Completeness Criteria for Group Cohomology

Simon King
joint work with G. Ellis (NUIG) and D. J. Green (Jena)

July 22, 2010

Outline

Brief history of group cohomology

Origin in number theory, algebra and topology;
led to homological algebra and algebraic K-theory.

- *H. Poincaré* [1895 ff]: Homology, duality
- *I. Schur* [1900s]: Schur multiplier
- *J.W. Alexander, S. Lefschetz* [1920s]: cochains
- *O. Schreier [1926], R. Baer* [1934]: group extensions
- *P.A. Smith [1930s]*: Group actions on spheres
- *H. Whitney, E. Čech* [1936/8]: Cup product
- *H. Hopf, W. Hurevicz, S. Eilenberg, S. MacLane, ...* [$\sim$1940]: Groups $\leftrightarrow$ Spaces
- *D. Quillen, J. Alperin, L. Evens, J. Carlson, D. Benson [1980s]*: Modular representations of groups

Modular Group Cohomology

G finite group, p prime dividing $|G|$. $H^*(G) := H^*(G; \mathbb{F}_p)$

- Finitely presentable graded commutative $\mathbb{F}_p$–algebra.
- $\phi : G_1 \to G_2 \rightsquigarrow \phi^* : H^*(G_2) \to H^*(G_1)$,
  $\rightsquigarrow$ restriction $\mathrm{r}_U^G : H^*(G) \to H^*(U)$ for $U \leq G$.
- G determines $H^*(G)$ up to isomorphism.

Wanted:

Software, using *general* methods, to compute

- minimal presentation of $H^*(G)$,
- depth, Poincaré series, a–invariants, ...
- higher structures (Massey products, Steenrod action)

for as many finite groups as possible.

Computational results with our optional SPKG

All 267 groups of order 64 and all 2328 groups of order 128

Order 64 first done by J. Carlson [1997-2001, 8 months comp. time].
We need about 30 minutes for order 64, about 2 months for order 128.

Interesting non prime power groups

http://www.nuigalway.ie/maths/sk/Cohomology/rings/
Modular cohomology for different primes of (among others)

- Co_3: $H^*(Co_3; \mathbb{F}_2)$ is Cohen-Macaulay.
- HS, Janko groups (not J_4), Mathieu groups (not M_{24})
- McL: correcting result of Adem-Milgram
- $Sz(8)$: minimal presentation of $H^*(Sz(8); \mathbb{F}_2)$ has 102 generators of maximal degree 29 and 4790 relations of maximal degree 58.

Approaches for computing cohomology

Topology

Construct *Classifying spaces*. Tailor made. Not algorithmic.

Spectral Sequences

- Lyndon–Hochschild–Serre: extrasp. 2–groups [Quillen 1971]
- Eilenberg–Moore: groups of order 32 [Rusin 1989]

But not general enough, and difficult to implement.

Ring approximations in increasing degree

- Prime power groups: Projective resolutions, general homological algebra.
- Otherwise: Stable element method.

Degree-wise approximation of cohomology

1. Case: G is a prime power group

D. Green [2001]: Initial segments of a minimal free $\mathbb{F}_p G$-resolution can be computed using n. c. Gröbner basis techniques for *finite* $\mathbb{F}_p G$-modules.

- *Negative* monomial orders (for minimality)
- *Two-speed* replacement rules: *Type I* precedes *Type II*

$\mathbb{F}_3 C_3 \cong \mathbb{F}_3[t]/\langle t^3\rangle$, $M = (\mathbb{F}_3 C_3 \cdot a \oplus \mathbb{F}_3 C_3 \cdot b)/\langle t \cdot a - t^2 \cdot a + t \cdot b\rangle$

Type I rule: $t^3 \rightsquigarrow 0$. Type II rule: $t \cdot a \rightsquigarrow t^2 \cdot a - t \cdot b$.
Reduce $t \cdot a + t \cdot b$: $\rightsquigarrow t^2 \cdot a - t \cdot b + t \cdot b = t^2 \cdot a$
$\rightsquigarrow t^3 \cdot a - t^2 \cdot b \rightsquigarrow -t^2 \cdot b$ (Type I precedes Type II!).

2. Case: Stable element method (Cartan–Eilenberg)

- For $U < G$: *Transfer* $\mathrm{tr}_U^G\colon\ H^*(U) \to H^*(G)$, with $\mathrm{tr}_U^G(\mathrm{r}_U^G(x)) = [G:U]\cdot x$ and $\mathrm{tr}_U^G(\mathrm{r}_G^U(y)\cdot x) = y\cdot \mathrm{tr}_U^G(x)$.
- If U contains a Sylow p-subgroup of G, then r_U^G is injective.
- Its image is characterised by *stability conditions* associated with representatives g for double cosets $U \setminus G / U$.

Stability under $g \in G$

Let $c_g : H^*(U) \to H^*(U^g)$ be induced by conjugation with g^{-1}.
$x \in H^*(U)$ is stable under g $:\iff$ $\mathrm{r}^U_{U\cap U^g}(x) = \mathrm{r}^{U^g}_{U\cap U^g}(c_g^*(x))$

Implementation

$\mathbb{F}_p P$-Resolutions: C/Cython. Double cosets: GAP.
$H^*(U)$, $H^*(U \cap U^g)$, maps: SINGULAR,
get $H^n(G)$ by solving linear equations (in Sage), for any n.

Problem

Can compute resolution for Sylow p-subgroup P of G,
thus get $H^{\leq n}(P)$ for any n,
get $H^{\leq n}(P) \supset ... \supset H^{\leq n}(U) \supset H^{\leq n}(G)$ with stable elements.
$\rightsquigarrow$ Degree n approximation $\alpha_n \colon \tau_n H^*(G) \to H^*(G)$.
For what n is α_n an isomorphism?

Completeness criteria

J. F. Carlson [$\sim$ 2000]

Complicated criterion that relies on a conjecture

D. J. Benson [2004]

- If n is big enough, *filter regular parameters* $\mathcal{P}$ for $H^*(G)$ can be constructed in $\tau_n H^*(G)$, using Dickson invariants in the cohomology of maximal p-elementary abelian subgroups.
- Using the *filter degree type* of $\mathcal{P}$, compute upper bound α for the regularity of $\tau_n H^*(G)$.
- If $n > \alpha + \sum_{\zeta \in \mathcal{P}} (|\zeta| - 1)$ then α_n is isomorphism.

Problems: Computation of filter degree type; high parameter degree

Completeness criteria

D. Green, S. K. [2009]

- For p-groups: Improved construction of filter regular parameters.
  Improve degree growth from $p^{\mathrm{rk}_p(G)}$ to $p^{\mathrm{rk}_p(G)-\mathrm{rk}(Z(G))}$.
- Existence result for filter regular parameters $\mathcal{P}'$ of $\tau_n H^*(G;k)$ in small degrees, for some finite extension field k of $\mathbb{F}_p$.
- If $n > \alpha + \sum_{\zeta\in\mathcal{P}'}(|\zeta|-1)$ then α_n is isomorphism.

Proof idea that α_n is isomorphism

$\tau_n H^*(G;k) \cong \tau_n H^*(G) \otimes_{\mathbb{F}_p} k$.
Filter degree types of $\mathcal{P}$ and $\mathcal{P}'$ coincide.
By Benson: $\tau_n H^*(G,k) \cong H^*(G,k)$, thus $\tau_n H^*(G) \cong H^*(G)$.

Proving existence of parameters

If $Q := \tau_n H^*(G)/\langle \zeta_1, ..., \zeta_i \rangle$ is finite over deg-d elements: There is a finite field extension k of $\mathbb{F}_p$, so that $H^*(G, k)$ has a f.r. HSOP formed by $\zeta_1, ..., \zeta_i$ and elements of degree d.

Proof idea

Start with *infinite* alg. extension $K/\mathbb{F}_p$; use Noether normalisation on $Q \otimes K$ and show: $\exists x \in Q \otimes K$ with finite dim. annihilator. Induction: Get filter regular parameters over K, and let $k < K$ contain the coefficients.

$G = P = \mathrm{Syl}_2(Co_3)$ (order 1024)

Benson: Parameter degrees 8, 12, 14, 15 (applies in degree 46). Our construction: Parameter degrees 8,4,6,7 (applies in degree 22). Existence proof: Parameter degrees 8,4,2,2 over finite extension field. We detect completion in degree 14, which is perfect.

Alternative completeness criteria

Problem

We still need to construct $\mathcal{P}$ and compute its filter degree type. Depending on the example, this can be very difficult!

P. Symonds [2009]

Let $\mathcal{P} \subset \tau_n H^*(G)$ yield parameters for $H^*(G)$.
If $n > \sum_{\zeta \in \mathcal{P}} (|\zeta| - 1)$ and $\tau_n H^*(G)$ is generated in degree $\leq n$ as module over $\langle\langle \mathcal{P} \rangle\rangle$ then α_n is isomorphism.

More freedom in construction of parameters

- Could start with *f. r.* parameters, and improve degrees.
- Prove that $\tau_n H^*(G)$ contains parameters for $H^*(G)$, and let $\mathcal{P}$ be formed by some generators.
  No algebraic independence needed!

... for non prime power groups

S. K. [2010]

1. If $H^*(U)$ is generated in degree $\leq n$ as a module over $\mathrm{im}(\mathrm{r}_{G,U} \circ \alpha_n)$ then α_n is surjective.

Benefit

No need to compute stable subspace in degree $> n$!

2. Let α_n be surjective, $\exists$ parameters $\mathcal{P}'$ of $\tau_n H^*(G; k)$, $n \geq N = \sum_{\zeta \in \mathcal{P}'} |\zeta| - depth(H^*(U))$.
α_n is isomorphism iff $p\left(\tau_n H^*(G), t\right) \cdot \prod_{\zeta \in \mathcal{P}'} (1 - t^{|\zeta|})$ is a polynomial of degree $\leq N$.

Benefit

No algebraic dependence needed, *and* can use field extension.

Proof sketch

Surjectivity

$x \in H^*(G)$ not in $\mathrm{im}(\alpha_n) \Rightarrow \deg(x) > n$ by definition.

- $\mathrm{r}_U^G(x) = \sum y_i \, \mathrm{r}_U^G(\alpha_n(z_i))$ with $y_i \in H^{\geq n}(U)$ by assumption.
- $x = \frac{1}{[G:U]} \mathrm{tr}_U^G(\mathrm{r}_U^G(x)) = \sum \mathrm{tr}_U^G(y_i)\alpha_n(z_i)$
- Since $\deg(y_i) \leq n$, we have $y_i \in \mathrm{im}\,\alpha_n$ and thus $x \in \mathrm{im}\,\alpha_n$.

Isomorphy — Basic idea due to P. Symonds

Param.degree $d_i \Rightarrow q(H^*(G)) := p(H^*(G); k) \cdot \prod(1 - t^{d_i})$ is polynomial.
$\deg(p(H^*(G))) \leq \mathrm{Reg}(H^*(G)) - \mathrm{depth}(H^*(G)) \leq -\mathrm{depth}(H^*(U))$.,
thus, $\deg q \leq N$.
If $\deg q(\tau_n H^*(G)) \leq N$: It determines $\tau_n H^{\leq N}(G)$.
$H^{\leq N}(G) = \tau_n H^{\leq N}(G)$, so, it determines $q(H^*(G))$.
Thus, $p(H^*(G)) = p(\tau_n(H^*(G)))$.

Comparing the Criteria

Symonds / improved Benson criteria work very well for p-groups. Non prime power groups: When using algebraically independent parameters, Hilbert–Poincaré is usually the best.

Consider $H^*(\mathfrak{S}_9; \mathbb{F}_2)$. The ring can be presented in degree 12.

- Benson: Filter-regular parameters in deg. 8, 12, 14, 15; applies in deg. 46 (actually 45).
- Green–K.: Exist filter-regular parameters in deg. 8, 12 14, 6 over extension field; applies in deg. 36.
- Symonds or Poincaré–Hilbert: Alg. indep. parameters in deg. 4, 12, 7, 6; applies in deg. 26.
- There are alg. *dependent* parameters in degrees 1, 2, 3, 3, 4, 6, 7. The Hilbert-Poincaré criterion applies in degree 23 = 1+2+3+3+4+6+7-3. The Symonds criterion applies in degree 21.

Thank you for listening, and for reviewing #8667...