

$$\begin{array}{ccccccc}
& 1 & & 1 & & 1 & & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & \frac{\mu_m(K)}{(\mu_{mn}(K))^n} & \longrightarrow & \frac{\mathcal{O}_{K,S}^*}{\mathcal{O}_{K,S}^{*n}} & \xrightarrow{m} & \frac{\mathcal{O}_{K,S}^*}{\mathcal{O}_{K,S}^{*mn}} & \longrightarrow & \frac{\mathcal{O}_{K,S}^*}{\mathcal{O}_{K,S}^{*m}} & \longrightarrow & 1 \\
& \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 \longrightarrow & \frac{\mu_m(K)}{(\mu_{mn}(K))^n} & \longrightarrow & K(\mathcal{S}, n) & \xrightarrow{m} & K(\mathcal{S}, mn) & \longrightarrow & K(\mathcal{S}, m) & \xrightarrow{\alpha_{m,n}} & \frac{\mathcal{C}_{K,S}[m]}{n\mathcal{C}_{K,S}[mn]} & \longrightarrow & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow \\
& 1 & \longrightarrow & \mathcal{C}_{K,S}[n] & \longrightarrow & \mathcal{C}_{K,S}[mn] & \xrightarrow{n} & \mathcal{C}_{K,S}[m] & \longrightarrow & \frac{\mathcal{C}_{K,S}[m]}{n\mathcal{C}_{K,S}[mn]} & \longrightarrow & 1 \\
& & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \frac{E(\mathbb{Q})[m]}{nE(\mathbb{Q})[mn]} & \longrightarrow & \frac{E(\mathbb{Q})}{nE(\mathbb{Q})} & \xrightarrow{m} & \frac{E(\mathbb{Q})}{mnE(\mathbb{Q})} & \longrightarrow & \frac{E(\mathbb{Q})}{mE(\mathbb{Q})} & \longrightarrow & 0 \\
& \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \frac{E(\mathbb{Q})[m]}{nE(\mathbb{Q})[mn]} & \longrightarrow & \text{Sel}^{(n)}(E/\mathbb{Q}) & \longrightarrow & \text{Sel}^{(mn)}(E/\mathbb{Q}) & \longrightarrow & \text{Sel}^{(m)}(E/\mathbb{Q}) & \longrightarrow & \frac{\text{III}(E/\mathbb{Q})[m]}{n\text{III}(E/\mathbb{Q})[mn]} & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow \\
& 0 & \longrightarrow & \text{III}(E/\mathbb{Q})[n] & \longrightarrow & \text{III}(E/\mathbb{Q})[mn] & \xrightarrow{n} & \text{III}(E/\mathbb{Q})[m] & \longrightarrow & \frac{\text{III}(E/\mathbb{Q})[m]}{n\text{III}(E/\mathbb{Q})[mn]} & \longrightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$