# Fibered K3 Surfaces using Sage

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March 11, 2009

# Outline

1. Reflexive Polytopes in Toric Geometry
   - Reflexive Polytopes
   - Homogeneous Coordinates
   - Calabi-Yau Hypersurfaces
   - Tools and Projects

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   - Calabi-Yau Hypersurfaces
   - Tools and Projects
2. Elliptically Fibered K3 Surfaces
   - Fibrations from Polytopes
   - Slicing the Cube
   - Splitting

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# Lattice and Reflexive Polytopes

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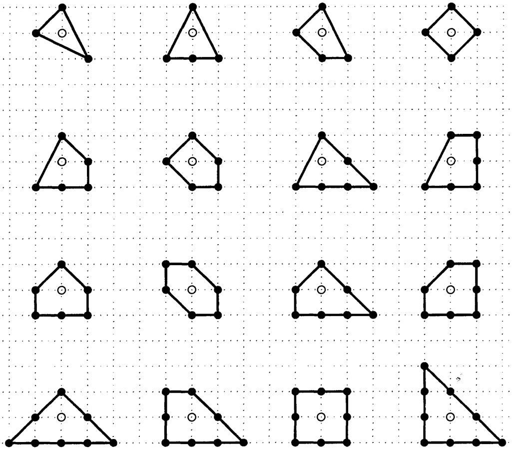
- A necessary condition for $\Delta$ to be reflexive: the origin is the only *interior* lattice point (sufficient for $d = 2$).
- Vertices of $\Delta^\circ$ are $w_1, \dots, w_m$, where $\langle w_i, v \rangle + 1 = 0$ are equations of facets of $\Delta$ and $w_i$ are normalized inner normals to them.
- $(\Delta^\circ)^\circ = \Delta$.

# Low-Dimensional Reflexive Polytopes

Up to the action of $GL(\mathbb{Z}^d)$, there are finitely many $d$-dimensional reflexive polytopes and there is a classification algorithm.



There are 16 reflexive polygons.

For $d = 3$: 4319.
For $d = 4$: 473,800,776.

The image is taken from Poonen, B. & Rodriguez-Villegas, F., Lattice polygons and the number 12, Amer. Math. Monthly 107 (2000), no. 3, 238–250.

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# Homogeneous Coordinates: Projective Space

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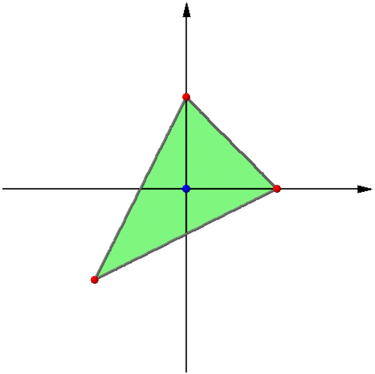

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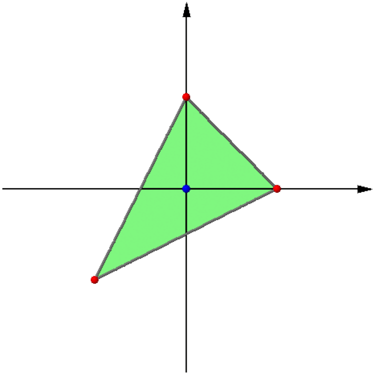

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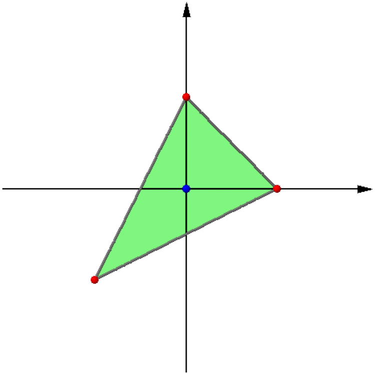

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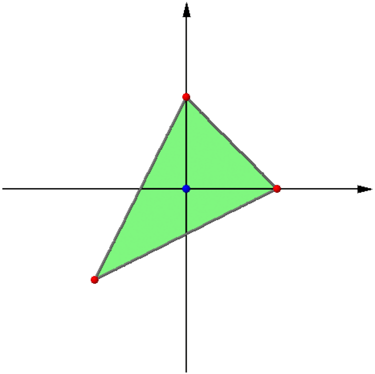

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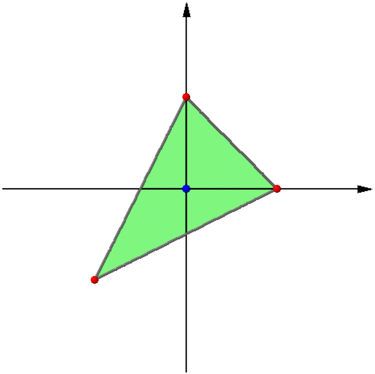

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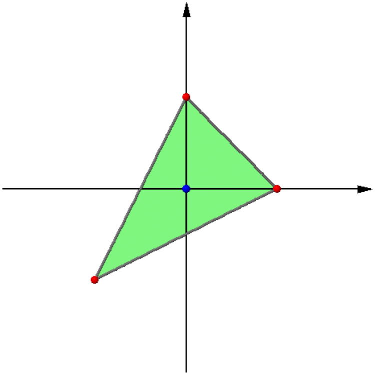

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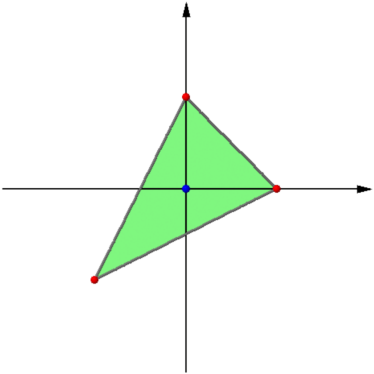

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This defines a toric variety $\mathbb{P}_\Delta$ as $(\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^\times \simeq \mathbb{P}^2$.

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- This defines the toric variety $\mathbb{P}_\Sigma$ as $(\mathbb{C}^n \setminus S)/G$, which is a good geometric quotient if $\Sigma$ is simplicial (e.g. if $\Sigma$ corresponds to a maximal lattice triangulation of $\partial\Delta$).

# Outline

# Anticanonical Hypersurfaces

- Batyrev's mirror pair of Calabi-Yau anticanonical hypersurfaces $X \subset \mathbb{P}_{\Sigma(\Delta)}$ and $X^\circ \subset \mathbb{P}_{\Sigma(\Delta^\circ)}$ is given by the equation

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- Q: What are the Hodge numbers of $X$ and $X^\circ$?
- A: For $d \geqslant 4$ they can be computed "in terms of $\Delta$:"

$$h^{1,1}(X) = \ell(\Delta) - 1 - d - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{F} \ell^*(F)\ell^*(F^*),$$

  where $\ell(\Delta) = |\Delta \cap N|$, $\Gamma$ runs over codimension-1 faces of $\Delta$, $\ell^*(\Gamma) = |\operatorname{int} \Gamma \cap N|$, $F$ runs over codimension-2 faces of $\Delta$, and $F^*$ is the dual to $F$ face of $\Delta^\circ$.

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# Geometry, Combinatorics, Computations

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can be dealt with explicitly using polytope data — facial structure and lattice points on faces. Perfect for computer experimentation!

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- Some extra code, which may be added to Sage once it is clean, if there will be general interest.

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Let’s look at elliptic fibrations closer.

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(Following works of E. Perevalov & H. Skarke, F. Rohsiepe, and others)

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- Moreover, exceptional fibers over zero and infinity can be directly seen from the edges of $\Delta$!
- With some care the method works for higher dimensions.

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- Let's see it in action in 3D using Sage!

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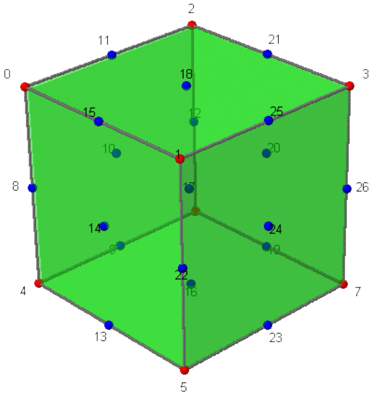
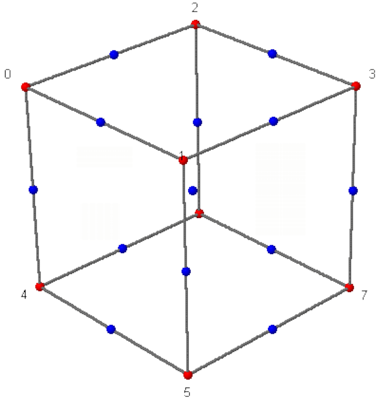
# Slicing the Cube

Take a cube:




Consider its skeleton:



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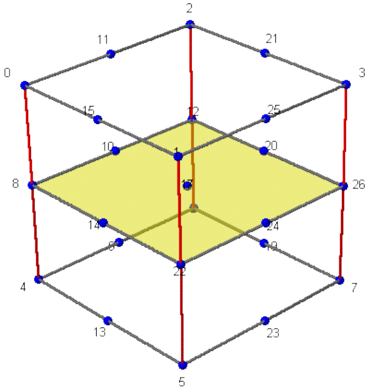


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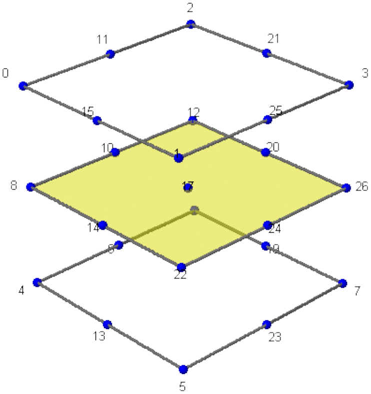

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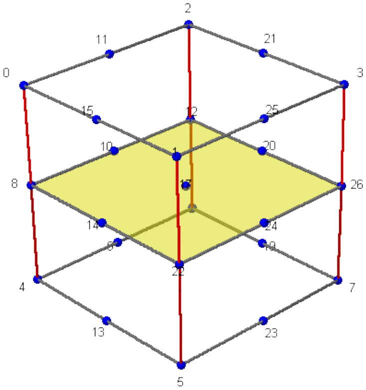
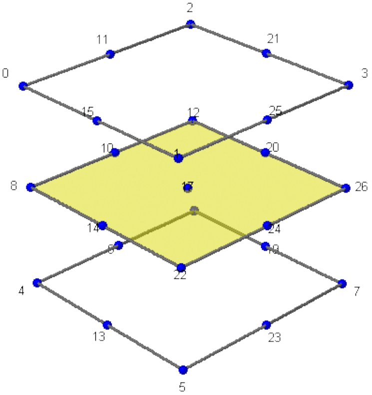



Remove “touching” parts:




Graphs on top and bottom are extended Dynkin diagrams corresponding to the exceptional fibers over zero and infinity of the induced fibration of $X$!

# Slicing the Cube — not by hands!!!

Using Sage, after some preparation we get

```
...
sage: ef.show()
A fibration of:
A polytope polar to An octahedron: 3-dimensional, 8 vertices.
Corresponding to:
...
Equation of a hypersurface:
```

$$a_3 z_0^2 z_2^2 z_4^2 z_6^2 z_8^2 z_9^2 + a_4 z_0^2 z_1^2 z_4^2 z_5^2 z_8^2 z_{10}^2 + a_2 z_0^2 z_1^2 z_2^2 z_3^2 z_8 z_9 z_{10} z_{11} + a_6 z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 z_9 z_{10} z_{11} + a_5 z_4^2 z_5^2 z_6^2 z_7^2 z_8 z_9 z_{10} z_{11} + a_1 z_2^2 z_3^2 z_6^2 z_7^2 z_9^2 z_{11}^2 + a_0 z_1^2 z_3^2 z_5^2 z_7^2 z_{10}^2 z_{11}^2$$

```
...
Fiber over (t^(1),1):
```

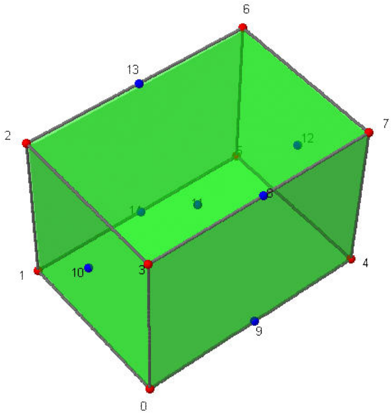
$$(a_3 t^2) z_8^2 z_9^2 + (a_4 t^2) z_8^2 z_{10}^2 + (a_2 t^2 + a_6 t + a_5) z_8 z_9 z_{10} z_{11} + a_1 z_9^2 z_{11}^2 + a_0 z_{10}^2 z_{11}^2$$

```
Top: ExtA7.
Bottom: ExtA7.
F_0: ('I', 8).
F_infinity: ('I', 8).
```

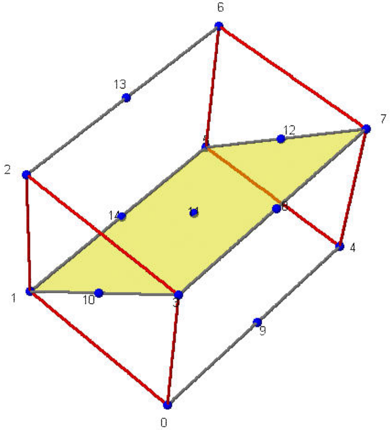
# Outline

# Another Example

Start with:




Fibration diagram:



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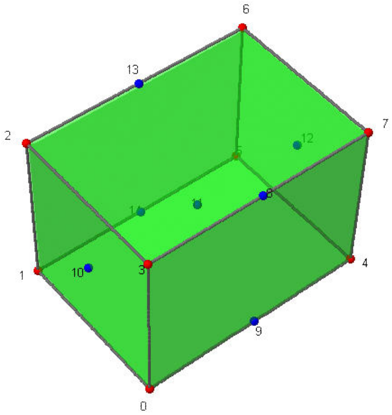


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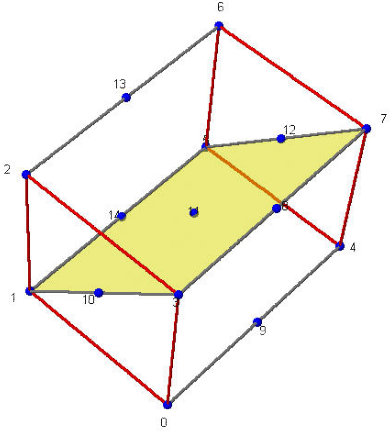


Not as satisfying as before — graphs of “top” and “bottom” are not extended Dynkin diagrams.

# Picard Lattice

- Up to linear equivalence, all divisors on $\mathbb{P}_{\Sigma(\Delta)}$ are toric-invariant, i.e. correspond to 1-dimensional cones of $\Sigma(\Delta)$.

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- Some of them may split. Namely, interior points of an edge $E$ of $\Delta$ “split” into $\ell^*(E^*) + 1$ components and the Picard rank of $X$ is, generically,

$$p(X) = \ell(\Delta) - 1 - 3 - \sum_{F,\text{ facets of }\Delta} \ell^*(F) + \sum_{E,\text{ edges of }\Delta} \ell^*(E)\ell^*(E^*).$$

(Same expression as for $h^{1,1}$ in higher dimensions.)

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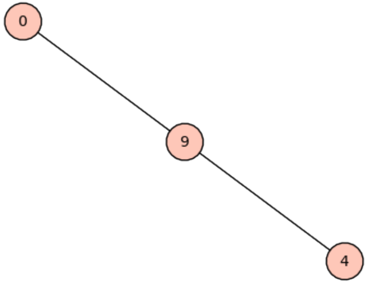
- After this "splitting" all points on graphs of the "top" and "bottom" correspond to divisors with self-intersection -2. "Edges split" as well and each line represents an intersection of divisors (without multiplicities except for the case o===o).

# Another Example — Splitting

Before splitting:

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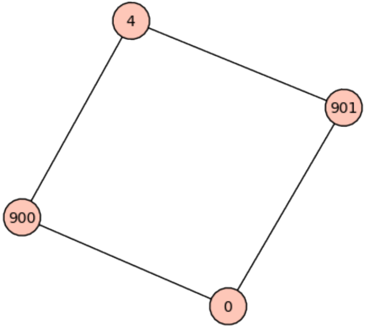

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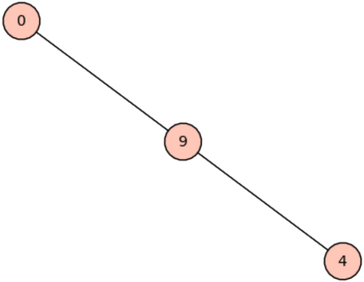


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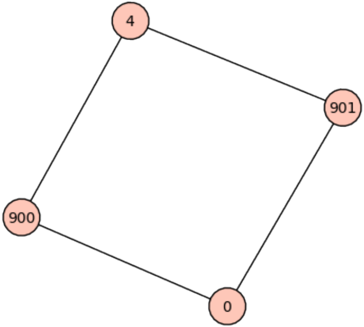


Now we do get an extended Dynkin diagram, as expected for an exceptional fiber!

# Splitting: Details

Intersection of a generic K3 surface and a torus-invariant divisor $\{z_j = 0\}$, where $v_j$ is an interior point of an edge $E$ gives

$$\sum_{w \in \Delta^\circ \cap M} a_w \prod_{i=1}^{n} z_i^{\langle w, v_i \rangle + 1} = \sum_{w \in E^* \cap M} a_w \prod_{i=1,\ldots,\hat{j},\ldots n} z_i^{\langle w, v_i \rangle + 1},$$

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Let $w_0, \dots, w_s$ be consecutive points along $E^*$. Since $w_k = w_0 + k(w_1 - w_0)$, we can rewrite the above as

$$\prod_{i=1}^{n} z_i^{\langle w_0, v_i \rangle + 1} \sum_{k=0}^{s} a_{w_k} \left( \prod_{t=1}^{n} z_t^{\langle w_1 - w_0, v_t \rangle} \right)^k$$

and the intersection splits into $s = \ell^*(E^*) + 1$ components.

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- Interfaces to commercial software are built-in (but you have to have these programs, of course), if a particular function is not supported in the sense of returning a Sage object, at least it is possible to get a string output for any command.
- This is convenient to get fast access to features that are not implemented in Sage and wait for someone to implement it.
- E.g. MAGMA was used for graph classification, but now there is no need for this, thanks to Robert Miller!

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- For classifying all exceptional fibers numerically it was important to find roots with specified precision to make sure that multiple roots are determined as multiple.
- For one of the projects it was important to have fast arithmetics for polynomials over $\mathbb{Z}$ and Sage is the fastest for the moment.

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```
def top3d(self,
        edge_thickness=3, edge_color=(0.5,0.5,0.5),
        point_size=10, point_color=(0,0,1),
        label_color=(0,0,0), label_shift=1.1):
    r"""
    Return the 3d plot of the top edge diagram.
    """
    g = self._s.top
    points = self.polytope().points().columns(copy=False)
    plot = None
    for v in g.vertices():
        pt = points[v]
        plot += point3d([pt], size=point_size, rgbcolor=point_color)
        plot += text3d(str(v), label_shift*pt, rgbcolor=label_color)
    for e in g.edges():
        plot += line3d([points[e[0]], points[e[1]]],
                    thickness=edge_thickness, rgbcolor=edge_color)
    return plot
```

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- Sage server/notebook is a really convenient feature for working on fast computers from slow laptops or random machines.
- If you do want to have a personal installation, you download the archive, follow simple instructions, and it works. If you have problems, it is clear where to get help.
- From personal experience, this can be more complicated (thus annoying) for much simpler free software.