Optimality and the Manin conjecture

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  - How to compute the isogenous curves (next time)
  - How to determine which curve in the isogeny class is "optimal" (and what that means)
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- that's not good enough when writing an appendix to a paper by Agashe, Ribet and Stein!

Some notation and terminology

- $f = \sum_{n=1}^{\infty} a_n q^n$, a normalised ($a_1 = 1$) cusp form of weight 2 for $\Gamma_0(N)$ which is a *rational newform*, in particular an eigenform for the Hecke algebra with rational integer eigenvalues

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- $L(f, s)$, the *L-series of f*, given for $\Re(s) > 3/2$ by the Dirichlet series $\sum a_n n^{-s}$ satisfying a *functional equation*

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- the *analytic rank* r of f is the order of vanishing of $\Lambda(f,s)$ at $s = 1$; the *sign of the functional equation* is $(-1)^r$ which gives the parity of r

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- $L(f, 1)/\Omega_0(f)$ is rational, and zero iff $r > 0$

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From the a_p we can

- determine the Fourier coefficients a_n
- compute the approximate value of the periods $\langle \gamma, f \rangle$ for each $\gamma \in \Gamma_0(N)$
- compute r and $L^{(r)}(f,1)$ (provided $r \leq 3$!)

From newform to curve, continued

The modular symbol method also provides the following crucial "period data", in the form of five integers and one bit:

- a matrix $\gamma \in \Gamma_0(N)$ (only the bottom row matters)
- nonzero integers m^+, m^- and $t \in \{1, 2\}$

such that if we set $\langle \gamma, f \rangle = x_\gamma + iy_\gamma$, then the period lattice Λ_f has $\mathbb{Z}$-basis

- $(2x, x + iy)$ if $t = 1$, or
- (x, iy) if $t = 2$

where $x = x_\gamma / m^+$ and $y = y_\gamma / m^-$. Given this basis of Λ_f (to some precision) we can compute the constants $c_4 = c_4(\Lambda_f)$ and $c_6 = c_6(\Lambda_f)$ which are known to be *integers* and the invariants of the elliptic curve E_f. Manin's "$c = 1$ conjecture" is equivalent to the easily checked statement that E_f is a *minimal model*.

Practical issues I

There are two different practical issues which affect the validity of this claim:

> *I compute the full period lattice of the newform f and hence the c_4 and c_6 invariants of this lattice. Since these are known to be integers, I can guarantee that they are correct. In each case I check that they are minimal invariants of an elliptic curve E_f, from which it follows that E_f is the optimal curve associated to f and that "$c = 1$" for this curve.*

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- It is hard to ensure that c_4,c_6 are computed to sufficient precision to guarantee (rigorously!) that they are correct. (They are integers, but can be quite large, requiring high precision.) Each is computed from the periods of f by summing a series, and the periods are also computed by summing a series.

Practical issues II

- Computing the complete “period data” requires working in the full modular symbol space representing $H_1(X_0(N), Z)$, which is expensive. It is much cheaper to work in the “plus space” $H_1^+(X_0(N), Z)$, which has half the dimension. This still gives all the information about f *except* that we can only determine the projection of Λ_f onto $\mathbb{R}$: we cannot determine either m^- or t.

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In the rest of this lecture, we will show how to handle the first issue rigorously.

The same methods also help us obtain as much information as possible when we only the “plus part” of the homology, which is currently the case for levels between 60000 and 130000.

Optimality and Manin's constant

Let E_f be the elliptic curve associated to the newform f as before. This elliptic curve is *optimal*, or an *optimal quotient* of the Jacobian $J_0(N)$ of $X_0(N)$. This means that the map $J_0(N) \to E_f$ has connected kernel.

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The Néron differential ω of E_f is uniquely defined up to sign: it is $dX/(2Y + a_1X + a_3)$ for a minimal Weierstrass equation. Pulling back to $X_0(N)$ gives a non-zero rational multiple of ω_f, and the absolute value of this rational number is called the *Manin constant* c.

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- Manin conjectured that $c = 1$ in all cases; no counterexamples are known.
- $c \in \mathbb{Z}$ (Edixhoven 1991); hence the invariants c_4 and c_6 of E_f are certainly integral.
- Many other results on c are known: see Agashe, Ribet, Stein (2006)

How to show that $c = 1$

We will show how to do two things:

1. obtain a provably correct equation for the optimal curve E_f
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using the following input:

1. a $\mathbb{Z}$-basis w_1, w_2 for Λ_f to some precision
2. the lattice type $t \in \{1, 2\}$
3. a complete isogeny class of elliptic curves $\{E_1, \ldots, E_m\}$ of conductor N, given by minimal models, such that $L(E_j, s)$ and $L(f, s)$ agree at the first few Euler factors

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It does not matter how this list of curves is obtained! In practice, E_1 will be our "approximate E_f" and the other E_j, if any, are computed from these (see next lecture).

Step 0

Proposition

$E_f \cong E_j$ *(over* $\mathbb{Q}$*) for some* $j \in \{1, \ldots, n\}$*, and* $L(E_j, s) = L(f, s)$*.*

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Proposition

$E_f \cong E_j$ (over $\mathbb{Q}$) for some $j \in \{1, \ldots, n\}$, and $L(E_j, s) = L(f, s)$.

Proof.

We use the modularity of elliptic curves defined over $\mathbb{Q}$! We have computed a full set of newforms f at level N, and the same number of isogeny classes of elliptic curves; the theory tells us that there is a bijection between these sets. Checking the first few terms of the L-series (i.e., comparing the Hecke eigenforms of the newforms with the Frobenius traces of the curves) allows us to pair up each isogeny class with a newform. □

Lattice normalization

Every lattice Λ in $\mathbb{C}$ which defined over $\mathbb{R}$ has a unique $\mathbb{Z}$-basis ω_1, ω_2 satisfying one of the following:

Type 1: ω_1 and $(2\omega_2 - \omega_1)/i$ are real and positive; or

Type 2: ω_1 and ω_2/i are real and positive.

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For each curve E_j, we compute (to some precision) a $\mathbb{Z}$-basis for its period lattice Λ_j, using the standard AGM method. Here, Λ_j is the lattice of periods of the Néron differential on E_j. The type of Λ_j is determined by the sign of the discriminant of E_j: type 1 for negative discriminant, and type 2 for positive discriminant.

Lattice labelling

We label the curves E_j so that Λ_1 and Λ_f are *approximately equal*. To be precise, we will require that

$$\left|\frac{\omega_{1,1}}{\omega_{1,f}} - 1\right| < \varepsilon \qquad \text{and} \qquad \left|\frac{\Im(\omega_{2,1})}{\Im(\omega_{2,f})} - 1\right| < \varepsilon \qquad (*)$$

for a specific $\varepsilon > 0$.

Here $\omega_{1,j}$, $\omega_{2,j}$ denote the normalised generators of Λ_j, and $\omega_{1,f}$, $\omega_{2,f}$ those of Λ_f.

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Here $\omega_{1,j}$, $\omega_{2,j}$ denote the normalised generators of Λ_j, and $\omega_{1,f}$, $\omega_{2,f}$ those of Λ_f.

The value of ε will be chosen later. For $N < 130000$ we never needed to use an $\varepsilon < \frac{1}{5}$, and in the vast majority of cases, $\varepsilon = 1$ was sufficient!

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Pulling back the Néron differential on E_{j_0} to $X_0(N)$ gives $c \cdot \omega_f$ where $c \in \mathbb{Z}$ is the Manin constant for f. Hence

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The way we have set things up, we will actually prove that $j_0 = 1$ and $c = 1$. As we will soon see, this will follow from (*), provided that ε is small enough.

Stevens' lattice inclusion lemma

A result of Glenn Stevens (1989) implies that in the isogeny class there is a curve, say E_{j_1}, whose period lattice Λ_{j_1} is contained in every Λ_j; this is the unique curve in the class with minimal Faltings height.

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For each j, we know therefore that there exist integers a_j, b_j such that

$$\omega_{1,j_1} = a_j \omega_{1,j} \qquad \text{and} \qquad \Im(\omega_{2,j_1}) = b_j \Im(\omega_{2,j}).$$

Let $B = \max(a_1, b_1)$.

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Remark: It is conjectured that E_{j_1} is the $\Gamma_1(N)$-optimal curve, but we do not need or use this fact. In many cases, the $\Gamma_0(N)$- and $\Gamma_1(N)$-optimal curves are the same, so we expect that $j_0 = j_1$ often; indeed, this holds for the vast majority of cases.

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We used this to establish the following:

Theorem

For all $N < 60000$, every optimal elliptic quotient of $J_0(N)$ has Manin constant equal to 1. Moreover, the optimal curve in each class is the one whose identifying number in the tables is 1 (except for class $990h$ where the optimal curve is $990h3$).

Proof of the Theorem

For all $N < 60000$ we used modular symbols to find all newforms f and their period lattices, and also the corresponding isogeny classes of curves. In all cases we verified that (*) held with the appropriate value of ε. (The case of $990h$ is only exceptional on account of an error in labelling the curves several years ago, and is not significant.)

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In the vast majority of cases, the value of B is 1, so the precision needed for the computation of the periods is very low. For $N < 60000$, out of 258502 isogeny classes, only 136 have $B > 1$: we found $a_1 = 2$ in 13 cases, $a_1 = 3$ in 29 cases, and $a_1 = 4$ and $a_1 = 5$ in one case each (for $N = 15$ and $N = 11$ respectively); $b_1 = 2$ in 93 cases; and no larger values. Class $17a$ is the only one for which both a_1 and b_1 are greater than 1 (both are 2). □

Proof of the key proposition

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For some $j(= j_0)$ we have $c\Lambda_f = \Lambda_j$.

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The idea is that

$$\operatorname{lcm}(a_1, b_1)\Lambda_1 \subseteq \Lambda_{j_1} \subseteq \Lambda_j = c\Lambda_f;$$

if $a_1 = b_1 = 1$, then the closeness of Λ_1 and Λ_f forces $c = 1$ and equality throughout.

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To cover the general case it is simpler to work with the real and imaginary periods separately.

We have

$$c = \frac{\omega_{1,j}}{\omega_{1,f}} = \frac{\omega_{1,1}}{\omega_{1,f}} \frac{\omega_{1,j}}{\omega_{1,1}} = \lambda \frac{a_1}{a_j} \in \mathbb{Z}.$$

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$$\frac{\Im(\omega_{2,j})}{\Im(\omega_{2,f})} = c \in \mathbb{Z}$$

and again we can conclude that $\Im(\omega_{2,1}) = \Im(\omega_{2,f})$, and hence $\omega_{2,1} = \omega_{2,f}$.

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Thus $\Lambda_1 = \Lambda_f = c^{-1}\Lambda_j$, from $j = 1 = c$ follows. □

The range $60000 < N < 130000$

Our results for $60000 < N < 130000$ are slightly weaker, and require more work.

In this range we do not know Λ_f precisely, but only its projection onto the real line. We argue as before, using $B = a_1$.

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Theorem

For all N in the range $60000 < N < 130000$, every optimal elliptic quotient of $J_0(N)$ has Manin constant equal to 1.

Recall that when we have only used the “plus part” of the homology of $X_0(N)$, we are only able obtain partial information.

Instead of knowing the lattice type and approximate basis for Λ_f, what we have is (to some precision) a positive real number $\omega_{1,f}^{+}$ such that

- *either* Λ_f has type 1 and $\omega_{1,f} = 2\omega_{1,f}^{+}$,
- *or* Λ_f has type 2 and $\omega_{1,f} = \omega_{1,f}^{+}$.

Curve E_1 has lattice Λ_1, and the ratio $\lambda = \omega_{1,1}^{+}/\omega_{1,f}^{+}$ satisfies $|\lambda - 1| < \varepsilon$.

In the range $60000 < N < 130000$ we have $a_1 = 1$ for all but three cases where $a_1 = 3$, and the inequality holds with $\varepsilon = \frac{1}{3}$, which suffices.

First assume that $a_1 = 1$.

If the type of Λ_f is the same as that of Λ_1 (for example, this must be the case if all the Λ_j have the same type, which will hold whenever all the isogenies between the E_j have odd degree) then from $c\Lambda_f = \Lambda_j$ we deduce as before that $\lambda = 1$ exactly, and $c = a_1/a_j = 1/a_j$, hence $c = a_j = 1$. So in this case we have that $c = 1$, though there might be some ambiguity in which curve is optimal if $a_j = 1$ for more than one value of j.

Assume next that Λ_1 has type 1 but Λ_f has type 2. Now $\lambda = \omega_{1,1}/2\omega_{1,f}$. The usual argument now gives $ca_j = 2$. Hence

- *either* $c = 1$ and $a_j = 2$,
- *or* $c = 2$ and $a_j = 1$.

To see if the case $c = 2$ and $a_j = 1$ could occur, we looked for classes in which $a_1 = 1$ and Λ_1 has type 1, while for some $j > 1$ we have $a_j = 1$ and Λ_j of type 2.

This occurs 28 times for $60000 < N < 130000$.

For 15 of these the level N is odd, so we know (Abbes–Ulmo 1996) that c must be odd. The remaining 13 are

$$62516a, 67664a, 71888e, 72916a, 75092a, 85328d, 86452a, 96116a,\\ 106292b, 111572a, 115664a, 121168e, 125332a,$$

all of which were eliminated by carrying out extra computations in $H_1(X_0(N), \mathbb{Z})$.

In all of these 13 cases, the isogeny class consists of two curves, E_1 of type 1 and E_2 of type 2, with $[\Lambda_1 : \Lambda_2] = 2$, so that E_2 rather than E_1 has minimal Faltings height.

Next suppose that Λ_1 has type 2 but Λ_f has type 1. Now $\lambda = 2\omega_{1,1}/\omega_{1,f}$. The usual argument now gives $2ca_j = 1$, which is impossible; so this case cannot occur.

Next suppose that Λ_1 has type 2 but Λ_f has type 1. Now $\lambda = 2\omega_{1,1}/\omega_{1,f}$. The usual argument now gives $2ca_j = 1$, which is impossible; so this case cannot occur.
Finally we consider the cases where $a_1 > 1$. There are only three of these for $60000 < N < 130000$: namely, $91270a$, $117622a$ and $124973b$, where $a_1 = 3$. In each case the Λ_j all have the same type (they are linked via 3-isogenies) and the usual argument shows that $ca_j = 3$. But none of these levels is divisible by 3, so $c = 1$ in each case. $\square$

Final remarks

1. In the vast majority of cases we can verify that $c = 1$ using only the plus part of homology. The remaining cases can be eliminated either by doing the extra work needed to obtain full homology information, or by using known results (of the form $p \mid c \implies p \mid N$).
2. It may be possible to obtain more information without resorting to the full H_1. For example, we could work in the "minus space" H_1^- which would give us all the information except that of the lattice type.