A short introduction to constructive algebraic analysis

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Introduction

- Mathematical systems theory aims at studying functional systems
  - ODEs, PDEs, difference equations, time-delay equations. . .
  - Determined, overdetermined and underdetermined systems.
  - Determined: integration (closed-forms & numerical analysis).
  - Overdetermined: integrability & compatibility conditions (Cartan, Riquier, Janet, Spencer. . . Gröbner/Janet bases).
  - Underdetermined: parametrizations, conservation laws
    1. Mathematical physics (field theory, variational problems)
    2. Control theory
    3. Differential geometry (e.g., Monge, Goursat, Gromov). . .

Simple examples

- **Determined:** $A \in \mathbb{R}^{n \times n}, \quad \dot{x}(t) = A\,x(t), \quad x(0) = x_0.$

$$\partial = \frac{d}{dt}, \quad (\partial\, I_n - A)\, x(t) = 0, \quad \det\,(\partial\, I_n - A) \neq 0.$$

- **Overdetermined:** $A \in \mathbb{R}^{n \times n}, \quad C \in \mathbb{R}^{p \times n}, \quad$ DAE:

$$\left\{ \begin{array}{l} \dot{x}(t) - A\,x(t) = 0, \\ C\,x(t) = 0, \end{array} \right. \quad \Leftrightarrow \quad \begin{pmatrix} \partial\, I_n - A \\ C \end{pmatrix} x(t) = 0.$$

$$\left\{ \begin{array}{l} \dot{x}(t) - A\,x(t) = z, \\ C\,x(t) = y, \end{array} \right. \quad \Rightarrow \quad P(\partial)\, z - Q(\partial)\, y = 0.$$

- **Underdetermined:** $x(0) = x_0, \quad B \in \mathbb{R}^{n \times m}$

$$\dot{x}(t) = A\,x(t) + B\,u(t) \;\Leftrightarrow\; (\partial\, I_n - A \quad -B) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = 0.$$

Computer algebra systems

• Remark 1: classical computer algebra systems are relatively good at integrating determined PD systems in closed-form (e.g., Maple).

• Remark 2: classical computer algebra systems are relatively bad at integrating overdetermined PD systems in closed-form.

• Example: $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $x = (x_1, x_2)$:

$$\left(\begin{array}{c} \partial_1^2 + \partial_1 \, \partial_2 - (x_1 + x_2) \, \partial_1 - 1 \\ \partial_2^2 + \partial_1 \, \partial_2 - (x_1 + x_2) \, \partial_2 - 1 \end{array} \right) y(x) = 0.$$

• Remark 3: classical computer algebra systems usually cannot integrate undetermined PD systems in closed-forms!

• Example: $(\partial_1 \quad \partial_2 \quad \partial_3) \, \vec{A} = \partial_1 \, A_1(x) + \partial_2 \, A_2(x) + \partial_3 \, A_3(x) = 0.$

Outline of the talk

1. Module theory & homological algebra (e.g., $\mathrm{ext}^i_D(N, D)$)

   $\Rightarrow$ Parametrization of underdetermined linear functional systems

2. Baer's extensions ($\mathrm{ext}^1_D(M, N)$)

   $\Rightarrow$ Monge problem of underdetermined linear functional systems

   $\Rightarrow$ Maple package OreModules (Chyzak, Robertz, Q.).

3. Purity filtration ($\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$, spectral sequences)

   $\Rightarrow$ equidimensional decomposition of linear PD systems

   $\Rightarrow$ Integration of over/underdetermined linear PD systems

   $\Rightarrow$ Maple package PurityFiltration (`homalg` in GAP4).

Matrices of differential operators

- Newton: Fluxion calculus (1666) ("dot-age")

$$\begin{cases} \ddot{x}_1(t) + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \ddot{x}_2(t) + \alpha\, x_2(t) - \alpha\, u(t) = 0, \end{cases} \qquad \alpha = g/l.$$

- Leibniz: Infinitesimal calculus (1676) ("d-ism")

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \dfrac{d^2 x_2(t)}{dt^2} + \alpha\, x_2(t) - \alpha\, u(t) = 0. \end{cases}$$

- Boole: Operational calculus (1859-60)

$$\begin{pmatrix} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} = 0.$$

$\Rightarrow$ ring of differential operators $D = \mathbb{Q}(\alpha)\left[\frac{d}{dt}\right]$:

$$\sum_{i=0}^{n} a_i \left(\frac{d}{dt}\right)^i \in D, \quad a_i \in \mathbb{Q}(\alpha), \quad \left(\frac{d}{dt}\right)^i = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

Functional operators

- Differential operator: $\left(\sum_{s=0}^{m} b_s(t)\,\partial^s\right)\left(\sum_{r=0}^{n} a_r(t)\,\partial^r\right)$

$$\partial\colon y \longmapsto \frac{dy}{dt}, \quad a = a(\cdot), \quad a\colon y \longmapsto a\,y,$$

$$(\partial\, a)(y) = \partial\,(a(y)) = \partial(a\,y) = \frac{d}{dt}(a\,y) = a\,\frac{dy}{dt} + \frac{da}{dt}\,y$$

$$= \left(a\,\partial + \frac{da}{dt}\right)(y).$$

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- Shift operator: $\partial \colon y_n \longmapsto \sigma(y_n) = y_{n+1}, \quad a \colon y_n \longmapsto a_n\,y_n,$

$$(\partial\, a)(y_n) = \partial(a(y_n)) = \partial(a_n\,y_n) = \sigma\,(a_n\,y_n) = a_{n+1}\,y_{n+1}$$
$$= (\sigma(a)\,\partial)(y_n).$$

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$$= (\sigma(a)\, \partial)(y_n).$$

- Time-delay operator: $\partial\colon y \longmapsto \delta(y) = y(\cdot - \tau),$

$$(\partial\, a)(y) = \partial(a(y)) = \partial(a\, y) = \delta\,(a\, y) = a(\cdot - \tau)\, y(\cdot - \tau)$$
$$= (\delta(a)\, \partial)(y).$$

Functional operators

- Other functional operators: difference, divided difference, Eulerian, Frobenius, q-dilation, q-shift, q-difference... operators.
- Unique expansion: $P = \sum_{i=0}^{n} a_i \, \partial^i$, $a_i \in A$: domain of coeffs.
- Degree condition: $\partial \, a = \alpha \, \partial + \beta = \alpha(a) \, \partial + \beta(a)$, a, b, $c \in A$.

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$$\left\{ \begin{array}{l} \partial \, (a + b) = \alpha(a + b) \, \partial + \beta(a + b), \\ \partial \, a = \alpha(a) \, \partial + \beta(a), \\ \partial \, b = \alpha(b) \, \partial + \beta(b), \end{array} \right.$$

$$\partial(a + b) = \partial \, a + \partial \, b \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \alpha(a + b) = \alpha(a) + \alpha(b), \\ \beta(a + b) = \beta(a) + \beta(b). \end{array} \right.$$

Functional operators

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- Degree condition: $\partial \, a = \alpha \, \partial + \beta = \alpha(a) \, \partial + \beta(a)$, a, b, $c \in A$.

$$\partial \, (a \, b) = \alpha(a \, b) \, \partial + \beta(a \, b)$$

$$\begin{aligned} \partial \, (a \, b) &= (\partial \, a) \, b = (\alpha(a) \, \partial + \beta(a)) \, b \\ &= \alpha(a) \, (\alpha(b) \, \partial + \beta(b)) + \beta(a) \, b, \end{aligned}$$

$$\Leftrightarrow \quad \left\{ \begin{array}{l} \alpha(a \, b) = \alpha(a) \, \alpha(b), \\ \beta(a \, b) = \alpha(a) \, \beta(b) + \beta(a) \, b. \end{array} \right.$$

- α is an endomorphism of A and β is a α-derivation of A.

Skew polynomial rings (Ore, 1933)

- Definition: A skew polynomial ring $A[\partial;\alpha,\beta]$ is a noncommutative polynomial ring in ∂ with coefficients in A satisfying

$$\forall\, a \in A, \quad \boxed{\partial\, a = \alpha(a)\,\partial + \beta(a)}$$

where $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ are such that:

$$\begin{cases} \alpha(1) = 1, \\ \alpha(a+b) = \alpha(a) + \alpha(b), \\ \alpha(a\,b) = \alpha(a)\,\alpha(b), \end{cases} \qquad \begin{cases} \beta(a+b) = \beta(a) + \beta(b), \\ \beta(a\,b) = \alpha(a)\,\beta(b) + \beta(a)\,b. \end{cases}$$

- $P \in A[\partial;\alpha,\beta]$ has a unique form $P = \sum_{i=0}^{n} a_i\,\partial^i$, $a_i \in A$.
  - Ring of differential operators: $A\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$.
  - Ring of shift operators: $A[\partial;\sigma,0]$, $A[\partial;\delta,0]$.
  - Ring of difference operators: $A[\partial;\tau,\tau-\mathrm{id}]$, $\tau a(x) = a(x+1)$.

Ore algebras (Chyzak-Salvy, 1996)

- We can iterate skew polynomial rings to get Ore extensions:

$$\boxed{A[\partial_1;\alpha_1,\beta_1]\ldots[\partial_n;\alpha_n,\beta_n]}$$

- Definition: An Ore extension $A[\partial_1;\alpha_1,\beta_1]\ldots[\partial_n;\alpha_n,\beta_n]$ is called an Ore algebra if the ∂_i's commute, i.e., if we have

$$1 \leq j < i \leq m, \quad \alpha_i(\partial_j) = \partial_j, \quad \beta_i(\partial_j) = 0,$$

and the $\alpha_{i\,|_A}$'s and $\beta_{j\,|_A}$'s commute for $i \neq j$.

- Ring of differential operators: $A\left[\partial_1;\mathrm{id},\frac{\partial}{\partial x_1}\right]\ldots\left[\partial_n;\mathrm{id},\frac{\partial}{\partial x_n}\right]$.
- Ring of differential delay operators: $A\left[\partial_1;\mathrm{id},\frac{d}{dt}\right][\partial_2;\delta,0]$.
- Ring of shift operators: $A[\partial_1;\sigma_1,0]\ldots[\partial_n;\sigma_n,0]$.

Matrix of functional operators

- The stirred tank model (Kwakernaak-Sivan, 72):

$$\begin{cases} \dot{x}_1(t) + \dfrac{1}{2\,\theta}\, x_1(t) - u_1(t) - u_2(t) = 0, \\ \dot{x}_2(t) + \dfrac{1}{\theta}\, x_2(t) - \left(\dfrac{c_1 - c_0}{V_0}\right) u_1(t-\tau) - \left(\dfrac{c_2 - c_0}{V_0}\right) u_2(t-\tau) = 0. \end{cases} \qquad (\star)$$

- We introduce the commutative Ore algebra:

$$D = \mathbb{Q}(\theta, c_0, c_1, c_2, V_0)\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right][\partial_2; \delta, 0].$$

- The linear functional system $(\star)$ can be rewritten as:

$$\begin{pmatrix} \partial_1 + \dfrac{1}{2\,\theta} & 0 & -1 & -1 \\ 0 & \partial_1 + \dfrac{1}{\theta} & -\left(\dfrac{c_1 - c_0}{V_0}\right)\partial_2 & -\left(\dfrac{c_2 - c_0}{V_0}\right)\partial_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u_1(t) \\ u_2(t) \end{pmatrix} = 0.$$

Matrix of functional operators

- Linearization of the Navier-Stokes $\sim$ a parabolic Poiseuille profile

$$\left\{\begin{array}{ll}\partial_t\, u_1 + 4\, y\, (1-y)\, \partial_x\, u_1 - 4\, (2\, y - 1)\, u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, u_1 + \partial_x\, p = 0, & \\ \partial_t\, u_2 + 4\, y\, (1-y)\, \partial_x\, u_2 - \nu\, (\partial_x^2 + \partial_y^2)\, u_2 + \partial_y\, p = 0, & (*) \\ \partial_x\, u_1 + \partial_y\, u_2 = 0. & \text{(e.g., Vazquez-Krstic, IEEE 07)}\end{array}\right.$$

- Let us introduce the so-called Weyl algebra $A_3(\mathbb{Q}(\nu))$

$$D = \mathbb{Q}(\nu)[t, x, y]\left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right]\left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right]\left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right].$$

$$(\partial_x\, y = y\, \partial_x,\ \partial_x\, x = x\, \partial_x + 1,\ \partial_x\, \partial_y = \partial_y\, \partial_x \ldots):$$

- The system $(*)$ is defined by the matrix of PD operators:

$$\left(\begin{array}{ccc}\partial_t + 4\, y\, (1-y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1-y)\, \partial_x - \nu\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0\end{array}\right).$$

Noncommutative Gröbner bases

- Let $D = A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_m; \alpha_m, \beta_m]$ be an Ore algebra.
- Theorem: (Kredel, 93) Let $A = k[x_1, \ldots, x_n]$ be a commutative polynomial ring ($k = \mathbb{Q}$, $\mathbb{F}_p$) and D an Ore algebra satisfying

$$\alpha_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain $0 \neq a_{ij} \in k$, $b_{ij} \in k$, $c_{ij} \in A$ and $\deg(c_{ij}) \leq 1$. Then, a non-commutative version of Buchberger's algorithm terminates for any term order and its result is a Gröbner basis.

- Implementation in the Maple package Ore_algebra (Chyzak)

  (Singular:Plural, Macaulay 2, NCAlgebra, JanetOre...).

- Gröbner bases can be used to effectively compute over $D^{1\times p}/F$.

Finitely presented left D-modules

- Let D be a left noetherian domain and $R \in D^{q \times p}$.
- Let us consider the left D-homomorphism (left D-linear map):

$$\begin{array}{ccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} \\ \lambda = (\lambda_1 \ \ldots \ \lambda_q) & \longmapsto & \lambda \, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$\boxed{M = \operatorname{coker}_D(.R) = D^{1 \times p} / \operatorname{im}_D(.R) = D^{1 \times p} / (D^{1 \times q} \, R).}$$

- M is formed by the equivalence classes $\pi(\mu)$ of $\mu \in D^{1 \times p}$ for the equivalence relation $\sim$ on $D^{1 \times p}$:

$$\mu_1 \sim \mu_2 \ \Leftrightarrow \ \exists \, \lambda \in D^{1 \times q} : \ \mu_1 - \mu_2 \in D^{1 \times q} \, R.$$

1. Number theory: $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$, $\mathbb{Z}[i\sqrt{5}] = \mathbb{Z}[x]/(x^2 + 5)$.
2. Algebraic geometry: $\mathbb{C}[x, y]/(x^2 + y^2 - 1, x - y)$.

- $M = D^{1\times p}/(D^{1\times q}\, R)$ can be defined by generators and relations:
- Let $\{e_k\}_{k=1,\ldots,p}$ be the standard basis of $D^{1\times p}$:

$$e_k = (0\ldots 1\ldots 0).$$

- Let $\pi : D^{1\times p} \longrightarrow M$ be the left D-morphism sending μ to $\pi(\mu)$.

$$\forall\, m \in M,\ \exists\, \mu = (\mu_1 \ \ldots \ \mu_p) \in D^{1\times p} :\ m = \pi(\mu) = \sum_{k=1}^{p} \mu_k\, \pi(e_k),$$

$\Rightarrow \{y_k = \pi(e_k)\}_{k=1,\ldots,p}$ is a family of generators of M.

Linear systems of equations

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$\Rightarrow \{y_k = \pi(e_k)\}_{k=1,\ldots,p}$ is a family of generators of M.

$$\pi((R_{l1} \ \ldots R_{\ lp})) = \pi\left(\sum_{k=1}^{p} R_{lk}\, e_k\right) = \sum_{k=1}^{p} R_{lk}\, y_k = 0,\ l = 1, \ldots, q,$$

$\Rightarrow y = (y_1 \ \ldots \ y_p)^T$ satisfies the relation $R\, y = 0$.

Duality & solution space

- Let $\mathcal{F}$ be a left D-module and $\hom_D(M, \mathcal{F})$ the abelian group:

$$\hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F} \mid f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)\}.$$

- Applying the contravariant left exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

we obtain the following exact sequence of abelian groups:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{\iota\circ\pi^\star} \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

- Theorem: $\boxed{\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}}$

- Remark: $\hom_D(M, \mathcal{F})$ intrinsically characterizes $\ker_{\mathcal{F}}(R.)$ as it does not depend on the embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$.

Linear functional systems

- Let $\mathcal{F}$ be a left D-module and $M = D^{1\times p}/(D^{1\times q}\, R)$.
- Let $f : M \longrightarrow \mathcal{F}$ be a left D-homomorphism. Then, we have:

$$\begin{array}{rcl} f : M & \longrightarrow & \mathcal{F} \\ y_k = \pi(e_k) & \longmapsto & \eta_k, \quad k = 1, \ldots, p, \end{array} \qquad f(0) = 0.$$

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$$\sum_{k=1}^{p} R_{lk}\, y_k = 0.$$

$$f\left(\sum_{k=1}^{p} R_{lk}\, y_k\right) = \sum_{k=1}^{p} R_{lk}\, f(y_k) = \sum_{k=1}^{p} R_{lk}\, \eta_k = 0, \quad l = 1, \ldots, q.$$

$$\Rightarrow \quad \eta = (\eta_1 \;\ldots\; \eta_p)^T \in \mathcal{F}^p : \; R\,\eta = 0.$$

Example: curl operator

- Let us consider $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right]$ and the curl operator defined by:

$$R = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \in D^{3\times 3}.$$

- Let us consider the D-homomorphism (D-linear map)

$$\begin{array}{ccc} D^{1\times 3} & \xrightarrow{.R} & D^{1\times 3} \\ \lambda & \longmapsto & (\lambda_2\,\partial_3 - \lambda_3\,\partial_2 \quad -\lambda_1\,\partial_3 + \lambda_3\,\partial_1 \quad \lambda_2\,\partial_2 - \lambda_2\,\partial_1), \end{array}$$

and the D-module $M = \mathrm{coker}_D(.R) = D^{1\times 3}/(D^{1\times 3}\,R)$.

- If $\mathcal{F} = C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\Omega)\ldots$ is a D-module, then:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^3 \mid \vec{\nabla} \wedge \eta = R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}).$$

Free resolutions

- Definition: A sequence of D-morphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a complex if $g \circ f = 0$, i.e., $\mathrm{im}\, f \subseteq \ker g$.

$\Rightarrow$ The defect of exactness at M is $H(M) = \ker g / \mathrm{im}\, f$.

$\Rightarrow$ The complex is exact at M if $\mathrm{im}\, f = \ker g$.

- Definition: A finite free resolution of a left D-module M is an exact sequence of the form:

$$\ldots \xrightarrow{.R_3} D^{1 \times l_2} \xrightarrow{.R_2} D^{1 \times l_1} \xrightarrow{.R_1} D^{1 \times l_0} \xrightarrow{\pi} M \longrightarrow 0,$$

$$R_i \in D^{l_i \times l_{i-1}}, \qquad \begin{array}{ccc} D^{1 \times l_i} & \xrightarrow{.R_i} & D^{1 \times l_{i-1}} \\ (P_1 \; \ldots \; P_{l_i}) & \longmapsto & (P_1 \; \ldots \; P_{l_i})\, R_i. \end{array}$$

- Algorithm: Find a basis of the compatibility conditions of the inhomogeneous system $R_i\, y = u$ by eliminating y (e.g., GB):

$$\forall\, P \in \ker_D(.R_i), \quad P\,(R_i\, y) = P\, u \;\Rightarrow\; P\, u = 0.$$

Example

- $D = \mathbb{Q}[x_1, x_2]$, $\quad R = \left(\begin{array}{c} x_1^2 \\ x_1\, x_2 \end{array} \right)$. We have the exact sequence:

$$0 \longrightarrow \ker_D(.R) \longrightarrow D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M = D/(x_1^2, x_1\, x_2) \longrightarrow 0$$

$$\begin{array}{rcl} \lambda = (\lambda_1 \quad \lambda_2) \in \ker_D(.R) & \Leftrightarrow & (\lambda_1\, x_1 + \lambda_2\, x_2)\, x_1 = 0 \\ & \Leftrightarrow & \lambda_1\, x_1 + \lambda_2\, x_2 = 0 \\ & \Leftrightarrow & \left\{ \begin{array}{l} \lambda_1 = \mu\, x_2, \\ \lambda_2 = -\mu\, x_1, \end{array} \right. \\ & \Leftrightarrow & \lambda = \mu\, (x_2 \quad -x_1). \end{array}$$

- If $R_2 = (x_2 \quad -x_1)$, then M admits the finite free resolution:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0.$$

$$R\, y = (u_1 \quad u_2)^T \;\Rightarrow\; x_2\, u_1 - x_1\, u_2 = 0.$$

Extension functor $\mathrm{ext}^i_D(\,\cdot\,,\mathcal{F})$

- We introduce the reduced free resolution of M by:

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \longrightarrow 0 \quad (\star).$$

- Let $\mathcal{F}$ be a left D-module.
- Applying the functor $\hom_D(\,\cdot\,,\mathcal{F})$ to $(\star)$, we obtain the complex:

$$\begin{array}{ccccccccc} \ldots & \xleftarrow{R_3.} & \mathcal{F}^{l_2} & \xleftarrow{R_2.} & \mathcal{F}^{l_1} & \xleftarrow{R_1.} & \mathcal{F}^{l_0} & \longleftarrow 0, & (\star\star) \\ & & & & R_1\,\eta & \longleftarrow\!\shortmid & \eta & & \\ & & R_2\,\zeta & \longleftarrow\!\shortmid & \zeta & & & & \end{array}$$

- We denote the defects of exactness of $(\star\star)$ by:

$$\begin{cases} \mathrm{ext}^0_D(M,\mathcal{F}) = \hom_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \mathrm{ext}^i_D(M,\mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)\,/\,\mathrm{im}_{\mathcal{F}}(R_i.), \quad i \geq 1. \end{cases}$$

- Theorem: The abelian group $\mathrm{ext}^i_D(M,\mathcal{F})$ depends only on M and $\mathcal{F}$ but not on the choice of the reduced free resolution $(\star)$.

Example

- $D=\mathbb{Q}[x_1,x_2], \quad R=\begin{pmatrix} x_1^2 \\ x_1\,x_2 \end{pmatrix}, \quad R_2=(x_2 \quad -x_1)$:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M=D/(x_1^2,x_1\,x_2) \longrightarrow 0.$$

- The reduced free resolution of M is the complex:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \longrightarrow 0. \quad (\star)$$

- Applying the functor $\hom_D(\,\cdot\,,D)$ to $(\star)$, we get the complex:

$$0 \longleftarrow D \xleftarrow{R_2.} D^2 \xleftarrow{R.} D \longleftarrow 0.$$

$$\begin{cases} \mathrm{ext}_D^0(M,D)=\hom_D(M,D)\cong \ker_D(R.)=0, \\ \mathrm{ext}_D^1(M,D)\cong \ker_D(R_2.)/\mathrm{im}_D(R.)=(R'\,D)/(R\,D)\cong D/(x_1)\neq 0, \\ \mathrm{ext}_D^2(M,D)\cong D/(R_2\,D^2)=D/(x_1,x_2)\neq 0, \end{cases}$$

where $\ker_D(R'.)=R'\,D, \quad R'=(x_1 \quad x_2)^T$.

Solving inhomogeneous linear systems

- Let $\mathcal{F}$ be a left D-module, $\zeta \in \mathcal{F}^q$ and $R \in D^{q\times p}$.

- Problem: Find necessary and sufficient conditions for the existence of $\eta \in \mathcal{F}^p$ such that $R\,\eta = \zeta$.

- Let $M = D^{1\times p}/(D^{1\times q}\,R)$ the left D-module finitely presented by R and the beginning of a finite free resolution of M:

$$D^{1\times r} \xrightarrow{.S} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0 \quad (\star).$$

- Applying the functor $\hom_D(\cdot\,, \mathcal{F})$ to $(\star)$, we get the complex:

$$\mathcal{F}^r \xleftarrow{S.} \mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0.$$

$\Rightarrow$ necessary conditions: $\zeta \in \ker_{\mathcal{F}}(S.)$, i.e., $S\,\zeta = 0$.

$\Rightarrow$ necessary and sufficient conditions:

$$0 = \overline{\zeta} \in \ker_{\mathcal{F}}(S.)/(R\,\mathcal{F}^p) \cong \mathrm{ext}^1_D(M, \mathcal{F}).$$

Injective modules over a left noetherian ring

- Definition: A left D-module $\mathcal{F}$ is injective if

$$\forall\, q \geq 1, \quad \forall\, R \in D^q, \quad \forall\, \zeta \in \ker_{\mathcal{F}}(S.),$$

where $\ker_D(.R) = D^{1\times r}\, S$, there exists $\eta \in \mathcal{F}$ satisfying $R\,\eta = \zeta$.

- Proposition: If $\mathcal{F}$ is a injective left D-module, then we have:

$$\mathrm{ext}^i_D(\,\cdot\,, \mathcal{F}) = 0, \quad \forall\, i \geq 1.$$

- Example: If Ω is an open convex subset of $\mathbb{R}^n$, then

$$C^\infty(\Omega), \quad \mathcal{D}'(\Omega), \quad \mathcal{S}'(\Omega), \quad \mathcal{A}(\Omega), \quad \mathcal{O}(\Omega), \quad \mathcal{B}(\Omega)$$

are injective $k\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$-modules ($k = \mathbb{R}$ or $\mathbb{C}$).

- Theorem: Injective left D-module always exists.

Parametrizations of linear systems

- Let D be a domain, $\mathcal{F}$ a left D-module and $R \in D^{q \times p}$.

- Problem: Find a set of compatibility conditions of $R\,\eta = \zeta$.
- Answer: $\ker_D(.R) = D^{1 \times r}\, S = \mathrm{im}_D(.S)$, $\quad S\,\zeta = 0$.

- If $\mathcal{F}$ is an injective left D-module, then $\ker_{\mathcal{F}}(S.) = \mathrm{im}_{\mathcal{F}}(R.)$:

$$\forall\, \zeta \in \ker_{\mathcal{F}}(S.), \quad \exists\, \eta \in \mathcal{F}^p : \quad \zeta = R\,\eta.$$

- Converse problem: When does $Q \in D^{p \times m}$ exist such that:

$$\ker_D(.Q) = D^{1 \times q}\, R = \mathrm{im}_D(.R)?$$

- If $\mathcal{F}$ is an injective left D-module, then $\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.)$:

$$\forall\, \eta \in \ker_{\mathcal{F}}(R.), \text{ i.e. } R\,\eta = 0, \quad \exists\, \xi \in \mathcal{F}^m : \quad \eta = Q\,\xi.$$

Module theory

- Definition: 1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$.

2. M is projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P such that:

$$M \oplus P \cong D^r.$$

3. M is reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall\, m \in M, \quad f \in \hom_D(M, D).$$

4. M is torsion-free if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq d \in D : d\, m = 0\} = 0.$$

5. M is torsion if $t(M) = M$.

Classification of modules

- Theorem: 1. We have the following implications:

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free.}$$

2. If D is a principal domain (e.g., $\mathbb{Q}(t)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{free}.$$

3. If D is a hereditary ring (e.g., $\mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{projective}.$$

4. If $D = k[x_1, \ldots, x_n]$ and k a field, then:

$$\text{projective} = \text{free} \quad \text{(Quillen-Suslin theorem)}.$$

4. If $D = A_n(k)$ or $B_n(k)$, k is a field of characteristic 0, then

$$\text{projective} = \text{free} \quad \text{(Stafford theorem)},$$

for modules of rank at least 2.

$M = D^{1\times p}/(D^{1\times q}\, R)$	$N = D^q/(R\, D^p)$	$\mathcal{F}$ **injective**
with torsion	$t(M) \cong \mathrm{ext}^1_D(N, D)$	$\emptyset$
torsion-free	$\mathrm{ext}^1_D(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^{l_1}$
reflexive	$\mathrm{ext}^i_D(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{l_2}$
projective = stably free	$\mathrm{ext}^i_D(N, D) = 0$ $1 \leq i \leq n = \mathrm{gld}(D)$	$\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\, \mathcal{F}^{l_2}$ $\ldots$ $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\, \mathcal{F}^{l_n}$
free	$\exists\ Q \in D^{p\times m},\ T \in D^{m\times p}$, $\ker_D(.Q) = D^{1\times q}\, R,\ T\, Q = I_m$	$\ker_{\mathcal{F}}(R.) = Q\, \mathcal{F}^m$, $\exists\ T \in D^{m\times p} :\ T\, Q = I_m$

A game with one matrix

$$\begin{array}{ccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} M \longrightarrow 0 \\ D^{q} & \xleftarrow{R.} & D^{p} & \end{array}$$

A game with one matrix

$$\begin{array}{ccccccccc} & & & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M \longrightarrow 0 \\ {\color{red}0} & {\color{red}\longleftarrow} & {\color{red}N} & {\color{red}\xleftarrow{\kappa}} & D^{q} & \xleftarrow{R.} & D^{p} & & \end{array}$$

A game with one matrix

$$
\begin{array}{ccccccccccc}
 & & & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
0 & \longleftarrow & N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \color{red}{\xleftarrow{Q.}} & \color{red}{D^{m}} & &
\end{array}
$$

A game with one matrix

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \| & & \| & & & & \\ D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \color{red}{\xrightarrow{.Q}} & \color{red}{D^{1\times m}} & & \end{array}$$

$$\begin{aligned} R\,Q = 0 \quad & \Rightarrow \mathrm{im}_D(.R) \subseteq \ker_D(.Q) \\ & \Rightarrow t(M) \cong \mathrm{ext}^1_D(N, D) \cong \color{red}{\ker_D(.Q)/\mathrm{im}_D(.R)}. \end{aligned}$$

A game with one matrix

$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\| & & \| & & & & \\
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & & \\
& & \| & & \| & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & &
\end{array}$$

$$\begin{aligned} t(M) \cong \mathrm{ext}^1_D(N, D) &\cong \ker_D(.Q)/\mathrm{im}_D(.R) \\ &= (D^{1\times q'}\, R')/(D^{1\times q}\, R). \end{aligned}$$

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0, \quad M = D^{1\times p}/(D^{1\times q}\, R),$$

$$\Rightarrow \; M/t(M) \cong D^{1\times p}/(D^{1\times q'}\, R').$$

A game with one matrix

$$\begin{array}{ccccccccc}
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \| & & \| & & & & \\
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & & \\
 & & \downarrow .R'' & & \| & & \| & & \\
D^{1\times r'} & \xrightarrow{.R_2'} & D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & &
\end{array}$$

$$D^{1\times q}\, R \subseteq D^{1\times q'}\, R' \quad \Rightarrow \quad \exists\, R'' \in D^{q\times q'} : \quad R = R''\, R'.$$

$$\begin{aligned}
t(M) \cong \mathrm{ext}^1_D(N, D) &\cong (D^{1\times q'}\, R')/(D^{1\times q}\, R) \\
&\cong (D^{1\times q'}\, R')/\left(D^{1\times (q+r')} \begin{pmatrix} R''\, R' \\ R_2'\, R' \end{pmatrix}\right) \\
&\cong D^{1\times q'}/\left(D^{1\times (q+r')} \begin{pmatrix} R'' \\ R_2' \end{pmatrix}\right)
\end{aligned}$$

$t(M)$ and $M/t(M)$

$$\operatorname{ext}^1_D(N, D) \cong t(M) = (D^{1\times q'}\, R')/(D^{1\times q}\, R),$$
$$M/t(M) \cong D^{1\times p}/(D^{1\times q'}\, R').$$

- $D^{1\times q}\, R \subseteq D^{1\times q'}\, R' \;\Rightarrow\; \exists\, R'' \in D^{q\times q'} : R = R''\, R'.$

- Since $(D^{1\times q'}\, R')/(D^{1\times q}\, R)$ is a torsion left D-module, then:

$$\exists\, P_i \in D : \; P_i\, \pi(R'_{i\bullet}) = 0 \;\Leftrightarrow\; \pi(P_i\, R'_{i\bullet}) = 0$$

$$\Rightarrow\; \exists\, \mu_i \in D^{1\times q} : \; P_i\, R'_{i\bullet} = \mu_i\, R \;\Leftrightarrow\; (P_i \quad -\mu_i) \begin{pmatrix} R'_{i\bullet} \\ R \end{pmatrix} = 0.$$

$\Rightarrow$ Find the compatibility conditions of

$$\left\{ \begin{array}{l} R'_{i\bullet}\, \eta = \tau_i, \\ R\, \eta = 0. \end{array} \right. \quad \stackrel{\text{GB}}{\Longrightarrow} \quad P_{ik}\, \tau_i = 0, \; k = 1, \ldots, m_i.$$

Involutions and adjoints

- Definition: A linear map $\theta : D \longrightarrow D$ is an involution of D if:

$$\forall\, P,\, Q \in D: \;\; \theta(P\,Q) = \theta(Q)\,\theta(P), \quad \theta^2 = \mathrm{id}.$$

- Example: 1. If D is a commutative ring, then $\theta = \mathrm{id}$.

2. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ is:

$$\forall\, a \in A, \quad \theta(a(x)) = a(x), \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$$

3. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right] [\partial_2; \delta, 0]$ is defined by:

$$\forall\, a \in A, \quad \theta(a(t)) = a(-t), \quad \theta(\partial_i) = \partial_i, \quad i = 1,\, 2.$$

- The adjoint of $R \in D^{q \times p}$ is defined by $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$.
- $\widetilde{N} = D^{1 \times q}/(D^{1 \times p}\,\theta(R))$ is called the adjoint of M.

Example

- $D = A_2(\mathbb{Q})$ and $R = (\partial_1 \quad \partial_2 \quad x_1\,\partial_1 + x_2\,\partial_2)$.

$$\begin{aligned}
\theta(R) &= (\theta(\partial_1) \quad \theta(\partial_2) \quad \theta(x_1\,\partial_1 + x_2\,\partial_2))^T \\
&= (-\partial_1 \quad -\partial_2 \quad \theta(\partial_1)\,\theta(x_1) + \theta(\partial_2)\,\theta(x_2))^T \\
&= (-\partial_1 \quad -\partial_2 \quad -\partial_1\,x_1 - \partial_2\,x_2)^T \\
&= -(\partial_1 \quad \partial_2 \quad x_1\,\partial_1 + x_2\,\partial_2 + 2)^T.
\end{aligned}$$

- $\theta(R)$ is the formal adjoint $\widetilde{R}$ of R in the theory of distributions.

Extension functor $\mathrm{ext}_D^1(N, D)$

$$4.\quad \theta(P)\, z = y \quad \Longrightarrow \quad R\, y = 0 \quad 1.$$

$$\Uparrow \text{ involution } \theta \Uparrow \qquad \Downarrow \text{ involution } \theta \Downarrow$$

$$3.\quad 0 = P\, \mu \quad \overset{\text{GB}}{\Longleftarrow} \quad \theta(R)\, \lambda = \mu \quad 2.$$

$$P \circ \theta(R) = 0 \quad \Longrightarrow \quad \theta(P \circ \theta(R)) = \theta^2(R) \circ \theta(P) = R \circ \theta(P) = 0.$$

$$5.\quad \theta(P)\, z = y \quad \overset{\text{GB}}{\Longrightarrow} \quad R'\, y = 0, \quad R' \in D^{q' \times p}.$$

$$\boxed{\mathrm{ext}_D^1(N, D) \cong (D^{1 \times q'}\, R')/(D^{1 \times q}\, R)}$$

6. Using GB, we can test whether or not $\mathrm{ext}_D^1(N, D) = 0$.

$$\mathrm{ext}_D^1(N, D) = 0 \;\Rightarrow\; R\, y = 0 \;\Leftrightarrow\; y = Q\, z, \quad Q = \theta(P).$$

Wind tunnel model (Manitius, IEEE TAC 84)

1. The w.t.m. is defined by the underdetermined system:

$$\begin{pmatrix} \partial_1 + a & -k\,a\,\partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{ \begin{array}{l} (\partial_1 + a)\;\lambda_1 = \mu_1, \\ -k\,a\,\partial_2\,\lambda_1 + \partial_1\,\lambda_2 + \omega^2\,\lambda_3 = \mu_2, \\ -\lambda_2 + (\partial_1 + 2\,\zeta\,\omega)\;\lambda_3 = \mu_3, \\ -\omega^2\,\lambda_3 = \mu_4. \end{array} \right. \qquad (2)$$

(2) is overdetermined $\stackrel{\mathrm{GB}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Wind tunnel model (Manitius, IEEE TAC 84)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$\begin{pmatrix} \omega^2\,k\,a\,\partial_2 & \omega^2\,(\partial_1 - a) & \omega^2\,(\partial_1^2 + a\,\partial_1) \\ (\partial_1^3 + 2\,\zeta\,\omega\,\partial_1^2 + a\,\partial_1^2 + \omega^2\,\partial_1 + 2\,a\,\zeta\,\omega\,\partial_1 + a\,\omega^2) & & \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_4 \end{pmatrix} = 0.$$

4. We consider the overdetermined system $P^T\,z = y$.

$$\left\{\begin{array}{l} \omega^2\,k\,a\,\partial_2\,z = x_1, \\ \omega^2\,(\partial_1 - a)\,z = x_2, \\ \omega^2\,(\partial_1^2 + a\,\partial_1)\,z = x_3, \\ (\partial^3 + (2\,\zeta\,\omega + a)\,\partial_1^2 + (\omega^2 + 2\,a\,\omega\,\zeta)\,\partial_1 + a\,\omega)\,z = u. \end{array}\right. \qquad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are exactly generated by $R\,y = 0$ and (4) is a parametrization of the w.t.m.

Moving tank (Petit, Rouchon, IEEE TAC 02)

1. The model of a moving tank is defined by:

$$\begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{ \begin{array}{l} \partial_1\,\lambda_1 + \partial_1\,\partial_2^2\,\lambda_2 = \mu_1, \\ -\partial_1\,\partial_2^2\,\lambda_1 - \partial_1\,\lambda_2 = \mu_2, \\ a\,\partial_1^2\,\partial_2\,\lambda_1 + a\,\partial_1^2\,\partial_2\,\lambda_2 = \mu_3. \end{array} \right. \qquad (2)$$

(2) is overdetermined $\stackrel{\mathrm{GB}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Moving tank (Petit, Rouchon, IEEE TAC 02)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$\begin{pmatrix} a\,\partial_1\,\partial_2 & -a\,\partial_1\,\partial_2 & -(1+\partial_2^2) \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = 0.$$

4. We consider the overdetermined system $P^T\,z = y$.

$$\left\{ \begin{array}{l} a\,\partial_1\,\partial_2\,z = y_1, \\ -a\,\partial_1\,\partial_2\,z = y_2, \\ -(1+\partial_2^2)\,z = y_3. \end{array} \right. \qquad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are $R'\,y = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & (1+\partial_2^2) & -a\,\partial_1\,\partial_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0.$$

Moving tank (Petit, Rouchon, IEEE TAC 02)

$$t(M) \quad \cong \quad \mathrm{ext}^1_D(N, D) \quad \cong$$

$$\left(D^{1\times 2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\partial_2^2 & -a\,\partial_1\,\partial_2 \end{pmatrix}\right) \Big/ \left(D^{1\times 2}\begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix}\right)$$

$$\begin{cases} y_1 + y_2 = z_1, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{cases} \quad \overset{\mathrm{GB}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_1 = 0.$$

$$\begin{cases} (1+\partial_2^2)\,y_2 - a\,\partial_1\,\partial_2\,y_3 = z_2, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{cases} \quad \overset{\mathrm{GB}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_2 = 0.$$

$\Rightarrow$ $z_1(t)$ and $z_2(t)$ are torsion elements.

Examples: reflexive modules

- div-curl-grad: $\vec{\nabla}.\,\vec{B} = 0 \;\Leftrightarrow\; \vec{B} = \vec{\nabla} \wedge \vec{A},\ \vec{\nabla} \wedge \vec{A} = \vec{0} \;\Leftrightarrow\; \vec{A} = \vec{\nabla} f.$
- First group of Maxwell equations:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}.\,\vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V = \vec{0}, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{A} = \vec{\nabla}\, \xi, \\ V = -\dfrac{\partial \xi}{\partial t}. \end{array} \right.$$

- 3D stress tensor: Maxwell, Morera parametrizations. . .
- Linearized Einstein equations (10×10 system of PDEs)?

$\Rightarrow$ OreModules (Chyzak, Robertz, Q.)

Duality and parametrizations of systems

$$\begin{array}{ccccccccc}
 & & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
0 \longleftarrow N & \xleftarrow{\kappa} & D^{q} & \xleftarrow{R.} & D^{p} & \xleftarrow{Q.} & D^{m} & & \\
 & & D^{1\times q} & \xrightarrow{.R} & {\color{red}D^{1\times p}} & \xrightarrow{.Q} & D^{1\times m} & & (\star) \\
 & & D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} & &
\end{array}$$

- If $\mathcal{F}$ is a left D-module, then the complex holds:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m, \quad \text{i.e.,} \quad Q\,\mathcal{F}^m \subseteq \ker_{\mathcal{F}}(R.).$$

- If $\mathcal{F}$ is injective, then the exact sequence holds:

$$\mathcal{F}^{q'} \xleftarrow{R'.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m, \quad \text{i.e.,} \quad \ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m.$$

- If $t(M) = (D^{1\times q'}\,R')/(D^{1\times q}\,R) = 0$ and $\mathcal{F}$ is injective, then:

$$\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m.$$

Monge parametrization

- $\ker_D(.R') = D^{1\times r'}\, R_2'$. If $\mathcal{F}$ be a left D-module then:

$$R\,\eta = 0 \quad \Leftrightarrow \quad R''\,(R'\,\eta) = 0 \quad \Leftrightarrow \quad \begin{cases} R'\,\eta = \theta, \\ R''\,\theta = 0, \\ R_2'\,\theta = 0. \end{cases}$$

Integration of $R\,\eta = 0$ in cascade:

1. Find a "general solution" $\overline{\theta} \in \mathcal{F}^p$ of:

$$\begin{cases} F\,\theta = 0, \\ T\,\theta = 0, \end{cases} \qquad \text{(over)determined system.}$$

2. Find a particular solution $\eta^\star \in \mathcal{F}^p$ of $S\,\eta = \overline{\theta}$.
3. If $\mathcal{F}$ is injective then $\ker_{\mathcal{F}}(S.) = Q\,\mathcal{F}^m$ and:

$$\forall\, \xi \in \mathcal{F}^m, \quad \eta = \eta^\star + Q\,\xi.$$

Example: Tank model (Petit-Rouchon, IEEE TAC, 02)

- We consider a model of the motion of a fluid in a 1-dimensional tank described by the OD time-delay system:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\,\ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\,\ddot{y}_3(t-h) = 0. \end{cases}$$

- Let $D = \mathbb{Q}(\alpha)[\partial, \delta]$ and $M = D^{1\times 3}/(D^{1\times 2}\,R)$, where:

$$R = \begin{pmatrix} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{pmatrix} \in D^{2\times 3}.$$

- Computing $\operatorname{ext}^1_D(N, D)$, where $N = D^2/(R\,D^3)$, we get $R'_2 = 0$,

$$Q = \begin{pmatrix} -\alpha\,\partial\,\delta \\ \alpha\,\partial\,\delta \\ 1+\delta^2 \end{pmatrix}, R' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1-\delta^2 & \alpha\,\partial\,\delta \end{pmatrix}, R'' = \begin{pmatrix} \partial & \partial \\ \partial\,\delta^2 & \partial \end{pmatrix}.$$

Example: Tank model (Petit-Rouchon, IEEE TAC, 02)

- We first integrate the torsion elements $R''\,\theta = 0$

$$\left\{\begin{array}{l} \dot{\theta}_1(t) + \dot{\theta}_2(t) = 0, \\ \dot{\theta}_1(t - 2\,h) + \dot{\theta}_2(t) = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \theta_1(t) = \psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t, \\ \theta_2(t) = -\psi(t) + c_1 - \dfrac{(c_1 - c_2)}{2\,h}\,t, \end{array}\right.$$

for all $c_1,\ c_2 \in \mathbb{R}$ and all arbitrary $2\,h$-periodic ψ of $\mathcal{F} = C^{\infty}(\mathbb{R})$.

- A particular solution of the inhomogeneous system $R'\,y = \theta$

$$\left\{\begin{array}{l} y_1(t) + y_2(t) = \psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t, \\ -y_2(t) - y_2(t - 2\,h) + \alpha\,\dot{y}_3(t - h) = -\psi(t) + c_1 - \dfrac{(c_1 - c_2)}{2\,h}\,t. \end{array}\right.$$

is:

$$\left\{\begin{array}{l} y_1(t) = \dfrac{1}{2}\left(\psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t + \dfrac{(c_1 + c_2)}{2}\right), \\ y_2(t) = \dfrac{1}{2}\left(\psi(t) + \dfrac{(c_1 - c_2)}{2\,h}\,t - \dfrac{(c_1 + c_2)}{2}\right), \\ y_3(t) = 0. \end{array}\right.$$

- The general solution of the homogeneous system $R' z = 0$ is:

$$\forall\, \xi \in \mathcal{F}, \quad \begin{cases} z_1(t) = -\alpha\, \dot{\xi}(t-h), \\ z_2(t) = \alpha\, \dot{\xi}(t-h), \\ z_3(t) = \xi(t) + \xi(t-2\,h). \end{cases}$$

- Finally, the general solution of $R\, y = 0$ is defined by

$$\begin{cases} y_1(t) = \dfrac{1}{2}\,(\psi(t) + C_1\, t + C_2) - \alpha\, \dot{\xi}(t-h), \\ y_2(t) = \dfrac{1}{2}\,(\psi(t) + C_1\, t - C_2) + \alpha\, \dot{\xi}(t-h), \\ y_3(t) = \xi(t) + \xi(t-2\,h), \end{cases}$$

for all $C_1, \; C_2 \in \mathbb{R}$, all arbitrary $2\,h$-periodic ψ of $\mathcal{F}$ and all $\xi \in \mathcal{F}$.

Monge parametrization

- Proposition:
  1. $0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0$ splits, i.e.:
  $$M \cong t(M) \oplus M/t(M).$$
  2. There exist $X \in D^{p \times q'}$, $Y \in D^{q' \times q}$ and $Z \in D^{q' \times r'}$ such that:
  $$R'\,X + (Y \quad Z) \begin{pmatrix} R'' \\ R'_2 \end{pmatrix} = I_{q'}. \quad (\star)$$
  3. There exist $X \in D^{p \times q'}$, $Y \in D^{q' \times q}$ s.t. $R' - R'\,X\,R' = Y\,R$.
- Applying $(\star)$ to $\overline{\tau}$, we get: $\quad \overline{\tau} = R'\,(X\,\overline{\tau}) \quad \Rightarrow \quad \eta^\star = X\,\overline{\tau}$.
- If $\mathcal{F}$ is injective, then $\eta = X\,\overline{\tau} + Q\,\xi, \ \forall\, \xi \in \mathcal{F}^m$.
- $M/t(M)$ is projective iff R' admits a generalized inverse:
$$R'\,X\,R' = R'.$$

Example: Tank model (Dubois-Petit-Rouchon, ECC 99)

- We consider another model of the motion of a fluid in a 1-dimensional tank described by the OD time-delay system:

$$\begin{cases} y_1(t-2\,h)+y_2(t)-2\,\dot{y}_3(t-h)=0, \\ y_1(t)+y_2(t-2\,h)-2\,\dot{y}_3(t-h)=0, \end{cases}$$

- Let $D=\mathbb{Q}[\partial,\delta]$ and $M=D^{1\times 3}/(D^{1\times 2}\,R)$, where:

$$R=\begin{pmatrix} \delta^2 & 1 & -2\,\partial\,\delta \\ 1 & \delta^2 & -2\,\partial\,\delta \end{pmatrix}\in D^{2\times 3},$$

- Computing $\mathrm{ext}^1_D(N,D)$, where $N=D^2/(R\,D^3)$, we get $R'_2=0$,

$$R'=\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1+\delta^2 & -2\,\partial\,\delta \end{pmatrix},\quad Q=\begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ 1+\delta^2 \end{pmatrix},\quad R''=\begin{pmatrix} \delta^2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Example: Tank model (Dubois-Petit-Rouchon, ECC 99)

- We first integrate the torsion elements $R''\,\theta = 0$

$$\left\{\begin{array}{l} \theta_1(t-2,h)+\theta_2(t)=0,\\ \theta_1(t)+\theta_2(t)=0, \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \theta_2(t)=-\theta_1(t),\\ \theta_1(t-2\,h)-\theta_1(t)=0, \end{array}\right.$$

for all arbitrary $2\,h$-periodic θ_1 of $\mathcal{F} = C^\infty(\mathbb{R})$.

- Let us find a particular solution $y^\star$ of $R'\,y = \theta$. The matrices

$$X = \frac{1}{2}\left(\begin{array}{cc} 1 & 0\\ -1 & 0\\ 0 & 0 \end{array}\right), \quad Y = \frac{1}{2}\left(\begin{array}{cc} 0 & 0\\ 1 & 1 \end{array}\right).$$

satisfy the identity $R' - R'\,X\,R' = Y\,R$

$$\Rightarrow y^\star = X\,\theta = \left\{\begin{array}{l} y_1^\star(t) = \frac{1}{2}\,\theta_1(t),\\ y_2^\star(t) = -\frac{1}{2}\,\theta_1(t),\\ y_3^\star(t) = 0. \end{array}\right.$$

Example: Tank model (Dubois-Petit-Rouchon, ECC 99)

- The general solution of the homogeneous system $R' z = 0$ is:

$$\forall\, \xi \in \mathcal{F}, \quad \begin{cases} z_1(t) = 2\,\dot{\xi}(t-h), \\ z_2(t) = 2\,\dot{\xi}(t-h), \\ z_3(t) = \xi(t) + \xi(t-2\,h), \end{cases}$$

- Finally, the general solution of $R\,y = 0$ is defined by

$$\begin{cases} y_1(t) = \frac{1}{2}\,\theta_1(t) + 2\,\dot{\xi}(t-h), \\ y_2(t) = -\frac{1}{2}\,\theta_1(t) + 2\,\dot{\xi}(t-h), \\ y_3(t) = \xi(t) + \xi(t-2\,h), \end{cases}$$

for all arbitrary $2\,h$-periodic θ_1 of $\mathcal{F}$ and all $\xi \in \mathcal{F}$.

Flexible rod (Mounier, Rudolph, Petitot, Fliess, ECC 95)

- We consider a model of a flexible rod with a torque described by:

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-h) - y_3(t) = 0, \\ 2\,\dot{y}_1(t-h) - \dot{y}_2(t) - \dot{y}_2(t-2\,h) = 0. \end{cases}$$

- Let $D = \mathbb{Q}[\partial, \delta]$ and $M = D^{1\times 3}/(D^{1\times 2}\,R)$, where:

$$R = \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(1+\delta^2) & 0 \end{pmatrix} \in D^{2\times 3}.$$

- Computing $\mathrm{ext}^1_D(N, D)$, where $N = D^2/(R\,D^3)$, we get

$$R' = \begin{pmatrix} -2\,\delta & 1+\delta^2 & 0 \\ -\partial & \partial\,\delta & 1 \\ \partial\,\delta & -\partial & \delta \end{pmatrix}, Q = \begin{pmatrix} 1+\delta^2 \\ 2\,\delta(1-\delta^2)\,\partial \end{pmatrix}, R'' = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -\delta & 1 \end{pmatrix},$$

and $R'_2 = (\partial \quad -\delta \quad 1)$.

Flexible rod (Mounier, Rudolph, Petitot, Fliess, ECC 95)

- We first integrate the torsion elements $(R''^T \quad R_2'^T)^T \, \theta = 0$:

$$\left\{\begin{array}{l} -\theta_2 = 0, \\ -\delta\,\theta_2 + \theta_3 = 0, \\ \partial\,\theta_1 - \delta\,\theta_2 + \theta_3 = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \partial\,\theta_1 = 0, \\ \theta_2 = 0, \\ \theta_3 = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \theta_1 = c \in \mathbb{R}, \\ \theta_2 = 0, \\ \theta_3 = 0. \end{array}\right.$$

- We can check that the D-module $M/t(M) = D^{1\times 3}/(D^{1\times 3}\,R')$ is projective and R' admits the following generalized inverse:

$$X = \frac{1}{2}\begin{pmatrix} \delta & 0 & 0 \\ 2 & 0 & 0 \\ -\partial\,\delta & 2 & 0 \end{pmatrix}, \quad \text{i.e.,} \quad R'\,X\,R' = R'.$$

$\Rightarrow$ $y^\star = X\,\theta = (c/2 \quad c \quad 0)^T$ is a particular solution of $R'\,y = \theta$.

Flexible rod (Mounier, Rudolph, Petitot, Fliess, ECC 95)

- The general solution of the homogeneous system $R'\,z = 0$ is:

$$\forall\, \xi \in \mathcal{F}, \quad \left\{ \begin{array}{l} z_1(t) = \xi(t) + \xi(t - 2\,h), \\ z_2(t) = 2\,\xi(t - h), \\ z_3(t) = \dot{\xi}(t) - \dot{\xi}(t - 2\,h), \end{array} \right.$$

- Finally, the general solution of $R\,y = 0$ is defined by

$$\left\{ \begin{array}{l} y_1(t) = \frac{1}{2}\,c + \xi(t) + \xi(t - 2\,h), \\ y_2(t) = c + 2\,\xi(t - h), \\ y_3(t) = \dot{\xi}(t) - \dot{\xi}(t - 2\,h), \end{array} \right.$$

where c is an arbitrary constant and ξ an arbitrary function of $\mathcal{F}$.

Purity filtration

- Let us consider the beginning of a finite free resolution of M:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} D^{1\times p_2} \xleftarrow{.R_3} D^{1\times p_3}.$$

- Let $R_{ii} = R_i$, $p_{ii} = p_i$ and the Auslander transposes:

$$N_{ii} = D_i^p/(R_i\, D^{p_{i-1}}) = D^{p_{ii}}/(R_{ii}\, D^{p_{(i-1)i}}).$$

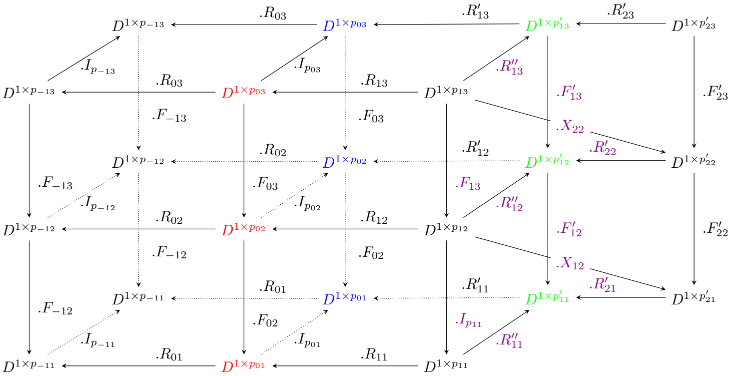
$$\begin{tikzcd}
D^{p_{-13}} \arrow[r, "R_{03}."] & D^{p_{03}} \arrow[r, "R_{13}."] & D^{p_{13}} \arrow[r, "R_{23}."] & D^{p_{23}} \arrow[r, "R_{33}."] & D^{p_{33}} \arrow[r, "\kappa_{33}"] & N_{33} \arrow[r] & 0 \\
D^{p_{-12}} \arrow[r, "R_{02}."] \arrow[u, "F_{-13}."'] & D^{p_{02}} \arrow[r, "R_{12}."] \arrow[u, "F_{03}."'] & D^{p_{12}} \arrow[r, "R_{22}."] \arrow[u, "F_{13}."'] & D^{p_{22}} \arrow[r, "\kappa_{22}"] \arrow[u, equal] & N_{22} \arrow[r] & 0 & \\
D^{p_{-11}} \arrow[r, "R_{01}."] \arrow[u, "F_{-12}."'] & D^{p_{01}} \arrow[r, "R_{11}."] \arrow[u, "F_{02}."'] & D^{p_{11}} \arrow[r, "\kappa_{11}"] \arrow[u, equal] & N_{11} \arrow[r] & 0 & & \\
0 \arrow[r] \arrow[u] & D^{p_{00}} \arrow[r, "\kappa_{00}"] \arrow[u, equal] & N_{00} \arrow[r] & 0, & & &
\end{tikzcd}$$

Purity filtration

$$\begin{array}{ccccc}
D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R_{13}} & D^{1\times p_{13}} \\
\downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F_{13} \\
D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R_{12}} & D^{1\times p_{12}} \\
\downarrow .F_{-12} & & \downarrow .F_{02} & & \| \\
D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R_{11}} & D^{1\times p_{11}}.
\end{array}$$

$$\begin{array}{ccccccc}
D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R'_{13}} & D^{1\times p'_{13}} & \xleftarrow{.R'_{23}} & D^{1\times p'_{23}} \\
\downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F'_{13} & & \downarrow .F'_{23} \\
D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R'_{12}} & D^{1\times p'_{12}} & \xleftarrow{.R'_{22}} & D^{1\times p'_{22}} \\
\downarrow .F_{-12} & & \downarrow .F_{02} & & \downarrow .F'_{12} & & \downarrow .F'_{22} \\
D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R'_{11}} & D^{1\times p'_{11}} & \xleftarrow{.R'_{21}} & D^{1\times p'_{21}}.
\end{array}$$

Purity filtration



Purity filtration

$$
\begin{tikzcd}
 & \mathcal{F}^{p_{-13}} \arrow[rr, "R_{03}."] \arrow[dl, "I_{p_{-13}}."] & & \mathcal{F}^{p_{03}} \arrow[rr, "R'_{13}."] \arrow[dl, "I_{p_{03}}."] & & \mathcal{F}^{p'_{13}} \arrow[r, "R'_{23}."] \arrow[dl, "R''_{13}."] & \mathcal{F}^{p'_{23}} \\
\mathcal{F}^{p_{-13}} \arrow[rr, "R_{03}."] & & \mathcal{F}^{p_{03}} \arrow[rr, "R_{13}."] & & \mathcal{F}^{p_{13}} & & \\
 & \mathcal{F}^{p_{-12}} \arrow[uu, dotted, "F_{-13}."] \arrow[rr, dotted, "R_{02}."] \arrow[dl, dotted, "I_{p_{-12}}."] & & \mathcal{F}^{p_{02}} \arrow[uu, dotted, "F_{03}."] \arrow[rr, dotted, "R'_{12}."] \arrow[dl, dotted, "I_{p_{02}}."] & & \mathcal{F}^{p'_{12}} \arrow[uu, "F'_{13}."] \arrow[r, "R'_{22}."] \arrow[dl, "R''_{12}."] & \mathcal{F}^{p'_{22}} \arrow[uu, "F'_{23}."] \arrow[ull, "X_{22}."] \\
\mathcal{F}^{p_{-12}} \arrow[uu, "F_{-13}."] \arrow[rr, "R_{02}."] & & \mathcal{F}^{p_{02}} \arrow[uu, "F_{03}."] \arrow[rr, "R_{12}."] & & \mathcal{F}^{p_{12}} \arrow[uu, "F_{13}."] & & \\
 & \mathcal{F}^{p_{-11}} \arrow[uu, dotted, "F_{-12}."] \arrow[rr, dotted, "R_{01}."] \arrow[dl, dotted, "I_{p_{-11}}."] & & \mathcal{F}^{p_{01}} \arrow[uu, dotted, "F_{02}."] \arrow[rr, dotted, "R'_{11}."] \arrow[dl, dotted, "I_{p_{01}}."] & & \mathcal{F}^{p'_{11}} \arrow[uu, "F'_{12}."] \arrow[r, "R'_{21}."] \arrow[dl, "R''_{11}."] & \mathcal{F}^{p'_{21}} \arrow[uu, "F'_{22}."] \arrow[ull, "X_{12}."] \\
\mathcal{F}^{p_{-11}} \arrow[uu, "F_{-12}."] \arrow[rr, "R_{01}."] & & \mathcal{F}^{p_{01}} \arrow[uu, "F_{02}."] \arrow[rr, "R_{11}."] & & \mathcal{F}^{p_{11}} \arrow[uu, "I_{p_{11}}."] & &
\end{tikzcd}
$$

Purity filtration

- Theorem: The following system equivalence holds:

$$R_{11}\,\eta = 0 \quad \Leftrightarrow \quad \left\{\begin{array}{l} R'_{11}\,\zeta = \tau_1, \\ \mathbf{F}'_{12}\,\tau_{\mathbf{1}} = \tau_2, \\ \mathbf{R}''_{11}\,\tau_{\mathbf{1}} = \mathbf{0}, \\ \mathbf{R}'_{21}\,\tau_{\mathbf{1}} = \mathbf{0}, \\ F'_{13}\,\tau_2 = \tau_3, \\ R''_{12}\,\tau_2 = 0, \\ R'_{22}\,\tau_2 = 0, \\ R''_{13}\,\tau_3 = 0, \\ R'_{23}\,\tau_3 = 0. \end{array}\right.$$

- $R_{11} = R, \quad R'_{11} = R', \quad R''_{11} = R'', \quad R'_{21} = R'_2.$

Purity filtration

$$S_0 = R'_{11}, \quad \mathbf{S_1} = \begin{pmatrix} F'_{12} \\ R''_{11} \\ R'_{21} \end{pmatrix}, \quad S_2 = \begin{pmatrix} F'_{13} \\ R''_{12} \\ R'_{22} \end{pmatrix}, \quad S_3 = \begin{pmatrix} R''_{13} \\ R'_{23} \end{pmatrix}.$$

1. $\ker_{\mathcal{F}}(S_3.)$ has dimension $\leq \dim(D) - 3$ when it is non-trivial $(= \dim(D) - 3$ when $\ker_D(.R_3) = 0)$,
2. $\ker_{\mathcal{F}}(S_2.)$ has dimension $\dim(D) - 2$ when it is non-trivial,
3. $\ker_{\mathcal{F}}(\mathbf{S_1}.)$ has dimension $\dim(\mathbf{D}) - \mathbf{1}$ when it is non-trivial,
4. $\ker_{\mathcal{F}}(S_0.)$ has dimension $\dim(D)$ when it is non-trivial.

• Example: If $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of $\mathrm{char}(k) = 0$, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then:

$$\dim \left(A \left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1} \right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n} \right] \right) = 2n.$$

Purity filtration

$$\begin{array}{rcl} \gamma : \ker_{\mathcal{F}}(P.) & \longrightarrow & \ker_{\mathcal{F}}(R_{11}.) \\ \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} & \longmapsto & \eta = \zeta, \end{array}$$

$$\begin{array}{rcl} \gamma^{-1} : \ker_{\mathcal{F}}(R_{11}.) & \longrightarrow & \ker_{\mathcal{F}}(P.) \\ \eta & \longmapsto & \begin{pmatrix} \zeta \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} I_{p_{01}} \\ R'_{11} \\ F'_{12}\, R'_{11} \\ F'_{13}\, F'_{12}\, R'_{11} \end{pmatrix} \eta. \end{array}$$

Example

- Let $D = \mathbb{Q}\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right]\left[\partial_2; \mathrm{id}, \frac{\partial}{\partial x_2}\right]\left[\partial_3; \mathrm{id}, \frac{\partial}{\partial x_3}\right] \cong \mathbb{Q}[\partial_1, \partial_2, \partial_3]$,

$$R = \begin{pmatrix} 0 & -2\,\partial_1 & \partial_3 - 2\,\partial_2 - \partial_1 & -1 \\ 0 & \partial_3 - 2\,\partial_1 & 2\,\partial_2 - 3\,\partial_1 & 1 \\ \partial_3 & -6\,\partial_1 & -2\,\partial_2 - 5\,\partial_1 & -1 \\ 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\ \partial_1 & -\partial_1 & -2\,\partial_1 & 0 \end{pmatrix},$$

and the D-module $M = D^{1\times 4}/(D^{1\times 6}\,R)$.

- Computing the purity filtration of M, we get;

$$M \cong N = D^{1\times 11}/(D^{1\times 23}\,P).$$

Example

$$P = \begin{pmatrix}
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \partial_1 - 2\,\partial_2 + \partial_3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{0} & \boldsymbol{-2\,\partial_1} & \boldsymbol{1} & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{0} & \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{0} & \boldsymbol{-2\,\partial_1} & \boldsymbol{1} & \boldsymbol{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{0} & \boldsymbol{-2\,\partial_1 + \partial_3} & \boldsymbol{-1} & \boldsymbol{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{\partial_3} & \boldsymbol{-6\,\partial_1} & \boldsymbol{1} & \boldsymbol{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{0} & \boldsymbol{-\partial_1 + \partial_2} & \boldsymbol{0} & \boldsymbol{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{\partial_2} & \boldsymbol{-\partial_1} & \boldsymbol{0} & \boldsymbol{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{\partial_1} & \boldsymbol{-\partial_1} & \boldsymbol{0} & \boldsymbol{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4\,\partial_1 - \partial_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4\,\partial_1 - \partial_3 & \partial_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1
\end{pmatrix}.$$

Example

$$\left\{\begin{array}{l} -\partial_2\,\tau_3=0,\\ -\partial_3\,\tau_3=0,\\ \partial_1\,\tau_3=0, \end{array}\right. \quad \Leftrightarrow \quad \tau_3=c_1\in\mathbb{R}.$$

$$\left\{\begin{array}{l} \tau_{23}-\tau_3=0,\\ \tau_{21}=0,\\ -\tau_{21}+(4\,\partial_1-\partial_3)\,\tau_{22}=0,\\ \tau_{21}+(4\,\partial_1-\partial_3)\,\tau_{22}+\partial_3\,\tau_{23}=0,\\ (\partial_1-\partial_2)\,\tau_{22}=0, \end{array}\right. \quad \Leftrightarrow \left\{\begin{array}{l} \tau_{23}=\tau_3=c_1,\\ \tau_{21}=0,\\ (4\,\partial_1-\partial_3)\,\tau_{22}=0,\\ (\partial_1-\partial_2)\,\tau_{22}=0, \end{array}\right.$$

$\Rightarrow\ \tau_{21}=0,\ \tau_{22}=f_1(x_3+\frac{1}{4}\,(x_1+x_2))$, where f_1 is an arbitrary smooth function, and $\tau_{23}=c_1$, where c_1 is an arbitrary constant.

Example

$$\left\{\begin{array}{l} -2\,\partial_1\,\tau_{12}+\tau_{13}-\tau_{21}=0,\\ -\tau_{12}-\tau_{22}=0,\\ \tau_{11}-\tau_{12}-\tau_{23}=0,\\ -2\,\partial_1\,\tau_{12}+\tau_{13}=0,\\ (-2\,\partial_1+\partial_3)\,\tau_{12}-\tau_{13}=0,\\ \partial_3\,\tau_{11}-6\,\partial_1\,\tau_{12}+\tau_{13}=0,\\ (-\partial_1+\partial_2)\,\tau_{12}=0,\\ \partial_2\,\tau_{11}-\partial_1\,\tau_{12}=0,\\ \partial_1\,\tau_{11}-\partial_1\,\tau_{12}=0, \end{array}\right.$$

$$\Leftrightarrow \quad \left\{\begin{array}{l} \tau_{12}=-\tau_{22}=-f_1(x_3+\frac{1}{4}\,(x_1+x_2)),\\ \tau_{11}=\tau_{12}+\tau_{23}=-f_1(x_3+\frac{1}{4}\,(x_1+x_2))+c_1,\\ \tau_{13}=2\,\partial_1\,\tau_{12}+\tau_{21}=-\frac{1}{2}\,\dot{f}_1(x_3+\frac{1}{4}\,(x_1+x_2)). \end{array}\right.$$

Example

$$\left\{\begin{array}{l}\zeta_1 - \zeta_3 - \tau_{11} = 0,\\ \zeta_2 + \zeta_3 - \tau_{12} = 0,\\ (\partial_1 - 2\,\partial_2 + \partial_3)\,\zeta_3 - \zeta_4 - \tau_{13} = 0,\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l}\zeta_1 - \zeta_2 = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1,\\ \zeta_2 + \zeta_3 = -f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)),\\ (\partial_1 - 2\,\partial_2 + \partial_3)\,\zeta_3 - \zeta_4 = -\frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)).\end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l}\zeta_1 = \xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1,\\ \zeta_2 = -\xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)),\\ \zeta_3 = \xi,\\ \zeta_4 = (\partial_1 - 2\,\partial_2 + \partial_3)\,\xi + \frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)),\end{array}\right.$$

Example

$$\begin{cases} -2\,\partial_1\,\eta_2 + \partial_3\,\eta_3 - 2\,\partial_2\,\eta_3 - \partial_1\,\eta_3 - \eta_4 = 0, \\ \partial_3\,\eta_2 - 2\,\partial_1\,\eta_2 + 2\,\partial_2\,\eta_3 - 3\,\partial_1\,\eta_3 + \eta_4 = 0, \\ \partial_3\,\eta_1 - 6\,\partial_1\,\eta_2 - 2\,\partial_2\,\eta_3 - 5\,\partial_1\,\eta_3 - \eta_4 = 0, \\ \partial_2\,\eta_2 - \partial_1\,\eta_2 + \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_2\,\eta_1 - \partial_1\,\eta_2 - \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_1\,\eta_1 - \partial_1\,\eta_2 - 2\,\partial_1\,\eta_3 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \eta_1 = \xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)) + c_1, \\ \eta_2 = -\xi - f_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \\ \eta_3 = \xi, \\ \eta_4 = (\partial_1 - 2\,\partial_2 + \partial_3)\,\xi + \frac{1}{2}\,\dot{f}_1(x_3 + \frac{1}{4}\,(x_1 + x_2)), \end{cases}$$

where ξ (resp., f_1, c_1) is an arbitrary function of $C^\infty(\mathbb{R}^3)$ (resp., $C^\infty(\mathbb{R})$, constant).