbf{Sym}}$ for any multiperm with $\alpha = \alpha_{\text{peak}}(w)$.

Multi-peak algebras

Restrict α to **peak compositions**: all parts but last must be at least two.

Can (indirectly) define $\bar{K}_\alpha$ by relation $\boxed{K_\alpha = \sum_{\delta \in \{0,1\}^{\ell(\alpha)}} 2^{\ell(\alpha)-|\delta|} \bar{K}_{\alpha+\delta}}$.

Let $\widehat{\Pi}\mathbf{Sym}_Q$ and $\widehat{\Pi}\mathbf{Sym}_P$ be infinite linear spans of $\{K_\alpha\}$ and $\{\bar{K}_\alpha\}$.

Theorem (Lewis-M., 2019)

$\widehat{\Pi}\mathbf{Sym}_Q$ and $\widehat{\Pi}\mathbf{Sym}_P$ are subalgebras of $\widehat{\mathbf{Q}}\mathbf{Sym}$ with bases $\{K_\alpha\}$ and $\{\bar{K}_\alpha\}$.

Define $\boxed{\pi p_\alpha := \sum_{\alpha_{\mathrm{peak}}(w)=\alpha} w \in \mathfrak{M}\mathbf{Perm}}$ (over multipermutations w).

Define $\boxed{\pi q_\alpha := \sum_{\delta \in \{0,1\}^{\ell(\alpha)}} 2^{\ell(\alpha)-|\delta|} \pi p_{\alpha-\delta} \in \mathfrak{M}\mathbf{Perm}}$.

Then $\{\pi p_\alpha\}$ and $\{\pi q_\alpha\}$ are bases for subalgebras $\mathfrak{M}\mathbf{Peak}_P$ and $\mathfrak{M}\mathbf{Peak}_Q$.

Theorem (Lewis-M., 2022+)

The algebras $\mathfrak{M}\mathbf{Peak}_P$ and $\widehat{\Pi}\mathbf{Sym}_Q$ (resp., $\mathfrak{M}\mathbf{Peak}_Q$ and $\widehat{\Pi}\mathbf{Sym}_P$) are dual Hopf algebras via the bilinear form with $[\pi p_\alpha, K_\gamma] = [\pi q_\alpha, \bar{K}_\gamma] = \delta_{\alpha\gamma}$.

Open problems

- Schur expansions of G_λ and g_λ are known (Lenart, 2000).
- Expansion of GP_λ into G_μ's is known. (M.-Pawlowski, 2020).
- Littlewood-Richardson rules to expand products $G_\lambda G_\mu$, $g_\lambda g_\mu$, and $GP_\lambda GP_\mu$ are known (Buch, 2000; Clifford-Thomas-Yong, 2014).
- Most antipode formulas known (in terms of certain conjugate bases).

- **Expansion of GQ_λ into G_μ's not yet known.**
  Expansion of gp_λ and gq_λ into g_μ's not yet known.
- **LR rule to expand $GQ_\lambda GQ_\mu \in \mathbb{N}$-span$\{GQ_\nu\}$ not yet known.**
  LR rule to expand $gp_\lambda gp_\mu \in \mathbb{N}$-span$\{gp_\nu\}$ not yet known.
  LR rule to expand $gq_\lambda gq_\mu \in \mathbb{N}$-span$\{gq_\nu\}$ not yet known.

- Great opportunities to implement these symmetric functions in Sage.

Thanks!