Algebraic geometry in Sage

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Outline

Algebraic schemes

In Sage, these are *schemes defined by means of polynomial equations*. (As opposed to **Schemes**, which are abstract schemes and for which we might not have explicit defining equations.)

Example

An **AlgebraicScheme**: $X \subset \mathbb{P}^2$ cut out by

$$y^2 z = x^3 - xz^2 \quad \text{and} \quad y = 0.$$

A **Scheme**: the Jacobian of a curve

Morphisms

In most situations, these are given by explicit equations.

Warning: the code is flimsy, buggy, not documented and does almost no checking that things make sense!

Points are treated as morphisms.

Curves

Can define affine/projective plane/space curves. Can compute their arithmetic/geometric genus.

For hyperelliptic curves over finite fields, there is a fast(er) point-counting algorithm.

Also for hyperelliptic curves: Monsky-Washnitzer cohomology (Coleman integrals, fast computation of zeta functions)

Elliptic curves

Huge amounts of functionality, especially for arithmetic applications (over $\mathbb{Q}$, number fields, finite fields).

Easy access to databases of elliptic curves (Cremona, Stein-Watkins).

Pieced together from Pari, Cremona's code, and quite a bit of native code.

Visualisation

Sage does 2D graphics (using matplotlib) and 3D graphics (interactive, using jmol; or static, using tachyon).

These capabilities are only partly tied in with the algebraic-geometric objects.

(Some) Relevant packages included in Sage

- Singular (commutative algebra)
- Polybori (polynomials over boolean rings)
- Frobby (commutative algebra with monomial ideals)
- Atlas, Blas, Iml, Lapack, Libm4ri, Linbox (linear algebra)
- Genus2reduction (reduction mod p of genus 2 curves)
- Gfan (Gröbner fans, tropical varieties)
- Symmetrica (symmetric functions, representations of symmetric groups)
- Gap (computational discrete algebra)

Native functionality

- number fields (some based on Pari)
- (algebraic) combinatorics
- graph algorithms
- open, accessible, supportive development environment

An example from Donu Arapura

Donu Arapura wrote Maple code for computing Hodge and Betti numbers for smooth complete intersections and for moduli spaces of stable vector bundles over curves. It uses formulas by Hirzebruch, respectively Zagier.

Donu ported his code to Sage, basically line-by-line.

There is more code of this type that's written in Maple, most likely because of John Stembridge's SF package for symmetric functions. Note that Sage includes Symmetrica, which provides the same functionality (and more).

Sage needs...

- more basic functionality for modules (tensor products, exterior powers, etc.)
- homological algebra
- serious spring cleaning of algebraic geometry code
- exposing more functionality from SINGULAR and other packages
- lots of other things
- YOU

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