Multiplication of binary polynomials

R. P. Brent, P. Gaudry, E. Thomé, P. Zimmermann

See paper at ANTS VIII, 2008

Multiplication of univariate binary polynomials

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Plan

1. **Introduction**
2. **Small sizes**
3. **Medium sizes**
4. **Large sizes**

1. Introduction
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Why ?

We focus on polynomial multiplication over $\mathbb{F}_2[x]$.

This is used in many contexts:

- polynomial factorization, irreducibility tests ;
- (some) crypto applications ;
- less obvious: sparse linear algebra over $\mathbb{F}_2$;
- and more.

How does data look like ?

Binary polynomial $x^3 + x^2 + 1 \quad \rightarrow \quad$ machine integer$(1101)_2$ (“dense”).

- up to degree 63: one machine word (64-bit).
- degree 64 to 127: two words.
- …

In hardware:

- `add` is trivial ;
- `mul` is easy ; much easier than integer `mul`.
- Not our business.

In software:

- `add` is trivial (`xor`) ;
- `mul` is tedious (no PCMULQDQ yet !).

What do we do ?

We are interested in:

- software.
- speed everywhere: from 64 to 2^{32} coefficients (think recursion).

Existing software

Existing software typically has:

- Possibly fast multiplication for $1, 2 \ldots$ up to a few words.
- Karatsuba multiplication above.

Main reference: Victor Shoup's NTL: `shoup.net/ntl`

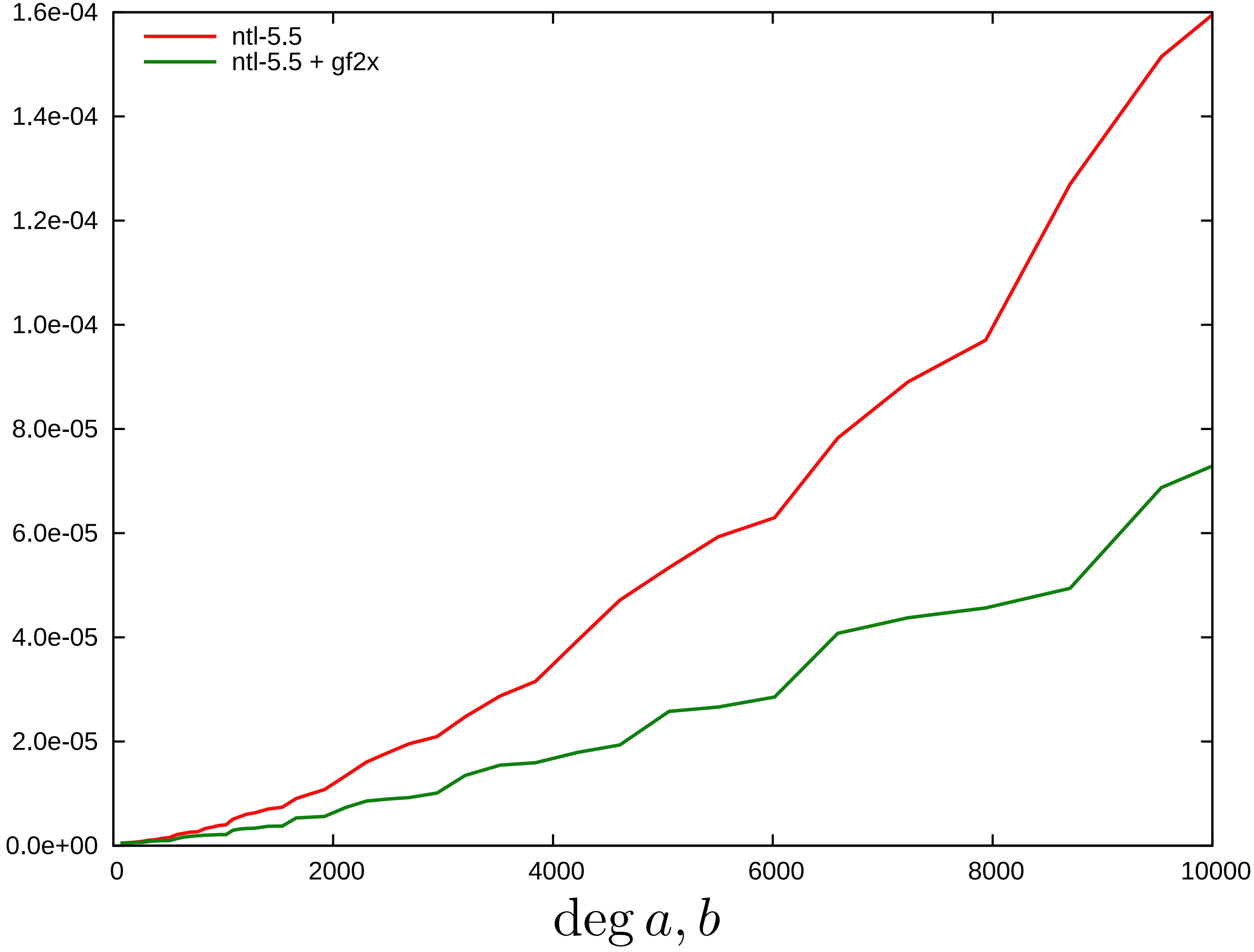
Very rarely (if ever), one finds:

- Code that takes advantage of CPU-specific instructions ;
- Toom-Cook multiplication ;
- Fast multiplication for unbalanced operands ;
- FFT (Schönhage ternary + Cantor additive).

All of this is in the `gf2x` software package.

It pays off !

Timings in seconds, Core2 2.4GHz



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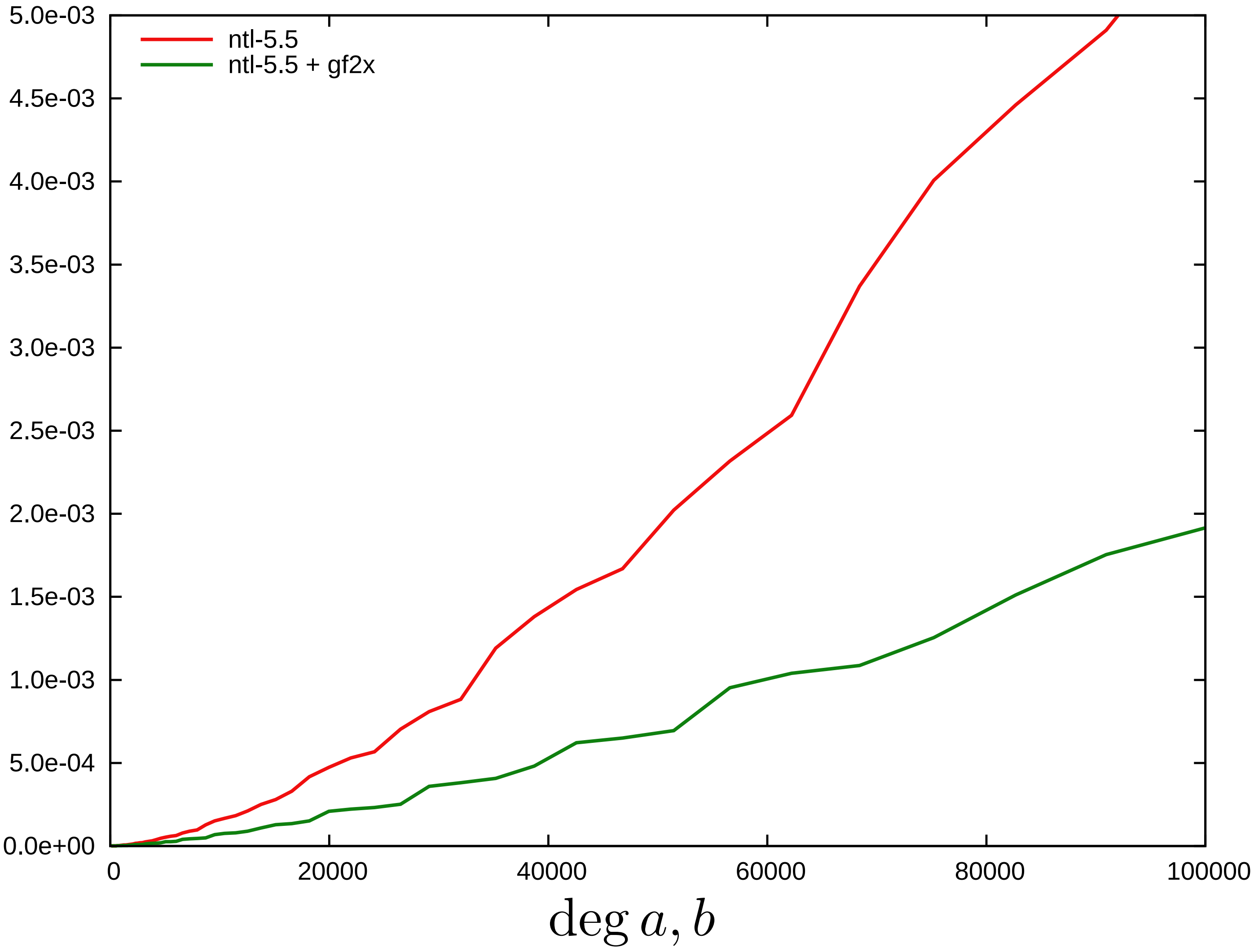

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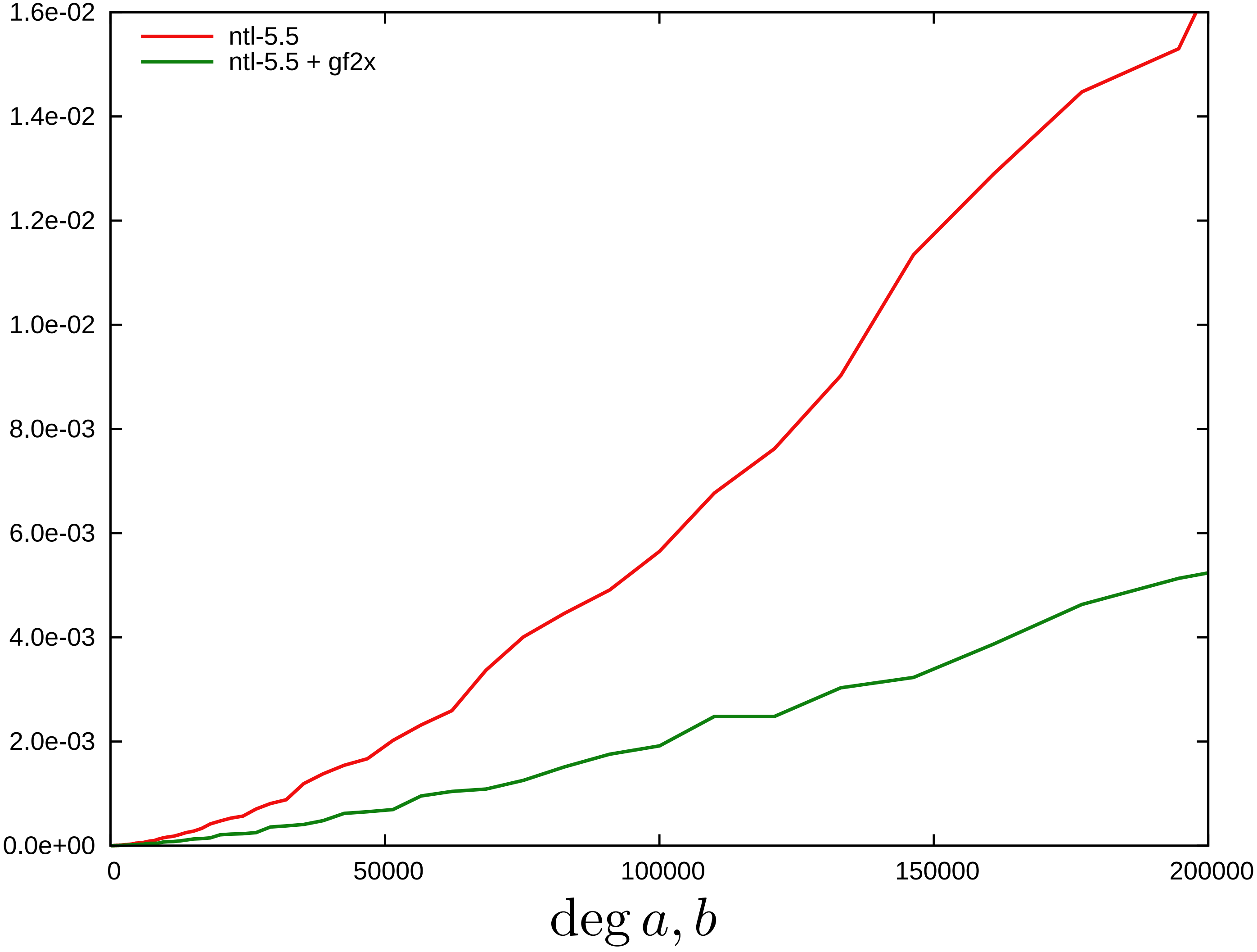

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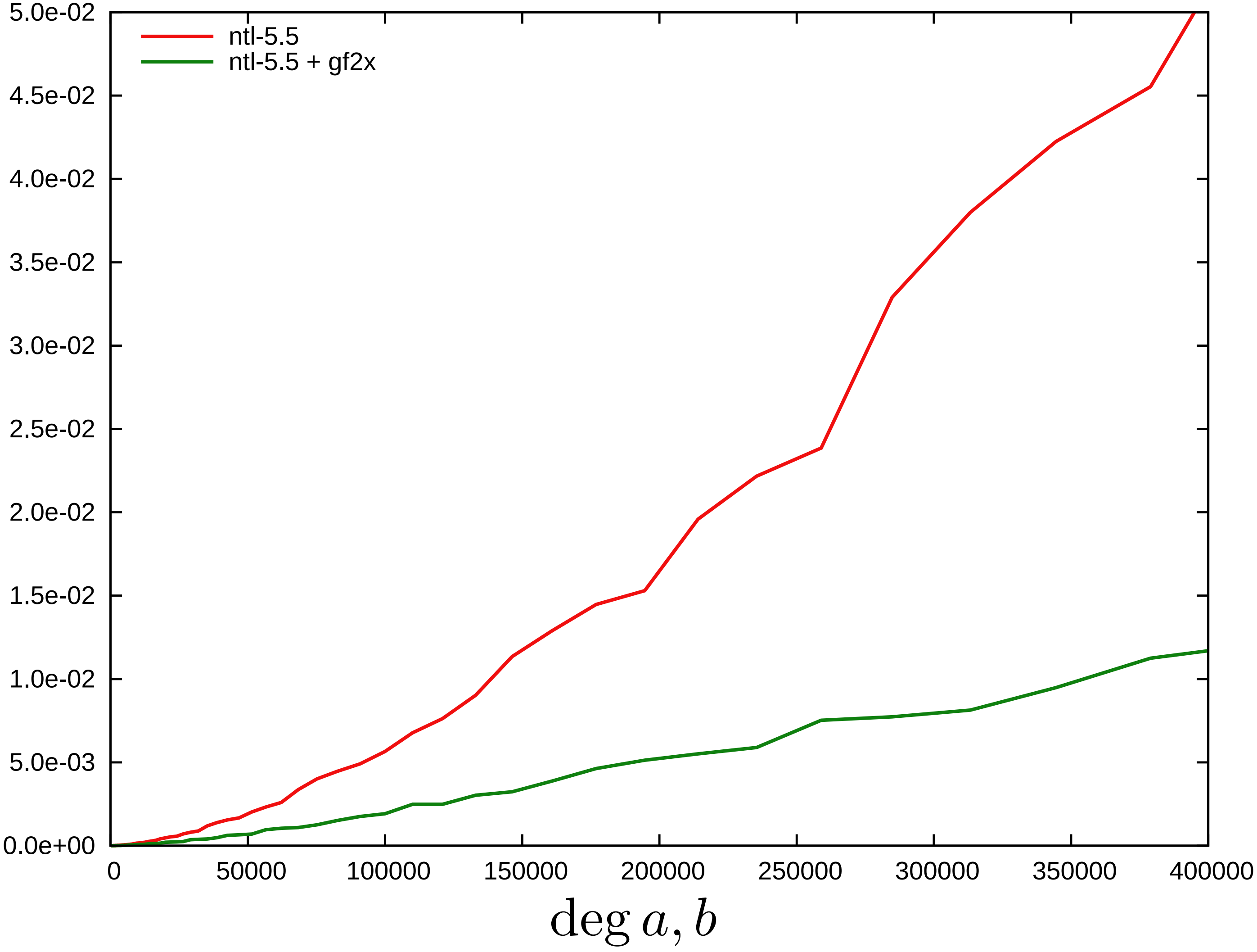

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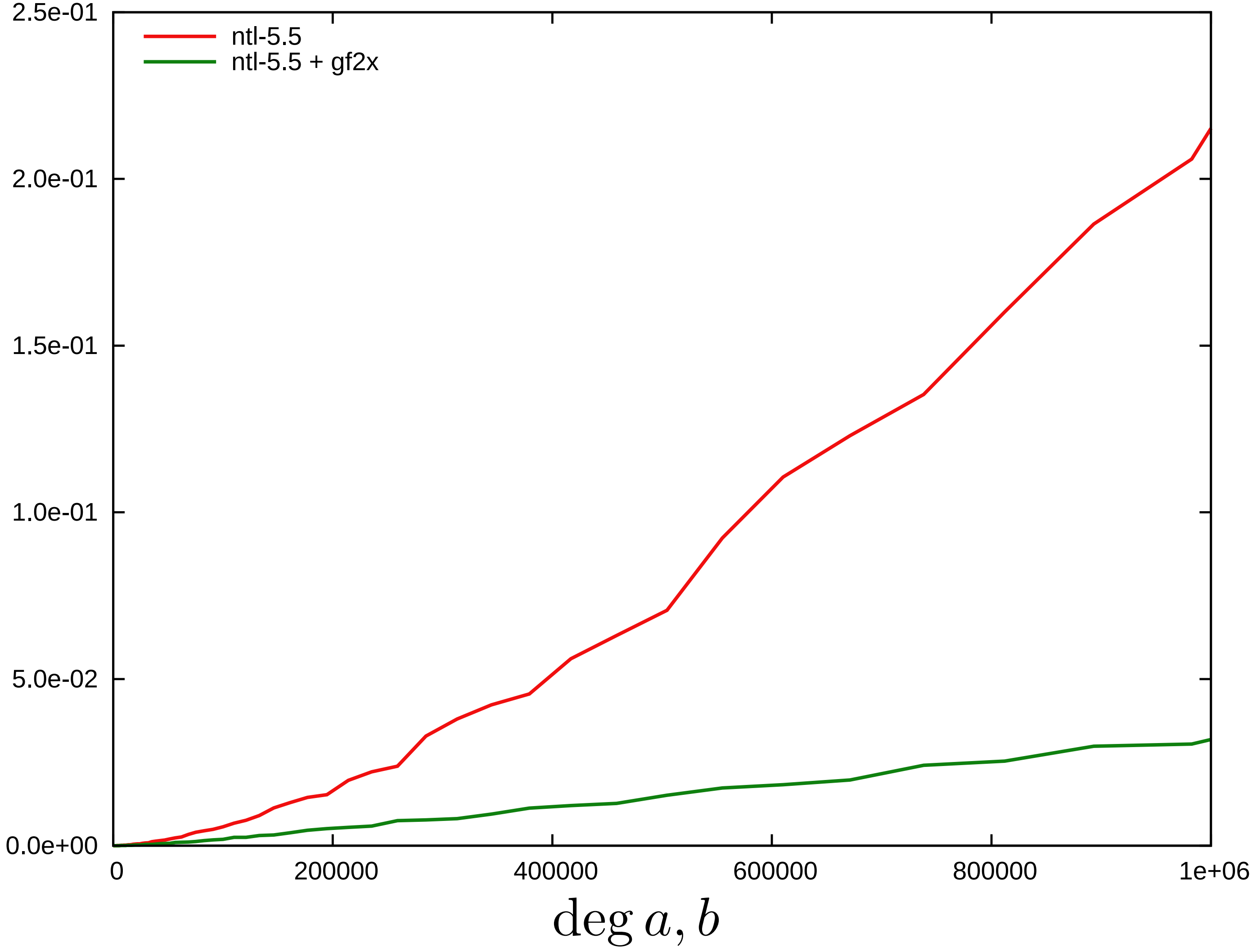

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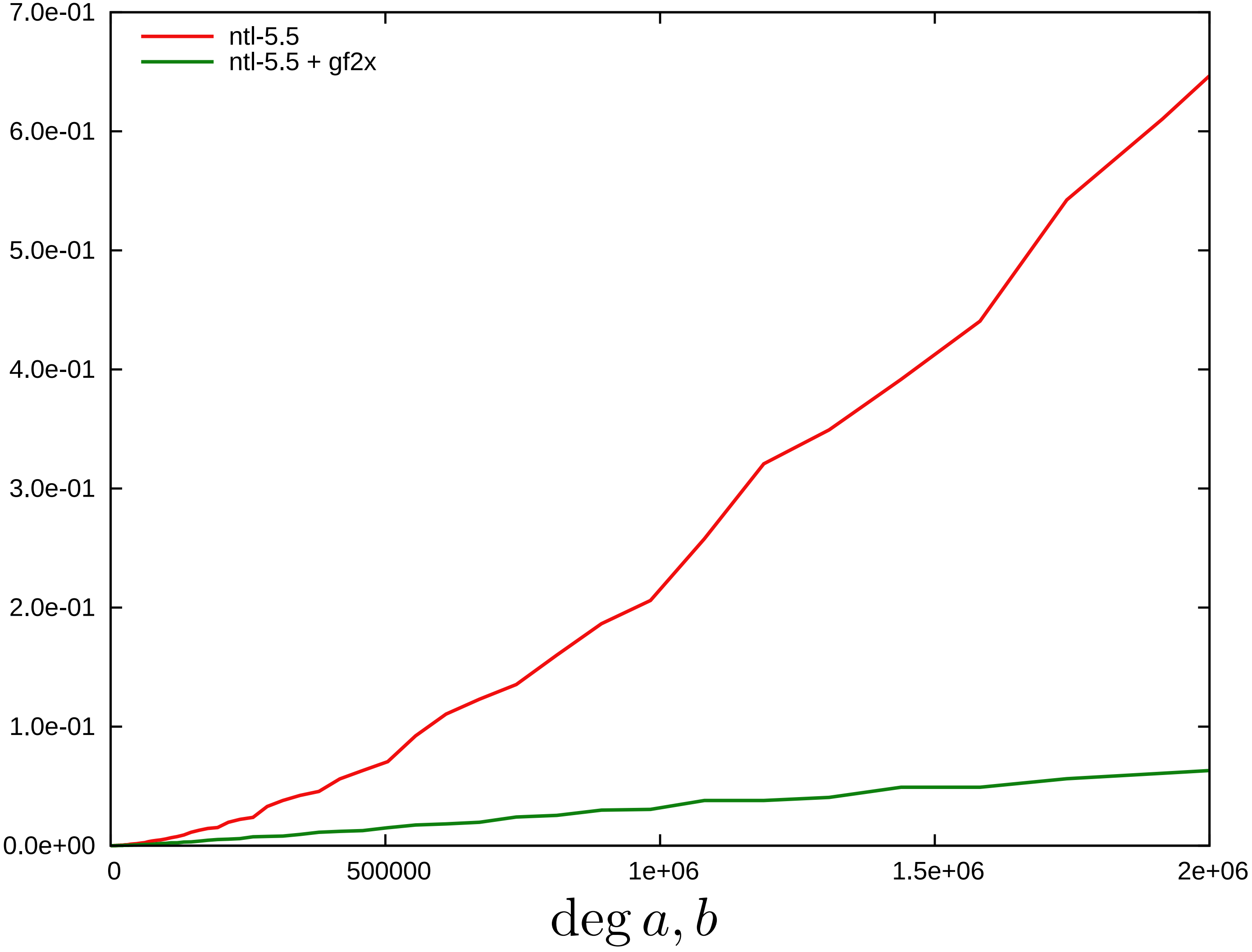

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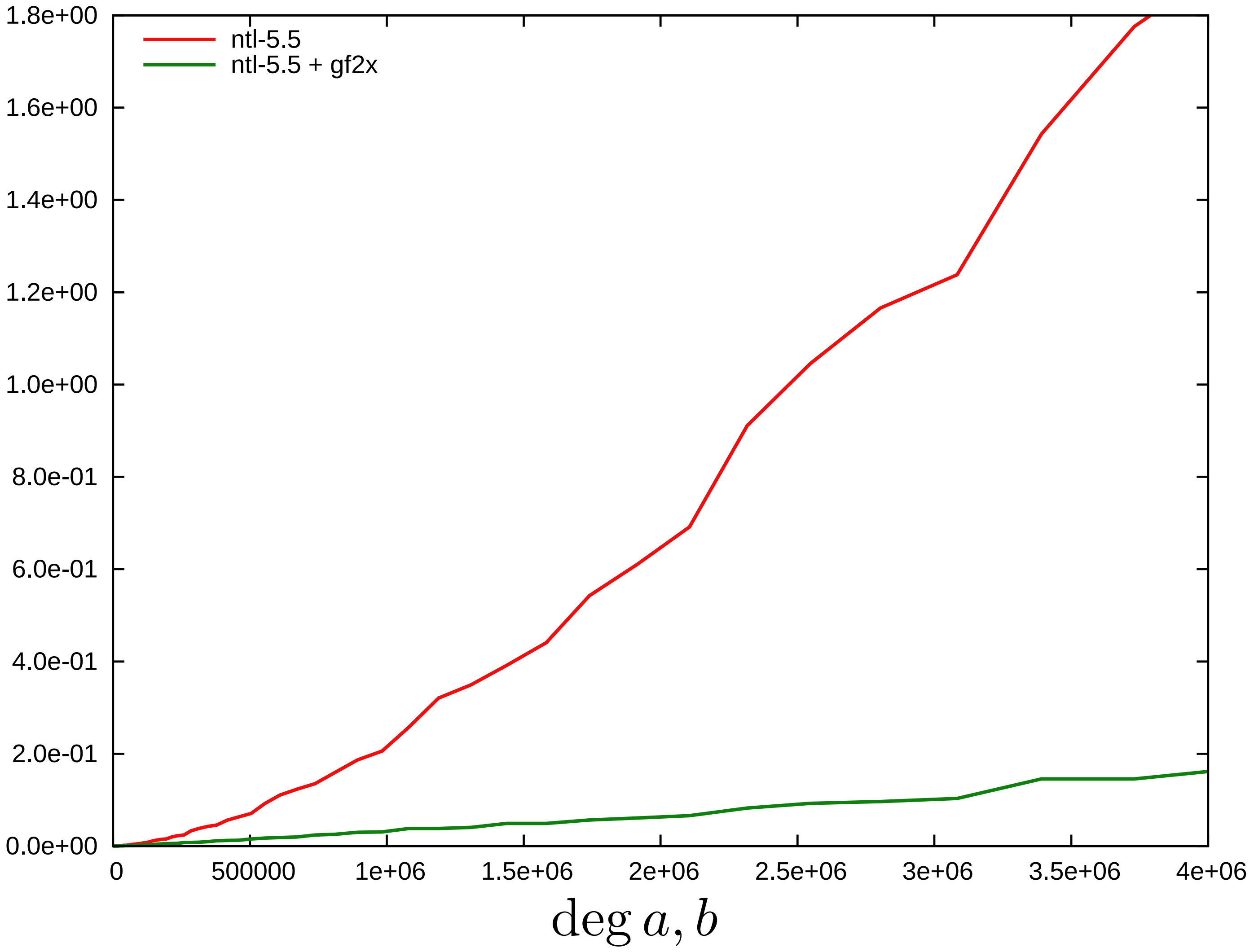

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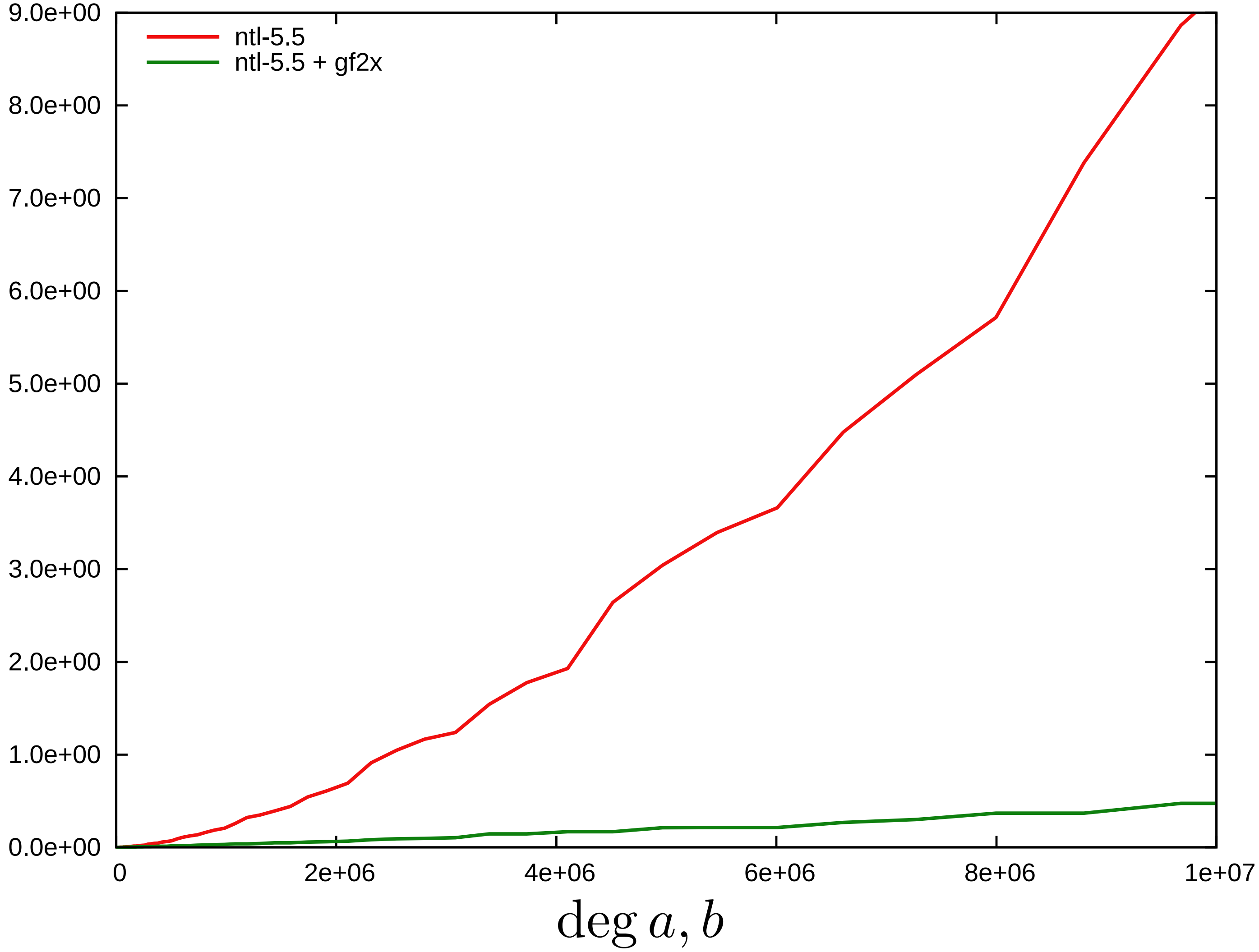

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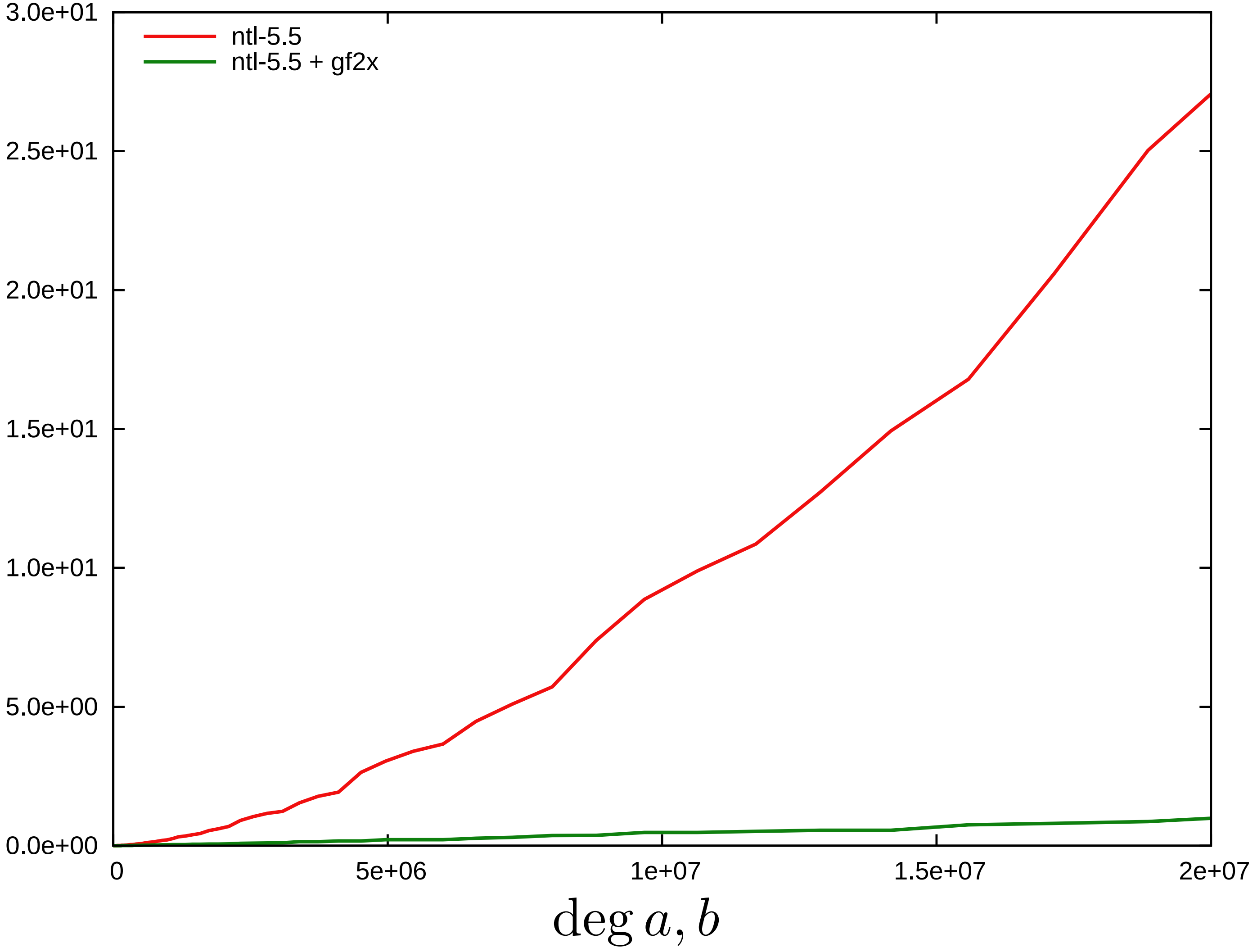

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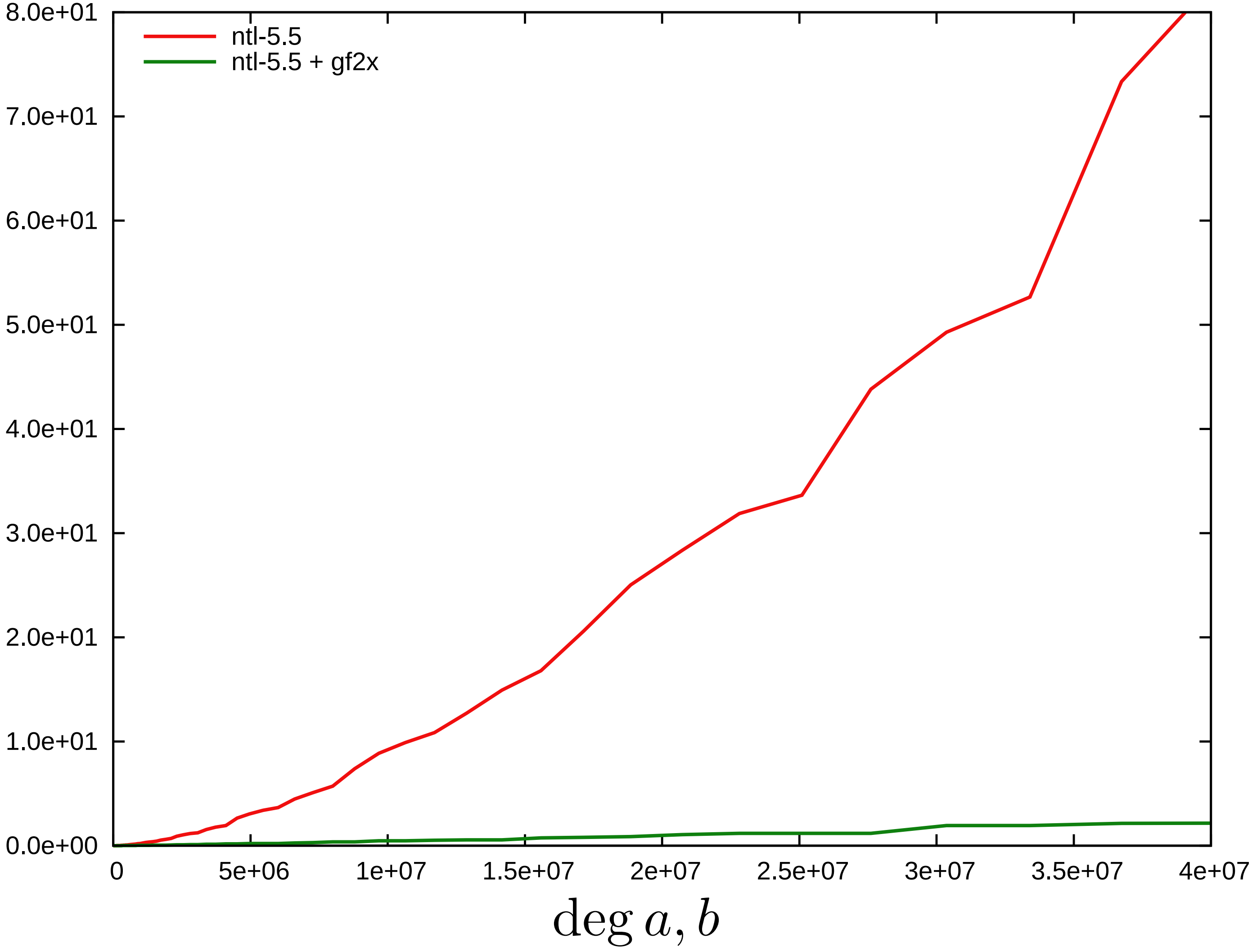

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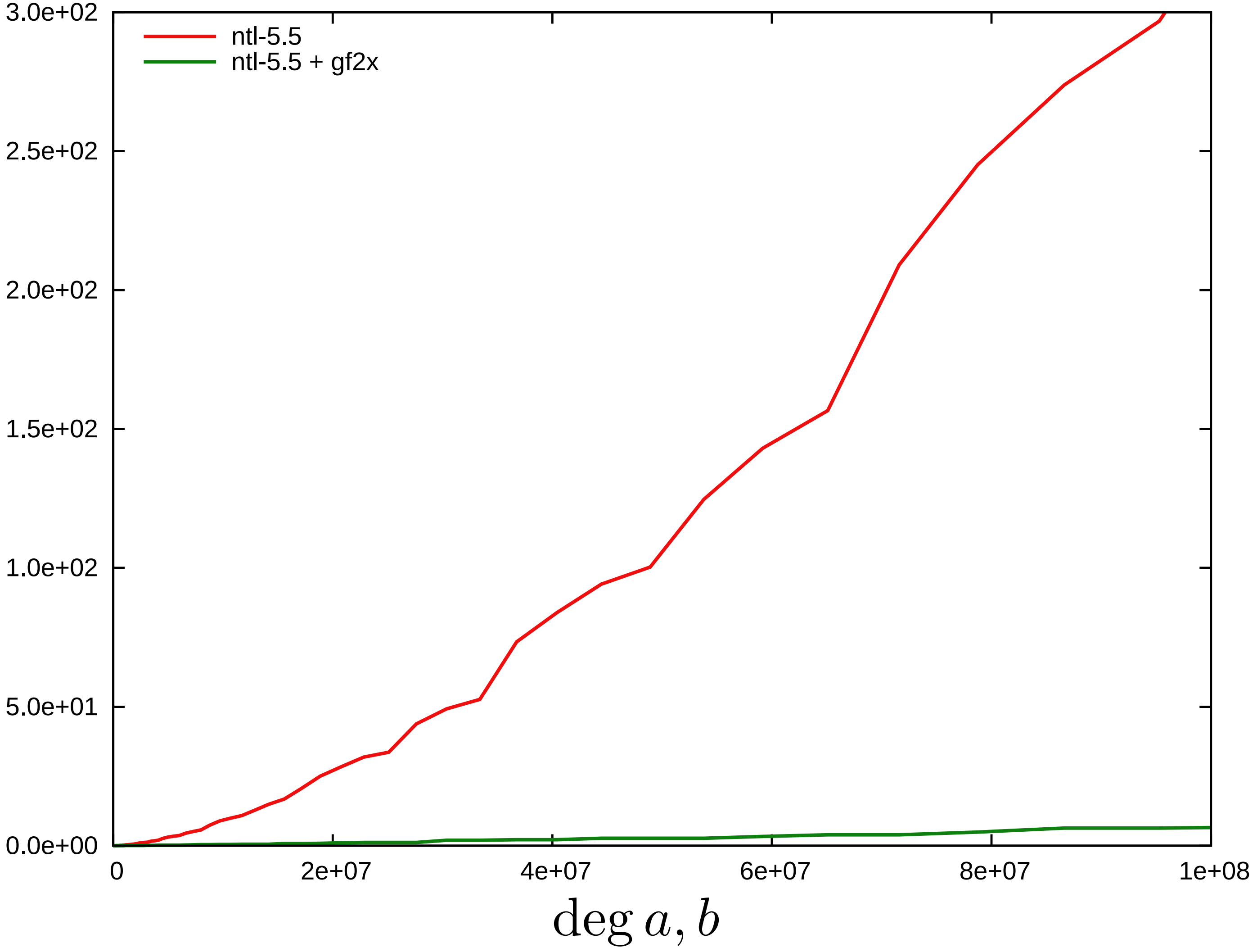

Notations

- Splitting of a polynomial $a(x)$ in s-bit slices:

$$\begin{aligned} a(x) &= A(x, x^s), \\ A(x,t) &= A_0(x) + A_1(x)t + A_2(x)t^2 + \cdots \\ &\text{and } \deg A_i < s. \end{aligned}$$

- Reconstruction: from $A(x,t)$, compute $a(x) = A(x, x^s)$.

Only of notational interest ; “computationally, nothing happens”.

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Below degree 64: `mul1`

Classical: $c = a \times b$ computed with a (fixed-) window method.

- Tabulate multiples $g \times b$, for $\deg g < s$ ($s =$ window size).
- Split $a = A_0 + A_1 x^s + A_2 x^{2s} + \cdots$.
- Accumulate $c = A_0 \times b + (A_1 \times b)x^s + (A_2 \times b)x^{2s} + \cdots$.

Operations required: shifts, XORs.

For degree below 64, we work with machine words only.

- ⚠ For $\deg b = 63$, the computation $A_i \times b$ overflows !
- Necessary “repair” step is rather easy (see paper).

`mul1` (cont'd)

What is the best window size ?

- Use trial and error. Typically 3 or 4.
- 64×64: ~ 75 CPU cycles on Intel `core2` and `i7` ;
  ~ 85 CPU cycles on AMD `k8`.

Note: This trivially extends to a routine for $64k \times 64$.

Using SIMD capabilities

What about 128×128 ?

- Karatsuba ⇒ three 64×64.
- Schoolbook requires $a \times b_{\text{low}}$ and $a \times b_{\text{high}}$ ⇒ two 128×64.
- BUT $a \times b_{\text{low}}$ and $a \times b_{\text{high}}$ can be computed in a SIMD-manner.
- SIMD instructions on `x86_64` provide the necessary shifts and XORs.
  - Accessible with compiler builtins (`gcc`, `icc`, MSVC).
  - Assembly is not an absolute necessity.

128×128:

- ~ 129 Intel `core2` cycles ;
- ~ 226 AMD `k8` cycles.
- Faster than Karatsuba here.

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Medium sizes

Classical: from 2 to 9 machine words, hard-code Karatsuba multiplication.

$\Rightarrow$ No branching.

Example for `mul4`:

```
mul2 (c, a, b);
mul2 (c + 4, a + 2, b + 2);
aa[0] = a[0] ^ a[2]; aa[1] = a[1] ^ a[3];
bb[0] = b[0] ^ b[2]; bb[1] = b[1] ^ b[3];
c24 = c[2] ^ c[4];
c35 = c[3] ^ c[5];
mul2 (ab, aa, bb);
c[2] = ab[0] ^ c[0] ^ c24; c[3] = ab[1] ^ c[1] ^ c35;
c[4] = ab[2] ^ c[6] ^ c24; c[5] = ab[3] ^ c[7] ^ c35;
```

Medium sizes

Classical: from 2 to 9 machine words, hard-code Karatsuba multiplication.

$\Rightarrow$ No branching.

Cycle counts, Intel `core2`.

$\deg$	NTL	LIDIA	ZEN	this paper
63	99	117	158	**75**
127	368	317	480	**132**
191	703	787	1 005	**364**
255	1 130	988	1 703	**410**
319	1 787	1 926	2 629	**806**
383	2 182	2 416	3 677	**850**
447	3 070	2 849	4 960	**1 242**
511	3 517	3 019	6 433	**1 287**

What comes next ?

Toom-3: $\deg a < 3k$, write $a = A(x, x^k)$, $A(x,t) = a_0(x) + a_1(x)t + a_2(x)t^2$.

- Evaluate $(A(x, x_i))_{i=0,1,2,3,4}$ and $(B(x, x_i))_{i=0,1,2,3,4}$
- Multiply: $C(x, x_i) = A(x, x_i)B(x, x_i)$.
- Interpolate: recover $C(x,t)$ from $(C(x, x_i))_{i=0,1,2,3,4}$

Misbelief:
- This is only for $\#K \geq 4\ldots$
- because we need 5 evaluation points (in $\mathbb{P}^1(K)$).

- We can use: $0, 1, \infty, x, x^{-1}$; call this TC3.
- Often better: $0, 1, \infty, x^{64}, x^{-64}$: avoids shifts ; call this TC3W.
- The degrees in recursive calls increase mildly.

Timings

$1 + \deg$	NTL	`gf2x`	method
$1\,536$	0.008	0.004	TC3
$4\,096$	0.033	0.017	K2
$8\,000$	0.098	0.046	TC3W
$10\,240$	0.160	0.080	TC3W
$16\,384$	0.295	0.140	TC3W
$24\,576$	0.567	0.240	TC3W
$32\,768$	0.887	0.395	TC4
$57\,344$	2.331	0.976	TC4
$65\,536$	2.667	1.067	TC4
$131\,072$	7.937	3.040	TC4

`ms`, Intel Core2 2.4GHz ; cycles:$\times 2.4 \cdot 10^6$.

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Large sizes

We are also interested in multiplication in the FFT range.

Several options:

- integer FFT and (huge) padding (Krönecker-Schönhage).
- Schönhage's ternary FFT algorithm.
- Cantor's additive FFT algorithm.
- . . .

Why is Krönecker's trick inefficient here ?

Assume we know how to multiply (fast) in $\mathbb{Z}$.

We use that in for multiplying in $\mathbb{F}_p[x]$.

If $a(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}$, and $b(x)$ similar:

- Choose $B = 2^k$ such that $Np^2 < 2^k$.
- Form the integer $\tilde{a} = a_0 + a_1 B + \cdots$. Same for $\tilde{b}$. We have:

$$\tilde{a} \cdot \tilde{b} = \sum_{\ell} \underbrace{\sum_{i+j=\ell} a_i b_j}_{\tilde{c}_\ell < Np^2 < B} B^\ell .$$

- The coefficients of $c = a \times b$ are recovered with $c_i = \tilde{c}_i \bmod p$.

Cost: $\mathsf{M}\left(N \log\left(Np^2\right)\right)$, to be compared to $\mathsf{M}(N \log p)$ (quasi-linearity w.r.t input size). This is very expensive for p small.

Other ways to do the FFT

Assume we are given one quotient ring R of $\mathbb{F}_2[x]$ of degree d with

- (reasonably) fast multiplication ;
- fast multipoint evaluation and interpolation on some subset W.

Then we can use this to compute products in $\mathbb{F}_2[x]$ up to $\frac{1}{2}d.\#W$ bits.

- Split a and b in blocks of $\frac{d}{2}$ coefficients: $a = A(x, x^{d/2})$. Map A to $R[t]$.
- Multi-evaluate A and B at W.
- Compute pointwise products $\{A(w) \cdot B(w), w \in W\}$.
- Interpolate: recognize C such that $\forall w \in W, C(w) = A(w) \cdot B(w)$.
  Note that $C = AB$ as long as $\deg(AB) < \#W$.
- Lift C to $\mathbb{F}_2[x, t]$. Recover $ab = C(x, x^{d/2})$.

Where to do the FFT ?

Several possibilities. Pitfall: W can not be a set of 2^n-th roots of 1 !

- Let $R = \mathbb{F}_{2^k}$ with $2^k - 1$ having a large smooth factor K.
  Let $W = \{K$-th roots of $1\}$.
- Let $R = \mathbb{F}_2[x]/x^{2L} + x^L + 1$, where $L = \lambda 3^{k-1}$.
  Then x^λ generates $W = \{3^k$-th roots of $1\}$.
- Let $R = \mathbb{F}_{2^{2^k}}$ and W be a sub-vector space.

“Butterflies” go “3-way”.

$$x, y \to x + \alpha y, x - \alpha y,$$

$$x, y, z \to x + \alpha y + \alpha^2 z, x + j\alpha y + j^2\alpha^2 z, x + j^2\alpha y + j\alpha z.$$

Where to do the FFT ?

Several possibilities. Pitfall: W can not be a set of 2^n-th roots of 1 !

- Let $R = \mathbb{F}_{2^k}$ with $2^k - 1$ having a large smooth factor K.
  Let $W = \{K$-th roots of $1\}$. not looked at.
- Let $R = \mathbb{F}_2[x]/x^{2L} + x^L + 1$, where $L = \lambda 3^{k-1}$.
  Then x^λ generates $W = \{3^k$-th roots of $1\}$. ternary FFT.
- Let $R = \mathbb{F}_{2^{2^k}}$ and W be a sub-vector space. additive FFT.

“Butterflies” go “3-way”.

$$x, y \to x + \alpha y, x - \alpha y,$$

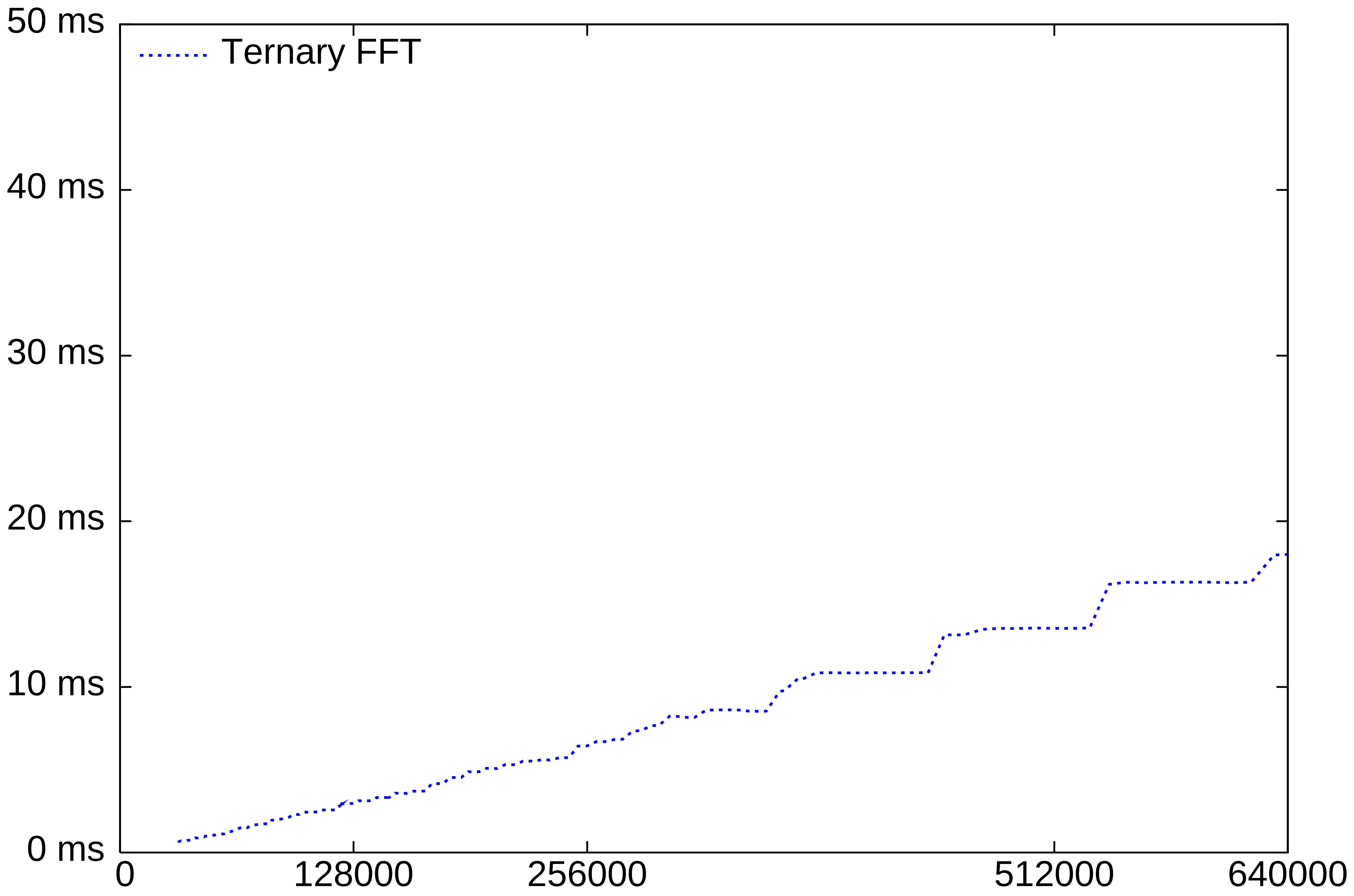
$$x, y, z \to x + \alpha y + \alpha^2 z, x + j\alpha y + j^2\alpha^2 z, x + j^2\alpha y + j\alpha z.$$

Ternary FFT (Schönhage)

The ternary FFT achieves $O(N \log N \log \log N)$ complexity if one uses it recursively also for the pointwise products, BUT:

- This requires computing the pointwise products modulo $X^{3L} + 1$.
- One doesn't *have* to. One top-level step gives already good results.
- Have to choose a sensible value for K.

Ternary FFT (Schönhage)



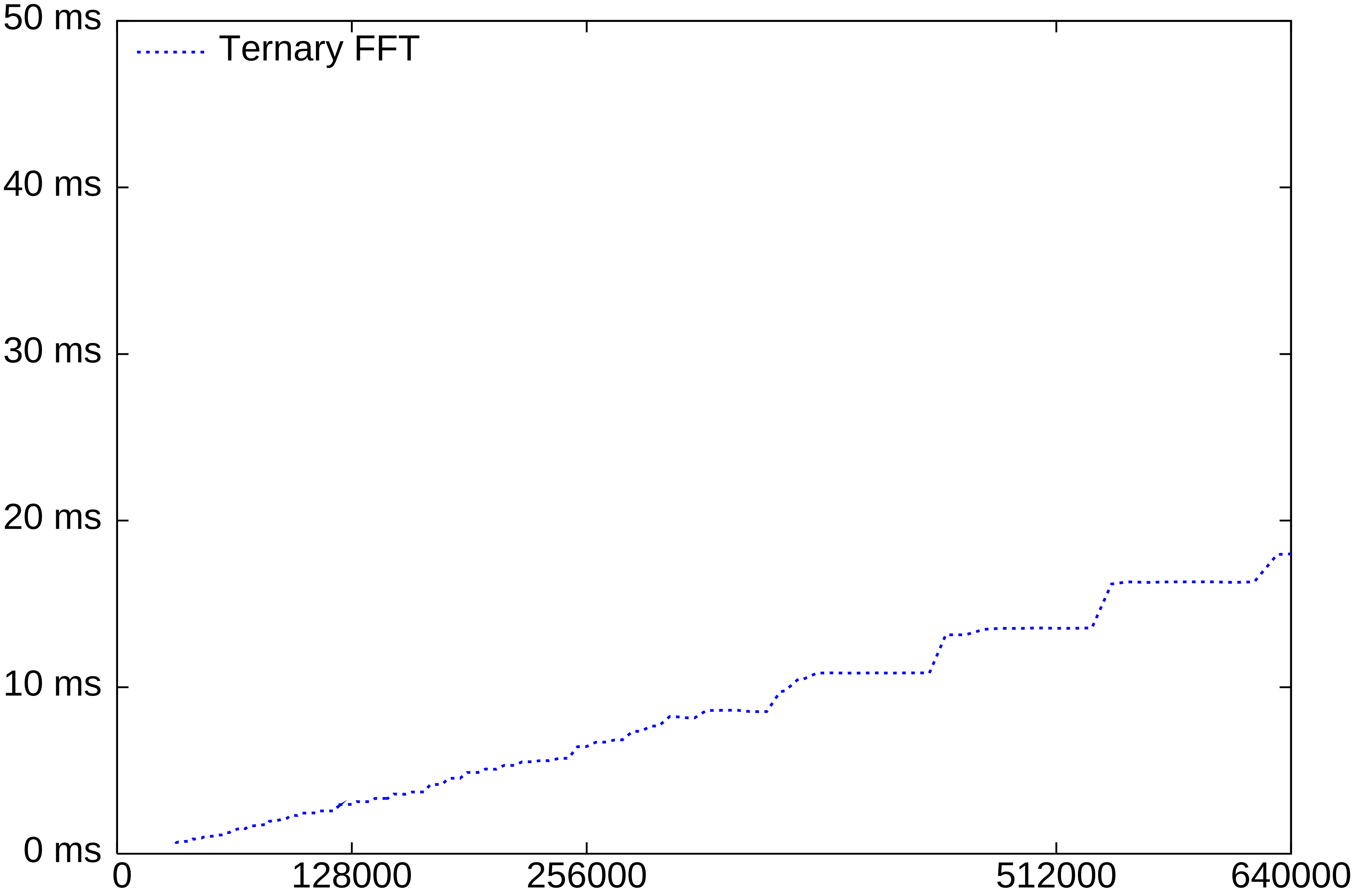
Splitting the ternary FFT

There is a (mild) staircase effect.

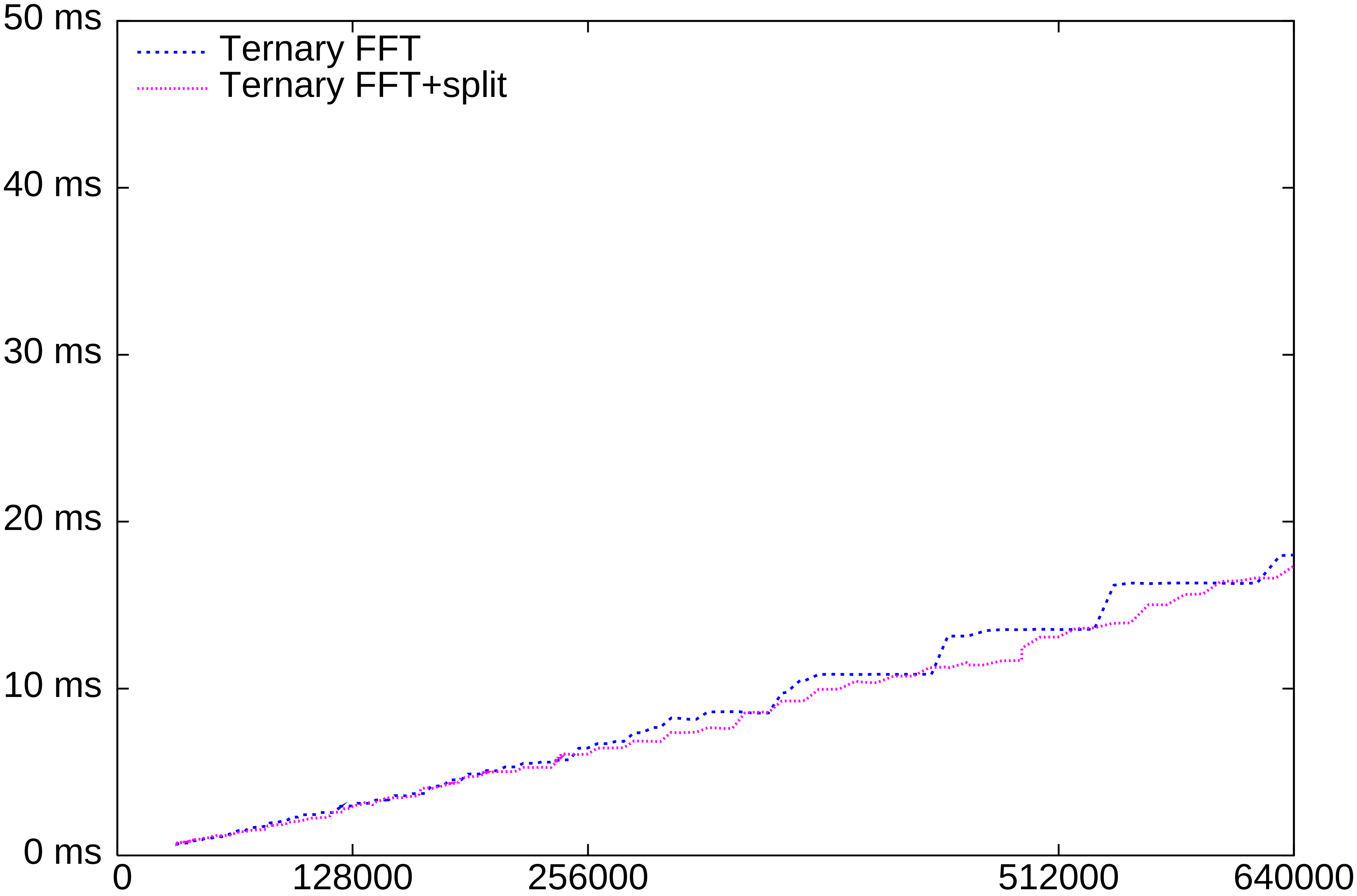
We can compute a product of degree $< N$ by splitting:

- Compute one product modulo $N' > N/2$.
- Compute another product modulo $N'' > N'$.
- Very simple XORs do the reconstruction.

Schönhage FFT + splitting



Schönhage FFT + splitting



Cantor's additive FFT

A completely different approach. Use a field $R = F_k = \mathbb{F}_{2^{2^k}} = \mathbb{F}_2[\gamma]$.

Which evaluation set W do we use ?

Let $s_1(x) = x^2 + x$, and $s_i(x) = \underbrace{s_1(s_1(\cdots s_1}_{i \text{ times}}(x)\cdots))$.

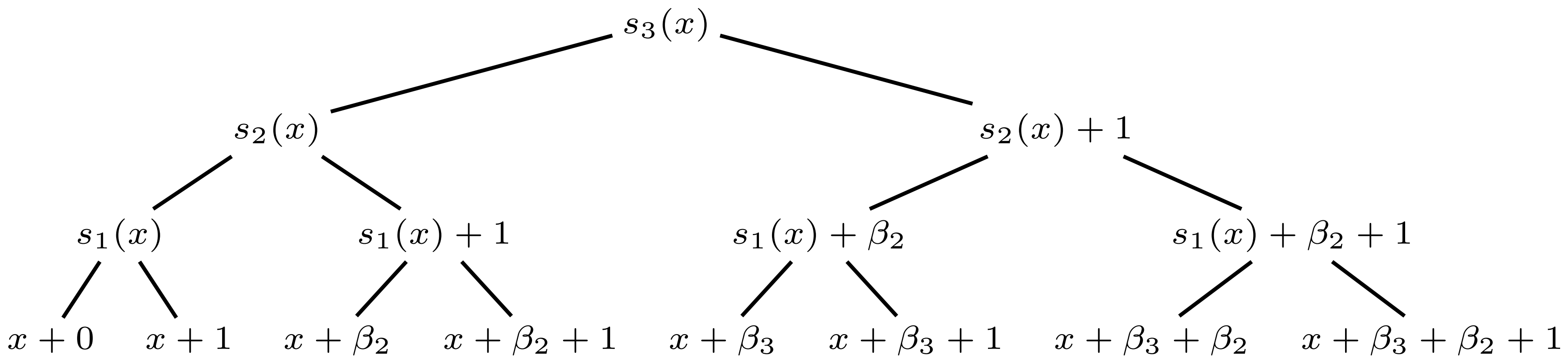
s_i satisfies many properties:

- s_i is sparse ; s_i is linear ; $s_{2^k} = x^{2^{2^k}} + x$.
- Let $2^k \geq i$ and $W_i = \{\alpha \in \mathbb{F}_{2^{2^k}} \mid s_i(\alpha) = 0\} = \operatorname{Ker} s_i$.
  W_i is a sub-vector space of $\mathbb{F}_{2^{2^k}}$; $\dim W_i = i$.

Multi-evaluation at W_i

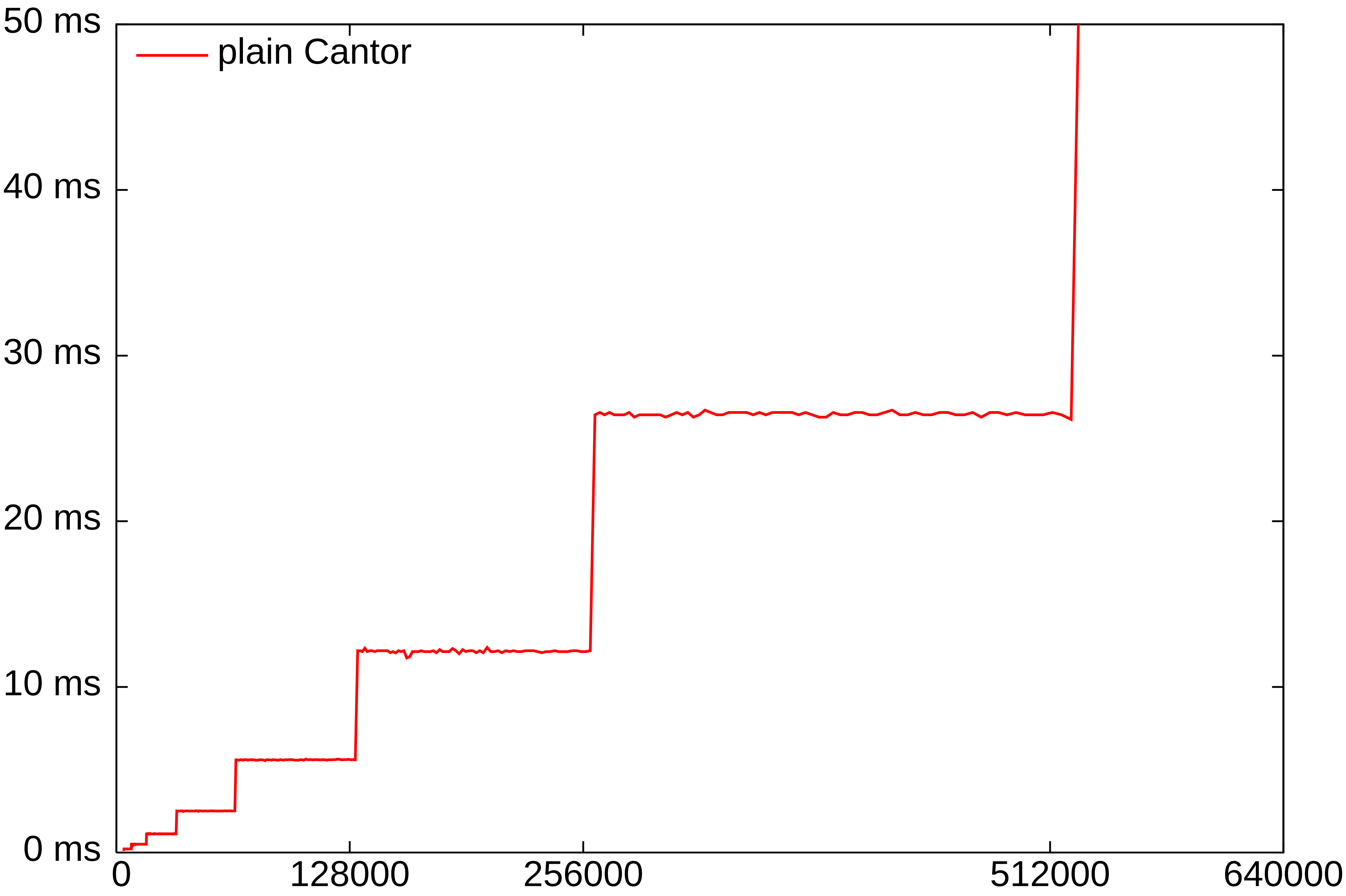
Use a sub-product tree:

$$\{f(\alpha),\ \alpha \in W_i\} = \{f \bmod (x+\alpha),\ \alpha \in W_i\}\,.$$



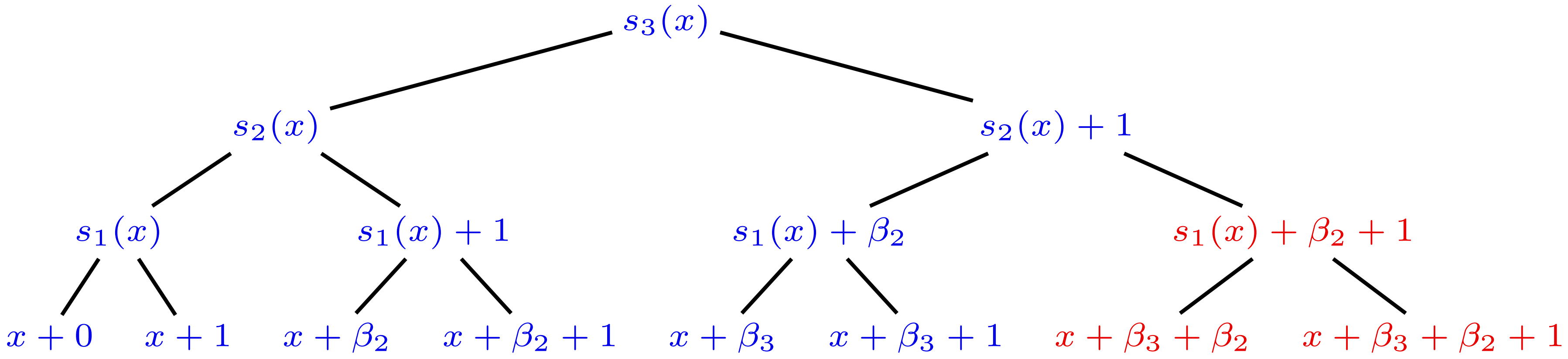
- right-child $= 1 +$ left-child.
- Only the constant coefficients are in extension fields.
- s_j is sparse, so reduction is cheap.

Cantor: staircase effect



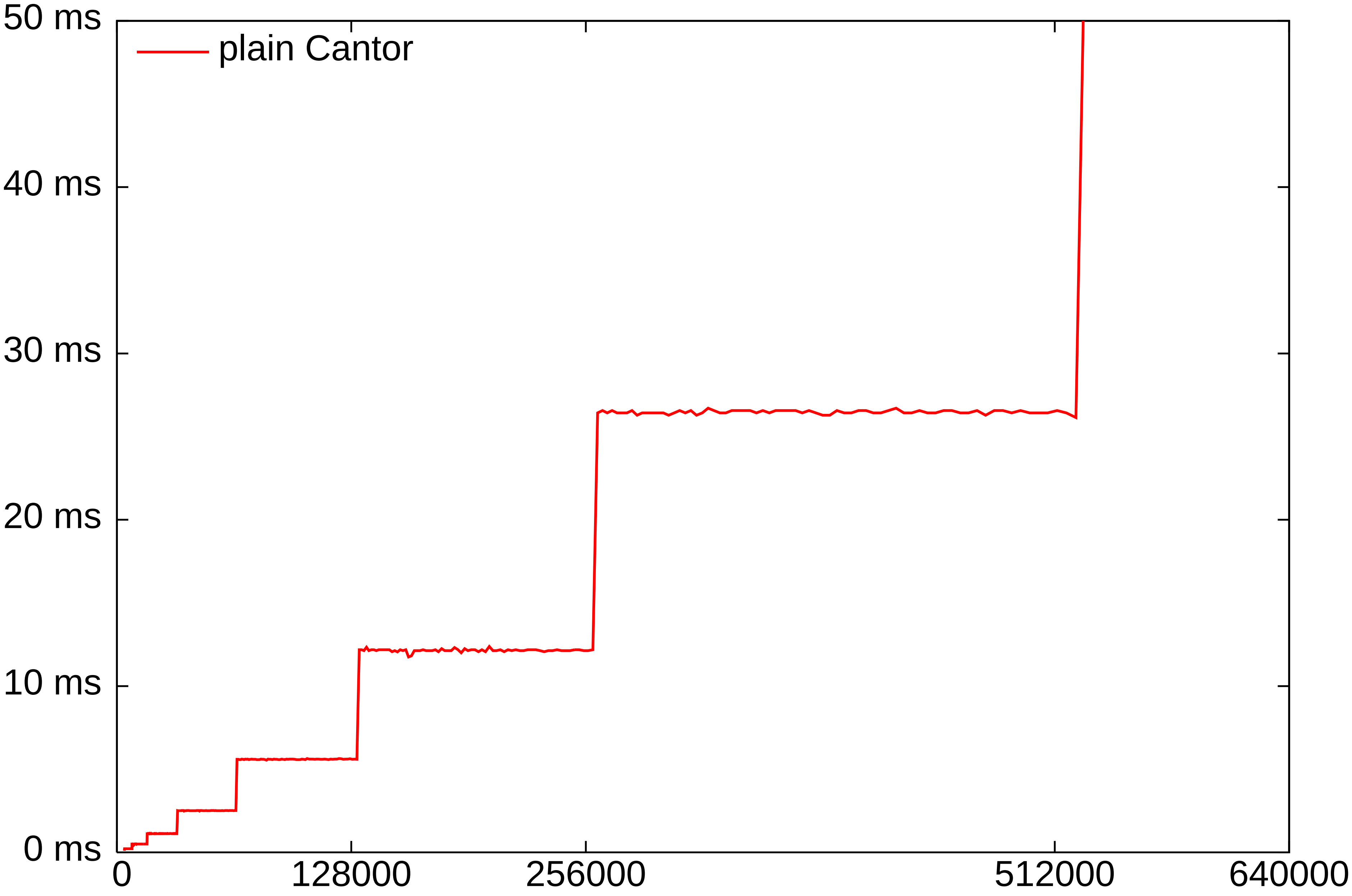
A truncated variant

- Evaluate at no more points than needed. Example for 6 points:



- Work modulo $s_2(x)\,(s_1(x)+\beta_2)$ instead of $s_3(x)$.
- Interpolation is tricky. Uses the sub-product tree twice. See paper.

Performance of additive FFT



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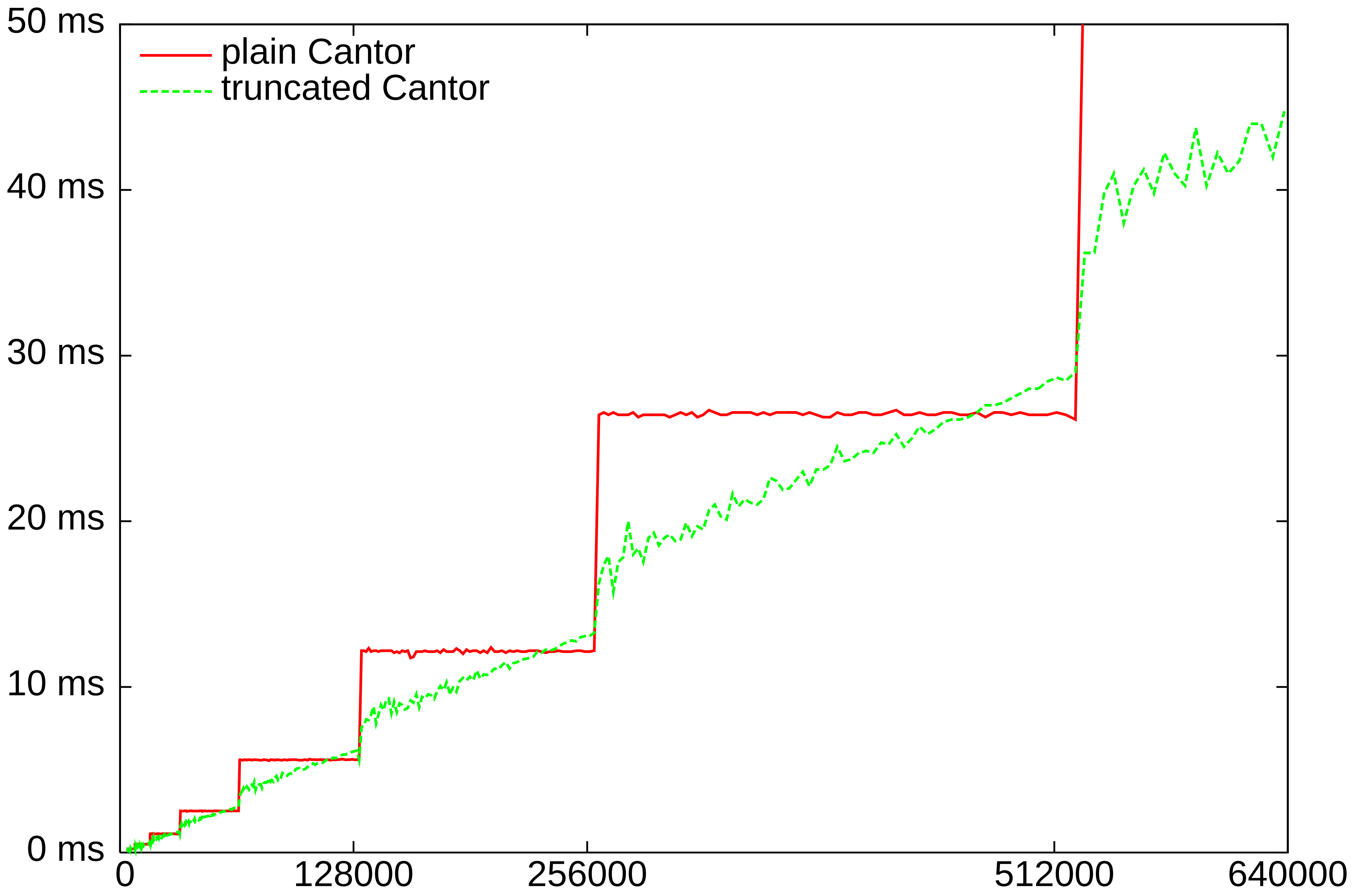

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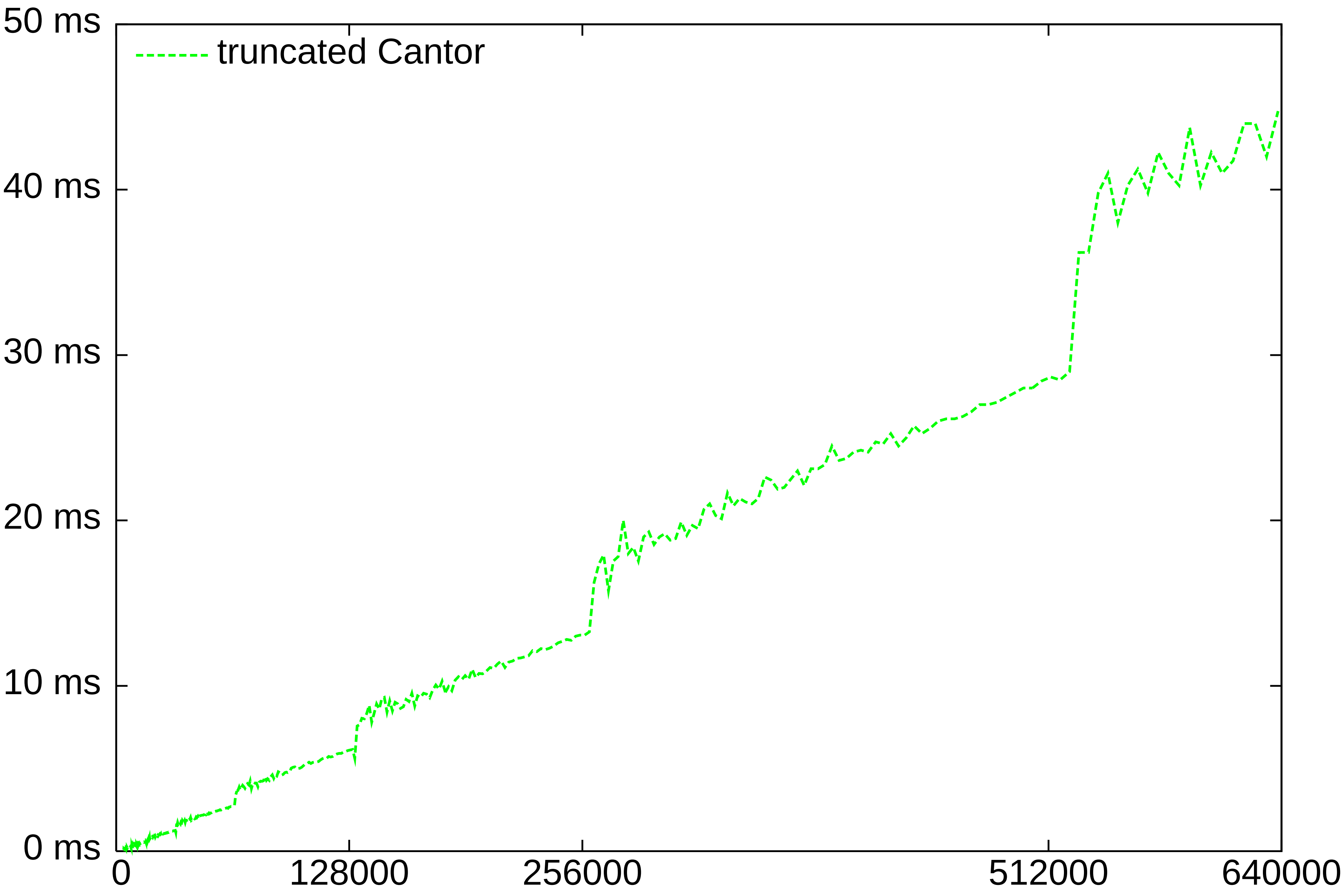
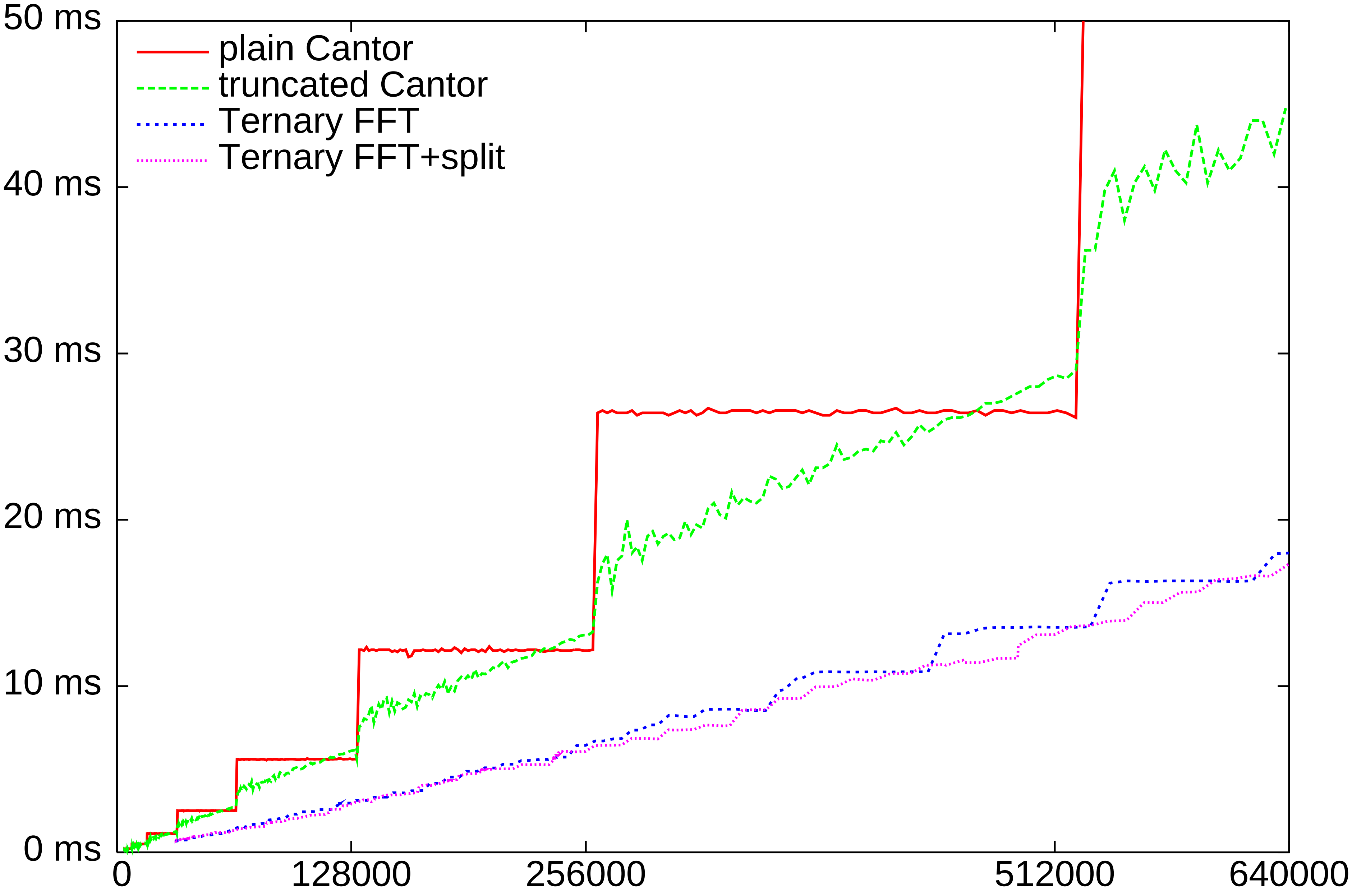


Comparison Cantor – Schönhage



Comparison Cantor – Schönhage

A word of caution:

- Additive FFT has cheap pointwise products.
- Ternary FFT has cheap evaluation / interpolation.

When transforms can be reused (matrices over $\mathbb{F}_2[x]$), additive FFT wins.
Example for $\deg ab < 2^{20}$:

- Additive FFT: 57 ms, 2.3 ms in pointwise mults.
- Ternary FFT: 28 ms, 18 ms in pointwise mults.
- $n \times n$ matrix mult: $c_{\text{eval/interp}} * n^2 + c_{\text{pointwise}} * n^3$
- Additive FFT faster for 3×3 matrices and above.

Conclusion

- Significant speed-ups over existing software.
- URL: `gf2x.gforge.inria.fr` (versions > 0.9 no longer have the additive FFT, because not routinely tested).
- Usable as an add-on to NTL 5.5: `NTL_GF2X_LIB=<path>`.

In the works: an update to expose the different steps of the transform, for algorithms where caching transforms is desired.