Computing L-functions

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$s = \sigma + it$

Given an L-function

- $$\boxed{L(s) = \sum_{1}^{\infty} \frac{b(n)}{n^s}}, \quad b(n) = O(n^\varepsilon)$$
  abs. conv $\sigma > 1$

- (Euler product)

- analytic or meromorphic continuation to $\mathbb{C}$

- functional eqn

  let $\Gamma_{\gamma,\lambda}(s) = \prod_{j=1}^{a} \Gamma(\gamma_j s + \lambda_j)$

  $$\Lambda(s) = Q^s \Gamma_{\gamma,\lambda}(s) L(s)$$

  where $Q, \gamma_j > 0$, $\mathrm{Re}\,\lambda_j \geq 0$

fnctnl eqn

$$\boxed{\Lambda(s) = \omega \overline{\Lambda(1-\bar{s})}, \quad |\omega| = 1}$$

$b(n)$'s are normalized so that critical line is $\sigma = 1/2$.

How to compute $L(s)$

Naive approach - use the Dirichlet series

Works better for larger σ.

$$\sum_{n \le x} \frac{b(n)}{n^s} \underset{\text{sum by parts}}{=} \frac{\sum_{n \le x} b(n)}{x^s} + s \int_1^x \sum_{n \le t} b(n) \frac{dt}{t^{s+1}}$$

Say $\sum_{n \le t} b(n) = O(t^{\sigma_0})$

Then for $\sigma > \sigma_0$, tail end equals

$$\boxed{\sum_{n > x} \frac{b(n)}{n^s}} = s \int_x^\infty \sum_{n \le t} b(n) \frac{dt}{t^{s+1}} - \frac{\sum_{n \le x} b(n)}{x^s}$$

$$= \boxed{O_s\left(x^{\sigma_0 - \sigma}\right)} \quad \text{as } x \to \infty$$

exs

1) $\zeta(s)$, $b(n) \equiv 1$, $\sum_{n \leq x} b(n) = O(x)$

tail: $O(x^{1-\sigma})$, $\sigma > 1$

2) $\zeta(s)\left(1 - \frac{1}{2^{s-1}}\right) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} \cdots$

$b(n) = (-1)^{n-1}$, $\sum_{n \leq x} b(n) = O(1)$

tail: $O(x^{-\sigma})$, $\sigma > 0$

3) $L(s, \chi)$, χ a non-trivial Dirichlet character mod q. $\sum_{n \leq x} \chi(n) = O_q(1)$

tail: $O(x^{-\sigma})$, $\sigma > 0$

Typically, what does one expect for a degree-k L-function? Degree-k means k Γ-factors all of the form $\Gamma\left(\frac{s}{2}+\lambda_j\right)$

Can make a guess based on a prototypical example of degree-k: $\zeta(s)^k$

Problem - $\zeta(s)^k$ is not typical. It has a k-th order pole at $s=1$.

For entire L-functions one expects to have cancellation in $\sum_{n \le x} b(n)$

Notice that the Dirichlet coefficients of $\zeta(s)^k$ are all positive, no cancellation.

$$\sum_{n\le x} b(n) = \frac{1}{2\pi i}\int_{(c)} L(s)\, x^s \frac{ds}{s}, \quad \begin{array}{l} c>1 \\ x>0,\ x\notin \mathbb{Z}. \end{array}$$

Inspiration

$$\zeta(s)^k = \sum \frac{d_k(n)}{n^s},$$

$d_k(n)$ - number of ways to express n as a product of d factors.

$$D_k(x) = \sum_{n\le x} d_k(n)$$

$$= \underbrace{x P_k(\log x)}_{\substack{\text{residue of } \zeta(s)^k \\ \text{at } s=1,\ P_k \text{ polynomial} \\ \text{of degree } k-1}} + \Delta_k(x)$$

example

$$D_2(x) = x\log x + (2\gamma-1)x + \Delta_2(x)$$

Old Conjecture (divisor problem)

$$\Delta_k(x) = O\left(x^{\frac{k-1}{2k}+\varepsilon}\right)$$

suggests: $\sum_{n\le x} b(n) = O\left(x^{\frac{k-1}{2k}+\varepsilon}\right)$ for L-functions without poles

So, Dirichlet series of primitive

- degree 2 L-functions (associated to cusp or Maass forms) should converge for $\sigma > 1/4$, with tail $O(x^{1/4+\varepsilon-\sigma})$

- degree 3 L-functions (ex symmetric square) should converge for $\sigma > \frac{1}{3}$, with tail $O(x^{\frac{1}{3}+\varepsilon-\sigma})$.

ex: $x = 10^6$ gives about 4 digits precision on the $\sigma = 1$ line. To get 16 digits, we'd need $x = 10^{24}$. Yikes!

Method 2 Euler Maclaurin Summation

Useful when ~~b~~ $a(n)$'s are periodic, for ex. $\zeta(s)$ or $L(s,\chi)$.

Euler-Mac formula

$K \in \mathbb{Z}, \geq 1$.

$g^{(K)}$ exists, continuous on $[a,b]$

$$\boxed{\sum_{a<n\leq b} g(n) = \int_a^b g(t)\,dt + \sum_{k=1}^{K} \frac{(-1)^k B_k}{k!}\left(g^{(k-1)}(b) - g^{(k-1)}(a)\right) + \frac{(-1)^{K+1}}{K!}\int_a^b B_K(\{t\})\, g^{(K)}(t)\,dt}$$

Bernoulli Polynomials / Numbers

$$B_0(t) = 1$$

$$B_k'(t) = k B_{k-1}(t), \quad k \geq 1$$

$$\int_0^1 B_k(t)\,dt = 0, \quad k \geq 1.$$

$1, \ t - 1/2, \ t^2 - t + 1/6, \ \text{etc}$

$$B_k = B_k(0)$$

$$B_k(\{t\}) = -k! \sum_{m \neq 0} \frac{e^{2\pi i m t}}{(2\pi i m)^k}, \quad k \geq 1$$

match $\pm m$ in $k=1$ case, and $t \notin \mathbb{Z}$ when $k=1$.

$$B_{2k} = (-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k), \quad k \geq 1$$

$$B_{2k+1} = 0, \quad k \geq 1$$

Apply to compute $\zeta(s)$

$$\zeta(s) = \sum_{1}^{N} n^{-s} + \sum_{N+1}^{\infty} n^{-s}$$

$$\sum_{N+1}^{\infty} \frac{1}{n^s} = \frac{N^{1-s}}{s-1} + \sum_{k=1}^{\underline{K}} \binom{s+k-2}{k-1} \frac{B_k}{k} N^{-s-k+1}$$

$$- \binom{s+\underline{K}-1}{\underline{K}} \int_N^\infty B_{\underline{K}}(\{t\})\, t^{-s-\underline{K}}\, dt,$$

$$\sigma > -\underline{K}+1$$

$\underline{K} = 2k_0$, even integer. Then:

$$|B_{\underline{K}}(\{t\})| \underset{\substack{\text{from}\\\text{Fourier}\\\text{series}}}{\leq} B_{\underline{K}}(0) = B_K$$

So, for $\sigma > -2k_0+1$

$$\left| \binom{s+2k_0-1}{k_0} \int_N^\infty B_{2k_0}(\{t\}) t^{-s-2k_0} \, dt \right|$$

$$\leq \frac{|s+2k_0-1|}{\sigma+2k_0-1} \cdot \left| \text{last term taken in sum} \right|$$

$$\leq \frac{\zeta(2k_0)}{\pi N^\sigma} \frac{|s+2k_0-1|}{\sigma+2k_0-1} \underbrace{\prod_{j=0}^{2k_0-2} \frac{|s+j|}{2\pi N}}$$

win if $2\pi N$ exceeds $|s|, |s+1|, \ldots |s+2k_0-2|$

If

$\sigma \geq 1/2$

$2\pi N \geq 10 |s+2k_0-2|$, so $N = O(|t|)$.

$2k_0-1 > \text{Digits} + \frac{1}{2}\log_{10}|s+2k_0-1|$

gives:

$$\boxed{< 10^{-\text{Digits}}}$$

so we simply ignore it.

<u>An overlooked fact</u>: this can be made much more efficient (competitive with Riemann-Siegel) if, instead of throwing away the $B_K(\{t\})$ integral, we expand $B_K(\{t\})$ into its fourier series, truncate, and integrate term by term:

Each term contributes

$$K! \binom{s+K-1}{K} \frac{1}{(2\pi i m)^K} \int_N^\infty e^{2\pi i m t}\, t^{-s-K}\, dt$$

Throw away terms $|m| > M$ at a total cost

$$< \frac{1}{N^\sigma} \frac{|s+K-1|}{\sigma+K-1} \prod_{j=0}^{K-2} \frac{|s+j|}{2\pi N} \underbrace{\left(\sum_{m+1}^{\infty} \frac{2}{m^K} \right)}_{< \frac{2}{(K-1)M^{K-1}}}$$

$$< \frac{2}{(K-1) N^\sigma} \frac{|s+K-1|}{\sigma+K-1} \underbrace{\prod_{j=0}^{K-2} \frac{|s+j|}{2\pi M N}}$$

win when $2\pi M N$ exceeds $|s|, |s+1|, \ldots |s+K-2|$

For example, for $\sigma \geq 1/2$,

choose $K > \text{Digits} + \log_{10}(|s+K-1|) + 1$,

$M = N$, with

$$2\pi M N \geq 10\,|s+K-2|$$

so $M = N$
$= O(|t|^{1/2})$

gives $< 10^{-\text{Digits}}$
for the neglected terms.

One can get closer
to $M = N \sim \frac{|s|^{1/2}}{(2\pi)^{1/2}}$
by choosing K larger.

Drawback: The individual terms summed get somewhat large compared to final result, since the binomial coefficients have numerator

$(s+k-2)\cdots(s+1)s$, while the denominator N^{+s+k-1} is now smaller

Leads to cancellation, so we need extra precision to capture the cancellation:

$O((\text{Digits} + \log(|s|))\log(|s|))$ working precision required

To evaluate terms $|m| \le M$, assume $\mathbb{K}$ is even, so that $\pm m$ together involve

$$\int_{N}^{\infty} \cos(2\pi m t)\, t^{-s-\mathbb{K}}\, dt$$

$$= (2\pi m)^{s+\mathbb{K}-1} \int_{2\pi m N}^{\infty} \cos(u)\, u^{-s-\mathbb{K}}\, du$$

But

$$\int_{w}^{\infty} \cos(u) u^{z-1} du = \frac{1}{2}\left(e^{-\frac{\pi i z}{2}} \Gamma(z, iw) + e^{\frac{\pi i z}{2}} \Gamma(z, -iw) \right)$$

with $\Gamma(z,w)$ the incomplete gamma function. more on this soon.

Paris (1994) does something related.

Riemann-Siegel formula

$$\tfrac{1}{2} \le \sigma \le 2 \quad , \quad m = \left\lfloor (t/2\pi)^{1/2} \right\rfloor$$

$$\zeta(s) = \sum_{1 \le n \le m} \frac{1}{n^s} + \frac{\chi(s)}{n^{1-s}}$$

$$+ (-1)^{m-1} (2\pi t)^{\frac{s-1}{2}} \exp\left(-\frac{i\pi(s-1)}{2} - \frac{it}{2} - \frac{i\pi}{8} \right) \Gamma(1-s) T_N(s)$$

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)$$

$$T_N(s) = \sum_{0}^{N-1} \sum_{r \le n/2} \frac{n!\, i^{r-n}}{r!(n-2r)!\, 2^n} \left(\frac{2}{\pi}\right)^{\frac{n}{2}-r} a_n(s)\, \psi^{(n-2r)}(2\nu)$$

$$+ O\left(t^{-N/6}\right)$$

Gabcke obtained a sharper bound with explicit constants when $\sigma = 1/2$

$$\nu = \left\{ (t/2\pi)^{1/2} \right\}$$

$$\psi(u) = \frac{\cos \pi \left(\frac{1}{2}u^2 - u - \frac{1}{8} \right)}{\cos \pi u}$$

$$a_0(s) = 1 \ , \ a_1(s) = \frac{\sigma - 1}{t^{1/2}} \ , \ a_2(s) = \frac{(\sigma-1)(\sigma-2)}{2t}$$

$$(n+1)t^{1/2} a_{n+1}(s) = (\sigma - n - 1) a_n(s) + i a_{n-2}(s) \ , \ \text{for } n \ge 2.$$

Smooth Approximate functional equation

Besides analytic continuation, functional equation, we need a very mild <u>growth condition</u> on $L(s)$:

for any $\alpha \leq \beta$, $L(\sigma+it) = O(\exp(t^A))$ for some $A>0$, as $|t| \to \infty$, $\alpha \leq \sigma \leq \beta$ the implied constant and A depending on α, β.

Then, in fact, Phragmen-Lindelöf thm:

$\boxed{L(s) = O(|t|^b)}$ for some $b>0$ depending on α, β.

Assume $\Lambda(s)$ is meromorphic with simple poles at $s_1, \dots s_\ell$ and corresponding residues $r_1, \dots r_\ell$. (multiple order poles can be dealt with too)

Let $g: \mathbb{C} \to \mathbb{C}$, entire, satisfying

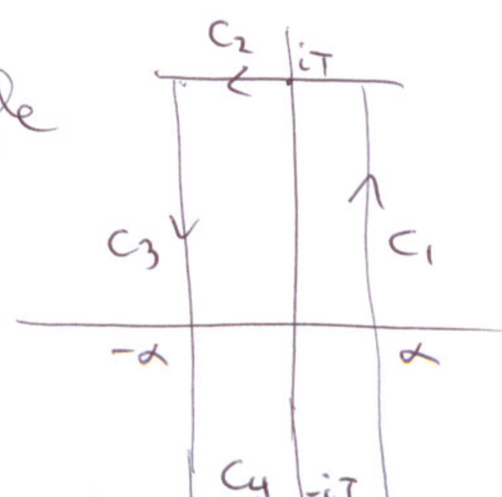
$$\frac{\Lambda(z+s)\, g(z+s)}{z} \to 0$$

as $|\mathrm{Im}\, z| \to \infty$ in vertical strips $-\alpha \le \mathrm{Re}\, z \le \alpha$.

Consider, for given s,

$$\frac{1}{2\pi i} \int_C \frac{\Lambda(z+s)\, g(z+s)}{z}\, dz$$

C: rectangle



α, T big enough so that C encloses all the poles, if any, of $\Lambda(z+s)$. Also $\alpha > 1$.

pole at $z=0$: $\Lambda(s)\, g(s)$

poles at $z = s_k - s$: $\dfrac{r_k\, g(s_k)}{s_k - s}$, $k = 1, \dots \ell$.

On the other hand:

$$\int_{C_2}, \int_{C_4} \to 0 \quad \text{as } T \to \infty$$

On C_1, expand $L(s+z) = \sum \dfrac{b(n)}{n^{s+z}}$ and interchange integration with summation.

On C_3, apply functional equation to throw us into region where we can apply Dirichlet series, and interchange order of integration and summation.

$$\boxed{\Lambda(s)g(s) = \sum_{k=1}^{\ell} \frac{r_k\, g(s_k)}{s - s_k} + Q^s \sum_{1}^{\infty} \frac{b(n)}{n^s} f_1(s,n) + \omega\, Q^{1-s} \sum_{n=1}^{\infty} \frac{\overline{b(n)}}{n^{1-s}} f_2(1-s,n)}$$

where

$$f_1(s,n) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \prod_{j=1}^{a} \Gamma\left(\gamma_j(z+s)+\lambda_j\right) \frac{g(s+z)}{z} \left(\frac{Q}{n}\right)^z dz$$

and

$$f_2(1-s,n) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \prod_{j=1}^{a} \Gamma\left(\gamma_j(z+1-s)+\overline{\lambda_j}\right) \frac{g(s-z)}{z} \left(\frac{Q}{n}\right)^z dz$$

with

ν to the right of the poles of the integrand:

$$\nu > \max\left\{0, -\mathrm{Re}\left(\frac{\lambda_1}{\gamma_1}+s\right), \ldots -\mathrm{Re}\left(\frac{\lambda_a}{\gamma_a}+s\right)\right\}$$

When $\text{Im}\, s$ is small, choose $g(s) \equiv 1$. However, as $|\text{Im}\, s|$ grows:

$$|\Gamma(s)| \sim (2\pi)^{1/2} |s|^{\sigma - 1/2} \underbrace{e^{-|t|\pi/2}}_{\text{decreases very quickly as } |t| \text{ increases}}$$

So, if we take $g(s) \equiv 1$, then $\Lambda(s)$ is very small, but terms on the right, though decreasing as $n \to \infty$, start off comparatively large.

Hence tremendous cancellation must occur on the r.h.s $\longrightarrow$ tremendous precision needed: $O(|t|)$ digits, ex millions of digits if $t \approx 10^6$.

Control for cancellation by setting

$$g(z) = \exp(irz)$$

where r depends on s, chosen to cancel out exponentially small size of each Γ-factor:

Let $c > 0$ (parameter that allows us to control amount of cancellation)

roughly $\quad t_j = \operatorname{Im}(\gamma_j s + \lambda_j)$

$$\varphi_j = \begin{cases} \pi/2 & \text{if } |t_j| \le \frac{2c}{a\pi} \\ \frac{c}{a|t_j|} & \text{if } |t_j| > \frac{2c}{a\pi} \end{cases}$$

$$r_j = -\operatorname{sgn}(t_j)\left(\pi/2 - \varphi_j\right)\gamma_j$$

$$r = \sum_1^a r_j$$

Gives

$$|\Lambda(s)g(s)| \sim * |L(s)| \cdot \prod_{|t_j| \le \frac{2c}{a\pi}} \exp\left(-|t_j|\frac{\pi}{2}\right)$$

$$\cdot \prod_{|t_j| > \frac{2c}{a\pi}} \exp\left(-\frac{c}{a}\right)$$

$$\geq * \cdot |L(s)| \exp(-c)$$

We have thus managed to control exponentially small size of $\Lambda(s)$ up to a factor of $\exp(-c)$ that we can regulate via the choice of c.

<u>Case a=1</u>

$$f_1(s,n) = \frac{\exp(irs)}{2\pi i} \int_{(\nu)} \frac{\Gamma(\gamma(z+s)+\lambda)}{z} \left(\frac{Q}{n}\right)^z \exp(irz)\, dz$$

subst $z = u/\gamma$

But

$$\frac{\Gamma(v+u)}{u} = \int_0^\infty \Gamma(v,t)\, t^{u-1}\, dt, \quad \begin{array}{l}\mathrm{Re}\, u>0 \\ \mathrm{Re}(v+u)>0\end{array}$$

where

$$\Gamma(z,w) = \int_w^\infty e^{-x} x^{z-1}\, dx, \quad |\arg w| < \pi$$

$$= w^z \int_1^\infty e^{-wx} x^{z-1}\, dx, \quad \mathrm{Re}\, w > 0$$

the incomplete gamma function.

Mellin inversion:

$$\boxed{f_1(s,n) = \exp(irs)\, \Gamma\left(\gamma s+\lambda, \left(\frac{n\exp(ir)}{Q}\right)^{\frac{1}{\gamma}}\right)}$$

likewise

$$f_2(1-s,n) = \exp(irs)\, \Gamma\left(\gamma(1-s)+\bar{\lambda}, \left(\frac{n\exp(-ir)}{Q}\right)^{\frac{1}{\gamma}}\right)$$

~~a>1~~

for simplicity, assume $\gamma_j = \frac{1}{2}$!

Γ-factors are $\prod_1^a \Gamma\left(\frac{s}{2}+\lambda_j\right)$

$$\boxed{f_1(s,n)} = \frac{\exp(irs)}{2\pi i}\int_{(\nu)} \frac{\prod_1^a \Gamma\left(\frac{s+z}{2}+\lambda_j\right)}{z}\left(\frac{Q}{n}\right)^z \exp(irz)\,dz$$

$$= \boxed{\exp(irs)\,\Gamma_\lambda\left(\frac{s}{2}+\mu,\left(\frac{n\exp(ir)}{Q}\right)^2\right)},\quad \mu = \frac{1}{a}\sum_1^a \lambda_j$$

and

$$f_2(1-s,n) = \exp(irs)\,\Gamma_{\bar\lambda}\left(\frac{1-s}{2}+\bar\mu,\left(\frac{n\exp(-ir)}{Q}\right)^2\right)$$

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where

$$\Gamma_\lambda(z,w) = \int_w^\infty E_\lambda(t)\,t^{z-1}\,dt$$

and

$$E_\lambda(t) = \int_{\mathbb{R}^{a-1}} \prod_{j=1}^{a-1}\left(u_j^{\lambda_{j+1}-\lambda_j} e^{-t^{\frac{1}{a}}\frac{u_{j-1}}{u_j}}\frac{du_j}{u_j}\right)$$

plays the role of e^{-t} in $a=1$ case

(set $u_0 = u_a = 1$)

is the inverse mellin transform of $\Gamma_\lambda(z) = \prod_1^a \Gamma(z-\mu+\lambda_j)$:

$$\Gamma_\lambda(z) = \int_0^\infty E_\lambda(t)\,t^{z-1}\,dt$$

$$w^{-z}\,\Gamma(z,w) \approx \exp(-\mathrm{Re}\, w)$$

$$w^{-z}\,\Gamma_\lambda(z,w) \approx \exp\left(-\mathrm{Re}\, w^{\frac{1}{a}}\right)$$

$a=1$ case

terms decrease rapidly once

$$\mathrm{Re}\left(\left(\frac{n\exp(ir)}{Q}\right)^{\frac{1}{\gamma}}\right) > 1$$

$$\sim \left(\frac{n}{Q}\right)^{\frac{1}{\gamma}} \frac{c}{|\gamma t|}$$

i.e. $\boxed{n > Q\,|t|^\gamma \cdot \frac{\gamma^\gamma}{c^\gamma}}$

to get $< 10^{-\text{Digits}}$ one should also throw in a factor $(\text{Digits} \cdot 2.3)^\gamma$

larger c - fewer terms needed but more loss of precision.

exs

1) $\zeta(s)$, $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$

$Q=\pi^{-1/2}$, $\gamma=\frac{1}{2}$, so $O\left(|t|^{1/2}\right)$ terms needed

2) $L(s,\chi)$: $\left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)$ (a: 0 or 1)

$Q=(q/\pi)^{1/2}$, $\gamma=1/2$, so $O\left(q^{1/2}|t|^{1/2}\right)$ terms needed

3) $L_E(s)$: $\left(\frac{\sqrt{N}}{2\pi}\right)^s\Gamma(s+1/2)L_E(s)$

$Q=\frac{\sqrt{N}}{2\pi}$, $\gamma=1$ so $O\left(\frac{\sqrt{N}}{2\pi}|t|\right)$ terms needed

$a \geq 1$ case, $\gamma_j = 1/2$

terms decay rapidly once

$$\mathbb{Re}\left(\left(\frac{n\exp(i\tau)}{Q}\right)^{\frac{2}{a}}\right) > 1$$

$$n > Q\left(\frac{t}{c}\cdot\frac{a}{2}\right)^{\frac{a}{2}} \approx Qt^{a/2}$$

So, for example, a degree 3 L-function takes, as $|t|$ increases, $O(|t|^{3/2})$ terms compared to $t^{1/2}$ for $\zeta(s)$.

How to compute $\Gamma(z,w)$ and $\Gamma_\lambda(z,w)$

$$\Gamma(z,w) = \int_w^\infty e^{-t} t^{z-1}\,dt \quad , |\arg w| < \pi$$

let $G(z,w) = w^{-z}\,\Gamma(z,w)$

$$\gamma(z,w) = \Gamma(z) - \Gamma(z,w) = \int_0^w e^{-t} t^{z-1}\,dt \quad , \operatorname{Re} z > 0, \; |\arg w| < \pi$$

$$g(z,w) = w^{-z}\gamma(z,w)$$

$\Gamma(z,w)$	useful when	truncation bounds	Analogous formulas for $\Gamma_\lambda(z,w)$ fully developed?
1) $g(z,w) = \sum \frac{(-1)^j}{j!}\frac{w^j}{z+j}$	$w = O(1)$	easy	yes (Tollis, Dokchitser)
2) $g(z,w) = e^{-w}\sum_0^\infty \frac{w^j}{(z)_{j+1}}$	$\left\lvert\frac{w}{z}\right\rvert < 1$	easy	—
3) $g(z,w) = \cfrac{e^{-w}}{z - \cfrac{zw}{z+1+\cfrac{w}{z+2-\cfrac{(z+1)w}{z+3-\cdots}}}}$	$\left\lvert\frac{w}{z}\right\rvert < 1$	harder: Akiyama-Tanigawa, Winitzki	—
4) (Nielsen) $\gamma(z,w+d) = \gamma(z,w) + w^{z-1}e^{-w}\sum_{j=0}^{\infty}\frac{(1-z)_j}{(-w)^j}\left(1-e^{-d}e_j(d)\right)$	$\lvert d\rvert < \lvert w\rvert$	easy	—

$\Gamma(z,w)$	useful when	truncation bounds	Analogous formulas for $\Gamma_\lambda(z,w)$?
5) $G(z,w) \sim \dfrac{e^{-w}}{w} \sum_{0}^{m-1} \dfrac{(1-z)_j}{(-w)^j}$	$\left\lvert \frac{z}{w} \right\rvert < 1$, w big	easy	yes (Dokchitser)
6) Temme-uniform asymptotics for $\dfrac{\Gamma(z,w)}{\Gamma(z)}$	$w, z \in \mathbb{C}$, z big	not explicit for complex parameters	in certain cases (Guthmann)
Paris-	$w \approx z$	explicit	—
7) $G(z,w) = \cfrac{e^{-w}}{w + \cfrac{1-z}{1 + \cfrac{1}{w + \cfrac{2-z}{1 + \cfrac{2}{\ddots}}}}}$	$\left\lvert \frac{z}{w} \right\rvert < 1$	Akiyama-Tanigawa Winitzki	—
8) compute $\dfrac{1}{2\pi i} \displaystyle\int_{\nu} \dfrac{\Gamma(u+z)}{u} w^{-u}\, du$ as a Riemann sum	$w, z \in \mathbb{C}$	harder, but doable	yes (Rubinstein, Booker)