Siegel Theta Series and Siegel Modular Forms

Handout version. Some tables not fitting into class *beamer* will be shown separately, some things will be on the blackboard only

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Outline

Representation numbers

Always: $A \in M_m^{\mathrm{sym}}(\mathbb{Z})$ positive definite with even diagonal,

$$Q_A(\mathbf{x}) = \frac{1}{2}{}^t\mathbf{x}A\mathbf{x} = \sum_{i,j=1}^{m} a_{ij}x_ix_j$$

the associated quadratic form.

Define the representation number

$$r(A,t) := r(Q_A,t) := |\{\mathbf{x} \in \mathbb{Z}^m \mid Q_A(\mathbf{x}) = t\}|,$$

Theta series

A symmetric of size m, positive definite with even diagonal.

Definition

The theta series (of degree 1) of A is

$$\begin{aligned}\vartheta(A,z) &:= \sum_{\mathbf{x}\in\mathbb{Z}^m} \exp(2\pi i Q_A(\mathbf{x})z) \\ &= \sum_{t=0}^{\infty} r(A,t)\exp(2\pi i t z), \quad z \in H = \{z \in \mathbb{C} \mid \Im(z) > 0\}\end{aligned}$$

It is a modular form of weight $k = \frac{m}{2}$ for the group $\Gamma_0(N)$ where NA^{-1} is integral with even diagonal:
$\vartheta(A,\cdot) \in M_k(\Gamma_0(N),\chi)$ with χ depending on $\det(A)$.

Siegel theta series extend this to the theta series of degree g, encoding representation numbers of $g \times g$-matrices.

Siegel theta series

Write $\mathfrak{H}_g = \{Z = X + iY \in M_g^{\mathrm{sym}}(\mathbb{C}) \mid X, Y \text{ real}, Y > 0\}$ (Siegel's upper half space).

Definition

The Siegel theta series of degree (or genus) g of A is

$$\begin{aligned} \vartheta^{(g)}(A, Z) &:= \sum_{X \in M_{m,g}(\mathbb{Z})} \exp(\pi i \operatorname{tr}({}^tXAX\, Z)) \\ &= \sum_T r(A, T) \exp(2\pi i \operatorname{tr}(TZ)) \quad Z \in \mathfrak{H}_g, \end{aligned}$$

where $2T$ runs over positive semidefinite symmetric matrices of size g with even diagonal.
It is a Siegel modular form of weight $k = \frac{m}{2}$ for the group $\Gamma_0^{(g)}(N)$ where NA^{-1} is integral with even diagonal

Special case

An often used special case of the above definition

$$\vartheta^{(g)}(A, Z) := \sum_{X \in M_{m,g}(\mathbb{Z})} \exp(\pi i \operatorname{tr}({}^tXAX\, Z)), \quad Z \in \mathfrak{H}_g$$

is the case $A = (2) \in M_1^{\mathrm{sym}}(\mathbb{Z})$, where it comes down to

$$\vartheta^{(g)}(Z) := \sum_{\mathbf{x} \in \mathbb{Z}^g} \exp(2\pi i \operatorname{tr}(\begin{pmatrix} x_1 \\ \vdots \\ x_g \end{pmatrix} (x_1, \dots, x_g)\, Z)), \quad Z \in \mathfrak{H}_g.$$

This Siegel modular form of degree g and weight $1/2$ is for $g > 1$ an example of a singular form: It has nonzero Fourier coefficients $r(A, T)$ only at degenerate (in fact: rank 1) matrices T.

Siegel modular forms

On $\mathfrak{H}_g$ the symplectic group

$$Sp_g(\mathbb{R}) = \{U \in M_{2g}(\mathbb{R}) \mid {}^tU \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} U = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}\}$$

acts by

$$\gamma Z = (AZ+B)(CZ+D)^{-1} \quad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Definition

A (scalar valued) Siegel modular form of weight k is a holomorphic function $F : \mathfrak{H}_g \to \mathbb{C}$ satisfying

$$F(\gamma Z) = (\det(CZ+D))^k F(Z)$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in a suitable congruence subgroup of $Sp_g(\mathbb{Z})$ or in a so called paramodular group.

The most common groups of are those of type

$$\Gamma_0^{(g)}(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{Z}) | C \equiv 0 \bmod N\}$$

(possibly with a character $\chi(\det D)$ inserted into the transformation condition: $F(\gamma Z) = \chi(\det(D))(\det(CZ+D))^{m/2}(CZ+D)F(Z))$ and in the case $g = 2$ the paramodular groups

$$K(N) = Sp_2(\mathbb{Q}) \cap \{\begin{pmatrix} * & * & *N^{-1} & * \\ *N & * & * & * \\ *N & *N & * & *N \\ *N & * & * & * \end{pmatrix}\} \quad \text{with } * \in \mathbb{Z}$$

Computations with Siegel modular forms

Skoruppa computed Siegel modular forms of degree 2 for the group $Sp_2(\mathbb{Z})$ (level 1 case) and Hecke eigenforms among these using the known (Igusa) structure of the graded ring in that case.

Recent computations have been done by several japanese groups, in particular Ibukiyama and his students. As far as I know these were isolated example computations.

Poor and Yuen have done many computations, enabled by their groundbreaking work on how many Fourier coefficients are needed to characterize a form.

Computations with Siegel modular forms, cont'd

Some of the problems for computations with Siegel modular forms are:

- One does not know to what extent "multiplicity one" is true: Knowing the Hecke eigenvalues of a Siegel modular form does not allow to compute all its Fourier coefficients. Also, the Hecke algebra is more complicated.
- In most cases it is difficult to determine the structure of the graded ring of Siegel modular forms of fixed level.
- There is no modular symbols method available.
- The index of a Fourier coefficient being a (reduced) quadratic form of g variables instead of just a number, one needs to care about efficient organization of the output of routines.
- Many interesting Siegel modular forms are "genuinely vector valued", transforming with $\rho(CZ + D)$ for some rational representation ρ of $GL_g(\mathbb{C})$ instead of $\det(CZ + D)^k$ as defined above.

In *Theta Series of Modular, Extremal, and Hermitian Lattices* we (Scharlau, Schiemann and S-P) computed genera of quadratic and hermitian forms and the associated theta series of degree 1 and 2 along with the action of Hecke operators on them.

The computation of the genera (i.e., enumeration of all isometry classes in them) was done with Kneser's neighbouring lattice method, extended to hermitian lattices by Detlev Hoffmann and further elaborated for that case by Schiemann.

I'll summarize some results and show some tables.

Class-list

We have **ternrep**, which takes as input a positive definite ternary form and outputs the unique reduced form in its class, and **ternclass**, which lists all ternaries in a given range of discriminants (with options to list only even, only primitives etc.) Both programs are in C, were written by Alexander Schiemann and are functional (recompiled under Linux using gcc 4.3 and run).

A similar tool is **rep-list**; this generates representatives of genera of forms of specified signature in a given discriminant range. I could compile this program but had runtime errors, probably due to the bigint-routines.

Isometry problem

It is necessary to test quadratic forms for equivalence efficiently. The programs **isom** and **auto(m)** of Bernd Souvignier (who is now, i. e. in 2009, in Nijmegen) have been used in our project, with marginal modifications made by Schiemann.

The program isom tests for equivalence, the program auto(m) determines the group of units (automorphims) of a given set of forms. The testing of sets of forms can be used to perform equivalence tests over rings of (quadratic) integers. As far as I know these programs are also still part of MAGMA.

Listing forms–Isolist

During the project a variety of routines was developed for computing various invariants of a set of quadratic forms (also called lattices in the sequel), for example dual lattice, root system, order of the automorphism group, list of short vectors, theta series of degree one and two or Jacobi theta series, dimension of the space of modular forms generated by these theta series, closure of this space under Hecke operators, mass of the set of forms at hand.

Schiemann wrote a C++-program **isolist** which takes as input a list of quadratic forms, extracts a set of representatives of the different classes and, depending on options handed to the program, computes various invariants of the forms and organizes the representatives into a list.

Neighbouring lattices

The method of neighbouring lattices has been introduced by Kneser. It can be used to determine all classes in a given genus, over the integers of number fields one has to be careful about obtaining all spinor genera in the genus.

Our project produced the routines **tn** calculating neighbours at the prime 2 of a $\mathbb{Z}$-lattice and thereby obtaining all classes in the genus, **pn, pn-sh** for doing the same using neighbours at some prime p, **qn** for doing the same over a real quadratic field, and **hn** for doing the same for hermitian forms over the integers of an imaginary quadratic field.

Of these, **hn** could not be recompiled (see section Problems). Some years ago Abshoff created an executable which was used on Linux PCs in Dortmund and Saarbrücken; I haven't tested recently whether it can still be used.

Neighbouring lattices, continued

The program **tn**, written by Boris Hemkemeier, has been maintained by him and is available in the web.
The programs **pn, pn-sh,qn** could be recompiled, running has not yet been tested, except for one or two runs with **pn-sh**.

The program **tn** has more utilities than **pn,qn** for generating additional information, e.g. root systems of lattices.
These facilities of **tn** have been collected by Schiemann in the program **invar** which takes as input a list of lattices generated by one of the other programs. **invar** could (with marginal changes) be recompiled but has not yet been tested for running.

Miscellaneous

Miscellaneous routines which have been programmed in the project:

decomp — The program **decomp**, written by Frank Vallentin, computes the decomposition of a given lattice into its irreducible components. I could recompile it, but the attempt to run it crashed. I refrained from attempts to debug it.

gsymbol — The program **gsymbol**, written by Schiemann computes the genus symbol of a given (definite or indefinite) integral quadratic form over $\mathbb{Z}$. It could be recompiled and ran in simple tests.

herm_mass — The program **herm_mass**, written by Schiemann, computes the mass of the hermitian genus of the sum of m squares over a given imaginary quadratic number field. It could not be recompiled.

More miscellaneous routines

perfrk — the program **perfrk**, written by Axel Pawellek, computes the prefection rank of a lattice. I could not compile it at first try, but the only problem seems to be a proper link to **Lapack**, being in a hurry I reserved that for later.

rootsystem — The program **rootsystem**, written by Axel Pawellek, computes the root system of a given lattice. I could not recompile it, due to LiDIA problems.

shvec — The program **shvec**, written by Frank Vallentin, computes lists of short vectors and the theta series of degree 2. According to Vallentin it has been “almost not” tested. I could recompile it but didn’t test it.

Theta series of degree n for hermitian forms are defined as functions on the space

$$\mathbf{H}_n = \{Z \in M_n(\mathbb{C}) \mid i({}^t\overline{Z} - Z) > 0\},$$

on which the group

$$U_1^{(n)} = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_{2n}(\mathbb{C}) \mid {}^t\overline{g} \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

operates by fractional linear transformations

$$Z \longmapsto (AZ + B)(CZ + D)^{-1}$$

Let $F = \mathbb{Q}(\sqrt{-\delta})$ with square free $\delta > 0$. For an even positive definite hermitian $\mathcal{O}_F$-lattice M as above we have

$$\vartheta^{(n)}(M, h, Z) := \vartheta^{(n)}(M, Z) := \sum_{\mathbf{x} \in M^n} \exp(\pi i \, \mathrm{tr}(h(\mathbf{x})Z))$$

with $h(\mathbf{x}) := (h(x_i, x_j))$ for $\mathbf{x} = (x_1, \ldots, x_n) \in M^n$.

$\vartheta^{(n)}(M, Z)$ is a holomorphic modular form of weight $k = \frac{\mathrm{rank}(M)}{2} = \frac{m}{2}$ for a congruence subgroup of $U_1^{(n)}(F)$

Theorem

a) *If (M, h) is integral unimodular and for $\delta \not\equiv 3 \mod 4$ even, the hermitian theta series of degree one of (M, h) and the ordinary theta series of its transfer (L, b) to $\mathbb{Z}$ coincide.*

b) *If the $\mathbb{Z}$-lattice (L, b) is the transfer to $\mathbb{Z}$ of the hermitian integral unimodular (even) lattice (M, h) as above, its theta series of degree n is a modular form for the Fricke group.*

Questions concerning spaces of modular forms generated by theta series of $\mathbb{Z}$-lattices and by theta series of lattices with hermitian structure:

a) Which part of a space of modular forms for the Fricke group containing theta series of a modular genus of $\mathbb{Z}$-lattices is generated by the theta series of modular lattices in that space?

b) Which part of a space of modular forms as in a) is generated (linearly or as a Hecke module) by theta series of lattices with hermitian structure?

c) Which part of some space of hermitian modular forms is generated by the hermitian theta series of hermitian lattices that are in that space? (Basis problem for hermitian modular forms)

Modification of one of the questions after our experimental results:

Given a congruence subgroup Δ of $U_1^{(n)}(F)$.

Is the Hecke closure of the space of Siegel modular forms that arise as restrictions to $\mathfrak{H}_n$ of the hermitian modular forms for Δ of given weight and multiplier equal to the space of all Siegel modular forms for $\Delta \cap \mathrm{Sp}_n(\mathbb{R})$ of corresponding weight and multiplier?