Cylindrical Algebraic Decomposition and Special Functions Inequalities

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Cylindrical algebraic decomposition

Algebraic decomposition (AD)

DEF: A finite set of polynomials $\{p_1, \ldots, p_m\} \subset \mathbb{R}[x_1, \ldots, x_n]$ induces a decomposition of $\mathbb{R}^n$ into maximal connected **cells** on which all the p_i are sign-invariant.

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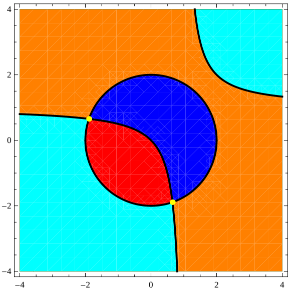
Example: $\{p_1(x, y) = x^2 + y^2 - 4, p_2(x, y) = (x - 1)(y - 1) - 1\}$



Quantifier elimination: Example

Given:

$$\phi \equiv \forall x \exists y : p_1(x,y) > 0 \Leftrightarrow p_2(x,y) > 0,$$

with $p_1(x,y) = x^2 + y^2 - 4$ and $p_2(x,y) = (x-1)(y-1) - 1$.

Find a quantifier free formula equivalent to ϕ

Consider the part of the AD for which the quantifier-free part is true:

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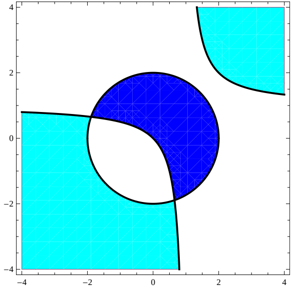

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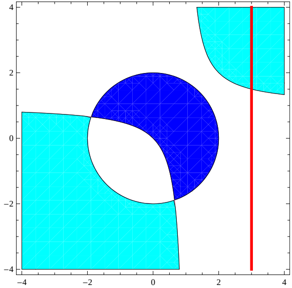

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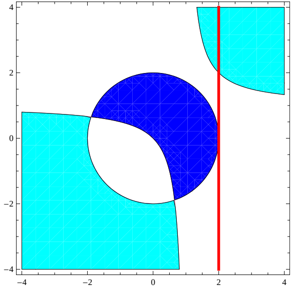

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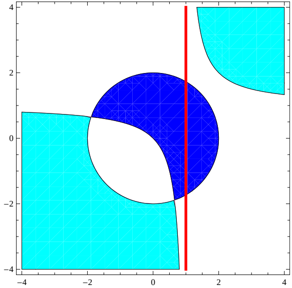

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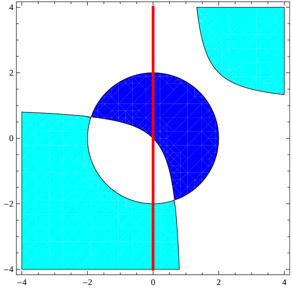

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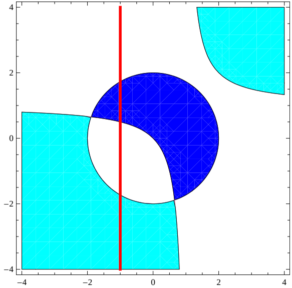

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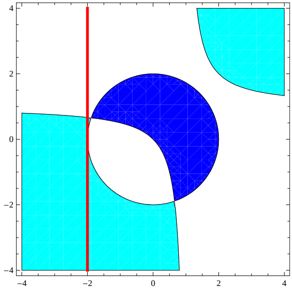

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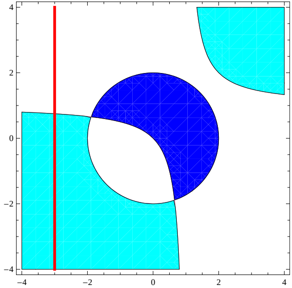


$$\phi \equiv \text{True}$$

Cylindrical algebraic decomposition (CAD)

- add polynomials to the given set such that the algebraic decomposition it defines is easier to handle.

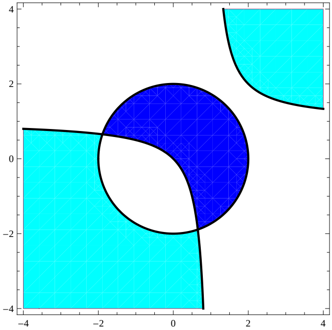
DEF: Let $p_1, \ldots, p_m \in \mathbb{Q}[x_1, \ldots, x_n]$. The algebraic decomposition of $\{p_1, \ldots, p_m\}$ is called **cylindrical** if

1. For any two cells c_1, c_2 in the algebraic decomposition: $\pi_n(c_1) = \pi_n(c_2)$ OR $\pi_n(c_1) \cap \pi_n(c_2) = \emptyset$
2. The algebraic decomposition of $\{p_1, \ldots, p_m\} \cap \mathbb{Q}[x_1, \ldots, x_{n-1}]$ is cylindrical

Base case: any algebraic decomposition of $\mathbb{R}$ is cylindrical.

Notation: $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ canonical projection

Cylindrical Algebraic Decomposition: Example



$$\{x^2 + y^2 - 4, (x-1)(y-1) - 1\}$$

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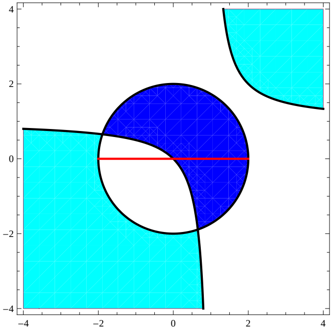

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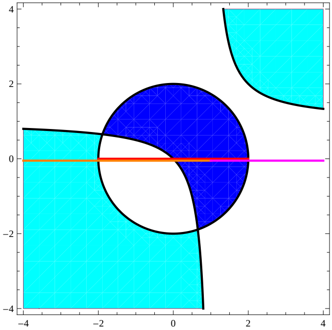

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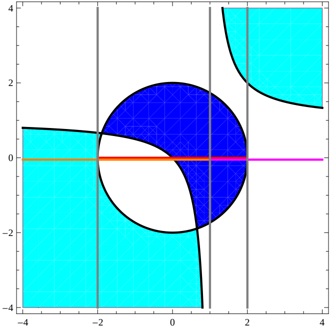

$$\{x^2 + y^2 - 4, (x-1)(y-1) - 1, x+2, x-1, x-2\}$$

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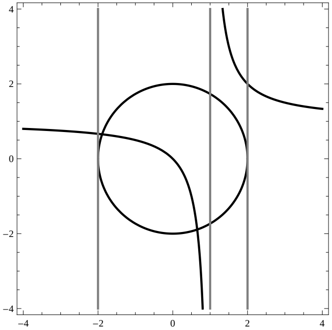

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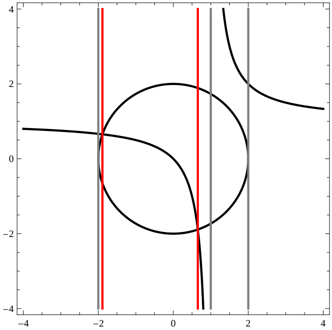
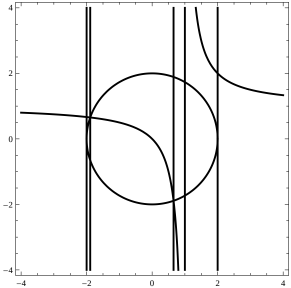


$$\{x^2 + y^2 - 4, (x-1)(y-1) - 1, x+2, x-1, x-2,$$
$$x^4 - 2x^3 - 2x^2 + 8x - 4\}$$

Basic steps for computing a CAD

- Projection (George Collins, Hoon Hong, Scott McCallum, Chris Brown)



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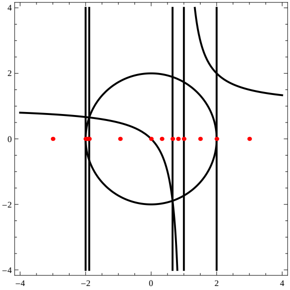
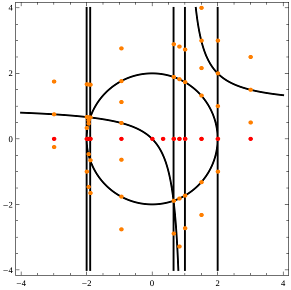


Basic steps for computing a CAD

- Projection (George Collins, Hoon Hong, Scott McCallum, Chris Brown)
- Base case (cylindrical decomposition of $\mathbb{R}$)
- Lifting (depends on the projection operator)



Quantifier elimination with CAD: Example

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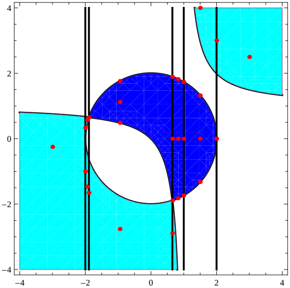


Only finitely many points need to be checked!

$$\phi \equiv \text{True}$$

Special functions inequalities:

The Gerhold/Kauers method

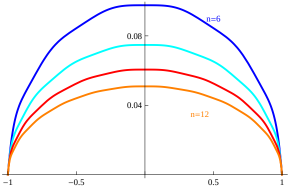
Turán's inequality for Legendre polynomials

Let $P_n(x)$ denote the nth Legendre polynomial defined for $n \geq 0$ by the three term recurrence

$$P_{n+1}(x) = \tfrac{2n+1}{n+1} x P_n(x) - \tfrac{n}{n+1} P_{n-1}(x), \qquad \begin{array}{l} P_{-1}(x) = 0, \\ P_0(x) = 1 \end{array} \quad .$$

Then

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \quad -1 \leq x \leq 1, \quad n \geq 0.$$



Proof by induction

Induction hypothesis:

$$H(n) \equiv P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0$$

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- By means of the Legendre three term recurrence, $P_{n+2}(x)$ can be expressed in terms of $x, n, P_n(x)$ and $P_{n+1}(x)$
- To be able to invoke CAD the induction step is generalized to a purely polynomial statement:

$$\begin{aligned} x &\mapsto t_0, \quad n &&\mapsto t_1, \\ P_n(x) &\mapsto p_0, \quad P_{n+1}(x) &&\mapsto p_1. \end{aligned}$$

This way, e.g., $P_{n+2}(x) \mapsto \frac{2t_1-1}{t_1} t_0 p_1 - \frac{t_1-1}{t_1} p_0$.

Proof by induction

Induction hypothesis:

$$H_0 \equiv p_1^2 - p_0 \left(\tfrac{2t_1 - 1}{t_1} t_0 p_1 - \tfrac{t_1 - 1}{t_1} p_0 \right) \geq 0$$

Induction step: Show

$$H_1 \equiv \frac{t_1^3 - 2t_0^2 t_1 + t_0^2}{t_1^2 (t_1 + 1)} p_1^2 - \frac{(t_1 - 1)(2t_1^2 + t_1 - 2)}{t_1^2 (t_1 + 1)} t_0 p_0 p_1 + \frac{(t_1 - 1)^2}{t_1^2} p_0^2 \geq 0$$

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- it might be necessary to extend the induction hypothesis, i.e., prove

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- no general criterion for termination known

Induction proof carried out by SumCracker

The Gerhold/Kauers method is implemented in the Mathematica package **SumCracker** (Manuel Kauers)

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- first practically applicable method for proving special functions inequalities automatically
- computationally expensive (underlying CAD computations)
- sometimes a reformulation by the user necessary

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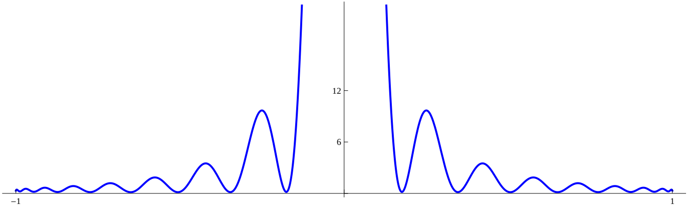
If $-1 \leq x \leq 1,\ n \geq 0$, then

$$S(2n,x) = \sum_{j=0}^{n} \tfrac{1}{2}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \geq 0.$$



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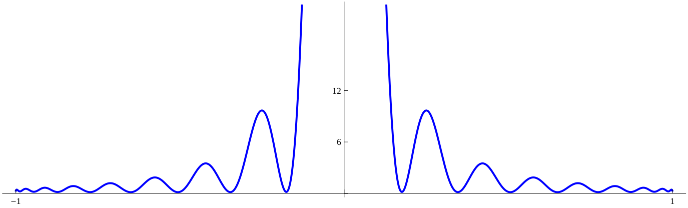
If $-1 \le x \le 1,\ n \ge 0,\ -\frac{1}{2} \le \alpha \le \frac{1}{2}$, then

$$\sum_{j=0}^{2n} \frac{c_j^\alpha}{x} \left(P_{j+1}^{(\alpha,\alpha)}(x) P_j^{(\alpha,\alpha)}(0) - P_j^{(\alpha,\alpha)}(x) P_{j+1}^{(\alpha,\alpha)}(0) \right) \ge 0.$$



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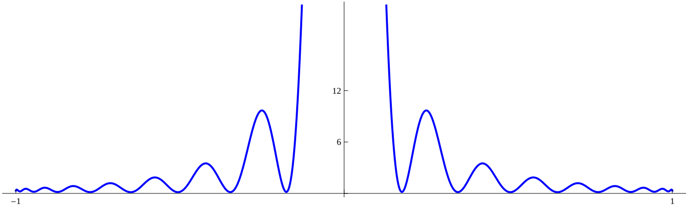

Recurrence relation for $S(n, x)$

The sums $S(n, x)$ satisfy a five term recurrence

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- the procedure, however, does not terminate

Recurrence relation for $S(n,x)$

The sums $S(n,x)$ satisfy a five term recurrence

$$\begin{aligned}4(4n+7)(n+4)^2S(n+4,x) = (2n+3)^2(4n+15)S(n,x)\\ +(4n+15)(16n^2x^2-8n^2+48nx^2-12n+35x^2+3)S(n+1,x)\\ -(-192n^2x^2+144n^2-1056nx^2+792n-1260x^2+943)S(n+2,x)\\ -(4n+7)(16n^2x^2-8n^2+128nx^2-76n+255x^2-173)S(n+3,x)\end{aligned}$$

- this representation makes it reasonable to try an application of the Gerhold/Kauers method
- the procedure, however, does not terminate
- a reformulation is needed!

Step 1: Decomposing $S(2n, x)$

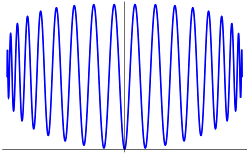
Using human insight we decompose

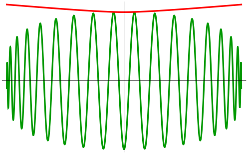
$$x^2 S(2n, x) = g(2n, x) + f(2n, x, 0),$$

where

$$g(2n, x) = \frac{2n+1}{2}\left(x P_{2n+1}(x) - \frac{4n+2}{4n+3} P_{2n}(x)\right) P_{2n}(0),$$

$$f(n, x, y) = -\sum_{j=0}^{n} \frac{1}{(2j-1)(2j+3)} P_j(x) P_j(y)$$





Step 2: Estimating $f(2n, x)$ from below

It is a basic exercise for a student to obtain the bound

$$f(2n, x, 0) \geq \frac{1}{2}\left(f(2n, x, x) + f(2n, 0, 0)\right) =: e(2n, x).$$

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It is a basic exercise for SumCracker to obtain the closed form

In[3]:= $\mathbf{Crack[SUM[\frac{1}{(2j-1)(2j+3)}LegendreP[j,x]^2,\{j,0,n\}]]}$

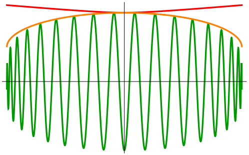
Out[3]= $-\frac{(n+1)^2}{2n+3}P_n(x)^2 + (n+1)xP_{n+1}(x)P_n(x) - \frac{(n+1)^2}{2n+1}P_{n+1}(x)^2$

for $f(n, x, x)$.

Step 3: Proving positivity of lower bound

Collecting the considerations above, the proof is completed if we can show positivity of $g(2n, x) + e(2n, x)$:

$$\begin{aligned} x^2 S(2n, x) &= g(2n, x) + f(2n, x, 0) \\ &\geq g(2n, x) + e(2n, x) \\ &\geq 0 \end{aligned}$$



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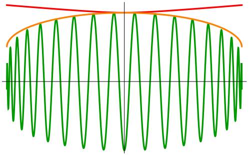

In[4]:= $\mathbf{ProveInequality}[g[2n,x]+e[2n,x]\geq 0,$
$\mathbf{Using}\rightarrow\{-1\leq x\leq 1\},\mathbf{Variable}\rightarrow n]$

Out[4]= True

CAD-input for Schöberl's inequality (general case)

$$\forall n,\alpha,x,y,z,w\big((n\geq 0\wedge -1\leq x\leq 1\wedge -1\leq 2\alpha\leq 1\wedge(2\alpha+4n+1)(y^2+z^2)(\alpha+2n+1)^2-(2\alpha+4n+1)(2\alpha+4n+3)wxz(\alpha+2n+1)+(2n+1)(2\alpha+2n+1)(2\alpha+4n+3)w^2\geq 0)\Rightarrow (2n+3)(\alpha+2n+1)^2(\alpha+2n+3)^2(2\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+2)^2+(\alpha+2n+1)^2(64n^5-256x^2n^4+160\alpha n^4+464n^4+144\alpha^2n^3-512\alpha x^2n^3-1184x^2n^3+928\alpha n^3+1344n^3+56\alpha^3n^2+628\alpha^2n^2-384\alpha^2x^2n^2-1776\alpha x^2n^2-1984x^2n^2+2016\alpha n^2+1944n^2+8\alpha^4n+164\alpha^3n+912\alpha^2n-128\alpha^3x^2n-888\alpha^2x^2n-1984\alpha x^2n-1434x^2n+1944\alpha n+1404n+12\alpha^4+120\alpha^3+441\alpha^2-16\alpha^4x^2-148\alpha^3x^2-496\alpha^2x^2-717\alpha x^2-378x^2+702\alpha+405)z^2(\alpha+2n+2)^2-w^2(-256n^7+4096x^4n^6-3072x^2n^6-896\alpha n^6-1728n^6+12288\alpha x^4n^5+25088x^4n^5-1216\alpha^2n^5-9216\alpha x^2n^5-19968x^2n^5-5184\alpha n^5-4864n^5+15360\alpha^2x^4n^4+62720\alpha x^4n^4+62464x^4n^4-800\alpha^3n^4-5872\alpha^2n^4-11008\alpha^2x^2n^4-49920\alpha x^2n^4-53120x^2n^4-12160\alpha n^4-7408n^4-256\alpha^4n^3+10240\alpha^3x^4n^3+62720\alpha^2x^4n^3+124928\alpha x^4n^3+81216x^4n^3-3104\alpha^3n^3-11072\alpha^2n^3-6656\alpha^3x^2n^3-47744\alpha^2x^2n^3-106240\alpha x^2n^3-74176x^2n^3-14816\alpha n^3-6592n^3-32\alpha^5n^2-752\alpha^4n^2+3840\alpha^4x^4n^2+31360\alpha^3x^4n^2+93696\alpha^2x^4n^2+121824\alpha x^4n^2+58320x^4n^2-4448\alpha^3n^2-10192\alpha^2n^2-2112\alpha^4x^2n^2-21696\alpha^3x^2n^2-76416\alpha^2x^2n^2-111264\alpha x^2n^2-57396x^2n^2-9888\alpha n^2-3424n^2-64\alpha^5n-736\alpha^4n+768\alpha^5x^4n+7840\alpha^4x^4n+31232\alpha^3x^4n+60912\alpha^2x^4n+58320\alpha x^4n+21978x^4n-2784\alpha^3n-4576\alpha^2n-320\alpha^5x^2n-4608\alpha^4x^2n-23296\alpha^3x^2n-53568\alpha^2x^2n-57396\alpha x^2n-23340x^2n-3424\alpha n-960n-32\alpha^5-240\alpha^4+64\alpha^6x^4+784\alpha^5x^4+3904\alpha^4x^4+10152\alpha^3x^4+14580\alpha^2x^4+10989\alpha x^4+3402x^4-640\alpha^3-800\alpha^2-16\alpha^6x^2-352\alpha^5x^2-2504\alpha^4x^2-8240\alpha^3x^2-13881\alpha^2x^2-11670\alpha x^2-3897x^2-480\alpha-112)(\alpha+2n+2)^2-2(\alpha+2n+1)wx(128n^6-1024x^2n^5+384\alpha n^5+1408n^5+448\alpha^2n^4-2560\alpha x^2n^4-5504x^2n^4+3520\alpha n^4+5592n^4+256\alpha^3n^3+3296\alpha^2n^3-2560\alpha^2x^2n^3-11008\alpha x^2n^3-11488x^2n^3+11184\alpha n^3+10888n^3+72\alpha^4n^2+1424\alpha^3n^2+7870\alpha^2n^2-1280\alpha^3x^2n^2-8256\alpha^2x^2n^2-17232\alpha x^2n^2-11688x^2n^2+16332\alpha n^2+11258n^2+8\alpha^5n+272\alpha^4n+2278\alpha^3n+7692\alpha^2n-320\alpha^4x^2n-2752\alpha^3x^2n-8616\alpha^2x^2n-11688\alpha x^2n-5814x^2n+11258\alpha n+5940n+16\alpha^5+220\alpha^4+1124\alpha^3+2669\alpha^2-32\alpha^5x^2-344\alpha^4x^2-1436\alpha^3x^2-2922\alpha^2x^2-2907\alpha x^2-1134x^2+2970\alpha+1257)z(\alpha+2n+2)^2\geq 0\big)$$