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On the Jacobian Varieties of Hyperelliptic Curves over Fields of Characteristic $p > 2$

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1. INTRODUCTION

It is well known that an Abelian variety X of dimension g defined over a field k of characteristic $p > 0$ yields a p-divisible group $X(p)$ of dimension g and of height $2g$. Let $\mathbf{\Gamma}$ be the formal group obtained by expansion into power series of the group law of X relative to some system of local parameters at the origin. Then $\mathbf{\Gamma}$ is nothing but the connected p-divisible group in $X(p)$ and $\mathbf{\Gamma}$ has any height between g and $2g$ (cf. Tate [14]).

In the present paper, we confine ourselves to the study of the Jacobian variety $J(C)$ of a hyperelliptic curve C over a field of characteristic $p > 2$. Our aims here are (i) to determine the structures of the p-divisible group $J(p)$ and of the formal group $\mathbf{\Gamma}$ of $J(C)$ (up to isogeny) with the help of the Cartier–Manin matrix A of C, and (ii) to investigate how much information about the algebraic (global) structure of $J(C)$ (up to isogeny) can be recovered from the formal (local) structure.

We shall give a brief survey of the paper here. In Section 2, we define the Cartier–Manin matrix A of a hyperelliptic curve C over a perfect field of characteristic $p > 2$ following Cartier [1] and Manin [8]. We then show that A coincides with the Hasse–Witt matrix of C. Some basic but important properties of A are also discussed. After this, throughout the forthcoming sections, we assume that k is a finite field with p^a ($a \geqslant 1$) elements. In Section 3, we give a complete characterization of the "ordinary" Jacobian variety $J(C)$ of C. When $J(C)$ is ordinary, the Cartier–Manin matrix A of C completely determines the formal structure $J(p)$, and in certain cases the algebraic structure as well (up to isogeny). In the rest of the paper, we study the Jacobian variety $J(C)$ of the hyperelliptic curve C whose Cartier–Manin matrix has determinant zero in k. In Section 4, we observe that the Cartier–Manin matrix A of C no longer provides enough information; it is the p-adic exponents of the eigenvalues of the characteristic polynomial of the Frobenius endomorphism of $J(C)$ relative to k that determine

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the isogeny class of $J(p)$. In Section 5, we characterize the "supersingular" Jacobian variety $J(C)$ of C. It is shown that, in this case, the formal group of $J(C)$ completely determines the algebraic structure of $J(C)$. We also show that the condition $A = (0)$ in k is sufficient but not necessary for $J(C)$ to be supersingular. In Section 6, we discuss the Jacobian variety $J(C)$ whose formal group Γ is isogenous to the symmetric formal group of dimension g. Finally, in Section 7, we consider the Jacobian variety $J(C)$ with the formal structure of the mixed type. It turns out that there is a k-simple Jacobian variety $J(C)$ with $J(p)$ isogenous to the mixed type $rG_{1,0} + (g - r)G_{1,1}$. We remark here that the Newton polygon $\mathfrak{N}(P_\pi)$ of the characteristic polynomial of the Frobenius endomorphism π of $J(C)$ relative to k, is a very useful tool for finding the local decomposition of $J(C)$ into isotypic (unfortunately not simple) components.

All formal groups and p-divisible groups discussed in this paper are commutative.

This paper is the result of my attempt to understand Manin's works [6, 7]. In the present paper, we deal only with the hyperelliptic curves, but we shall consider more general cases (algebraic curves) in the forthcoming paper [16].

2. The Cartier–Manin Matrix of a Hyperelliptic Curve over a Perfect Field of Characteristic $p > 2$

Let k be a perfect field of characteristic $p > 2$ and let C be a complete non-singular curve over k defined by the equation

$$C : y^2 = f(x), \tag{1}$$

where $f(x)$ is a polynomial over k without multiple roots of degree $2g + 1$.

Denote by $K = k(x, y)$ the algebraic function field of C of one variable over k. Then K has the unique subfield $K^p = k^p(x^p, y^p) = k(x^p, y^p)$ over which K is separably generated, e.g., $K = K^p(x)$ for a separably generating transcendental element $x \in K - K^p$. Let $\Omega^1(K)$ be the space of all differential forms of degree 1 on K and $d : K \to \Omega^1(K)$ the canonical derivation of K. Since $dx \neq 0$ for a separating element x, every element $\omega \in \Omega^1(K)$ can be expressed uniquely in the form

$$\omega = d\phi + \psi^p\, dx/x \qquad \text{with} \quad \phi, \psi \in K,\ \psi^p \in K^p. \tag{2}$$

Definition 2.1. Let $\Omega^1(K^p)$ be the space of all differential forms of degree 1 on K^p and $d^p : K^p \to \Omega^1(K^p)$ the corresponding derivation of K^p to d. We define *the Cartier operator* $\mathscr{C} : \Omega^1(K) \to \Omega^1(K^p)$ by letting, for ω given as (2),

$$\mathscr{C}(\omega) = \psi^p(d^p x^p/x^p).$$

$\mathscr{C}$ is a well-defined K^p-linear operator and $\mathscr{C}(d\phi) = 0$.

Sometimes it is convenient to use the following expression for $\omega \in \Omega^1(K)$:

$$\omega = d\phi + \eta^p x^{p-1}\, dx \qquad \text{with} \quad \phi, \eta \in K, \eta^p \in K^p. \tag{2'}$$

Definition 2.1′. *The modified Cartier operator* $\mathscr{C}' : \Omega^1(K) \to \Omega^1(K)$ is defined for ω given as (2′) by

$$\mathscr{C}'(\omega) = \eta\, dx.$$

Proposition 2.1. *The basic properties of the modified Cartier operator $\mathscr{C}'$ are summarized as follows:*

(a) $\mathscr{C}'(\omega + \omega') = \mathscr{C}'(\omega) + \mathscr{C}'(\omega')$.

(b) $\mathscr{C}'(\phi^p \omega) = \phi \mathscr{C}'(\omega)$ *for* $\phi \in K$.

(c) $\mathscr{C}'(\phi^{n-1}\, d\phi) = d\phi$ *if* $n = p$, *and* 0 *otherwise, for* $\phi \in K$.

(d) $\mathscr{C}'(\omega) = 0 \Leftrightarrow \omega = d\phi$ *with some* $\phi \in K$.

If this is the case, ω *is called exact.*

(e) $\mathscr{C}'(\omega) = \omega \Leftrightarrow \omega = d\phi/\phi$ *with some* $\phi \in K$.

If this is the case, ω *is called logarithmic.*

Proof. They are immediately derived from the definition except (e). For (e) see Cartier [1]. Q.E.D.

Now the differential forms of degree 1 and of the first kind on K form a k-vector space, denoted, $\mathfrak{D}_0(K)$, of dimension g with a system of the canonical basis

$$\mathscr{B} = \left\{\omega_i = \frac{x^{i-1}\, dx}{y}, i = 1,..., g\right\}. \tag{3}$$

The images of the ω_i's under the modified Cartier operator $\mathscr{C}'$ are determined in the following way due to Manin [8]. Rewrite ω_i as

$$\omega_i = \frac{x^{i-1}\, dx}{y} = x^{i-1} y^{-p} y^{p-1}\, dx = y^{-p} x^{i-1} \sum_{j=0}^{N} c_j x^j\, dx,$$

where the coefficients $c_j \in k$ are obtained from the expansion

$$f(x)^{(p-1)/2} = \sum_{j=0}^{N} c_j x^j, \qquad N = \frac{p-1}{2}(2g+1).$$

Then we get for $i = 1,\dots, g$,

$$\omega_i = y^{-p}\Bigl(\sum_{\substack{j\\ i+j\not\equiv 0\ (\mathrm{mod}\, p)}} c_j x^{j+i-1}\, dx\Bigr) + \sum_l c_{(l+1)p-i}\,\frac{x^{(l+1)p}}{y^p}\,\frac{dx}{x}$$

$$= d\Bigl(y^{-p}\sum_{\substack{j\\ i+j\not\equiv 0\ (\mathrm{mod}\, p)}} \frac{c_j x^{j+i}}{j+i}\Bigr) + \sum_l c_{(l+1)p-i}\,\frac{x^{lp}}{y^p}\,x^{p-1}\, dx.$$

Note here that

$$0 \leqslant l \leqslant \frac{N+i}{p} - 1 = \frac{((p-1)/2)(2g+1)+i}{p} - 1 < g - \frac{1}{2}.$$

Thus we have

$$\mathscr{C}'(\omega_i) = \sum_{l=0}^{g-1} c^{1/p}_{(l+1)p-i}\,\frac{x^l}{y}\, dx.$$

This shows that $\mathfrak{D}_0(K)$ is closed under the modified Cartier operator $\mathscr{C}'$. Thus we can represent $\mathscr{C}'$ by a matrix. Indeed, if we write $\boldsymbol{\omega} = (\omega_1,\dots,\omega_g)$, we have

$$\mathscr{C}'(\boldsymbol{\omega}) = A^{(1/p)}\boldsymbol{\omega},$$

where A is the $(g \times g)$ matrix with elements in k given as

$$A = \begin{pmatrix} c_{p-1} & c_{p-2} & \cdots & c_{p-g} \\ c_{2p-1} & c_{2p-2} & \cdots & c_{2p-g} \\ & \cdots & & \\ c_{gp-1} & c_{gp-2} & \cdots & c_{gp-g} \end{pmatrix}.$$

(Correspondingly, under the Cartier operator $\mathscr{C}$, we have

$$\mathscr{C}(\omega_i) = \sum_{l=0}^{g-1} c_{(l+1)p-i}\,\frac{x^{(l+1)p}}{y^p}\,\frac{d^p x^p}{x^p}$$

and hence

$$\mathscr{C}(\boldsymbol{\omega}) = A\boldsymbol{\omega}^p.)$$

DEFINITION 2.2. The matrix A obtained above is called *the Cartier–Manin matrix* of the hyperelliptic curve C of genus g defined over k (with respect to the canonical basis $\boldsymbol{\omega}$ of $\mathfrak{D}_0(K)$). We denote it by $H(C, \boldsymbol{\omega})$.

PROPOSITION 2.2. *The Cartier–Manin matrix A of C is determined up to transformation of the form $S^{(p)}AS^{-1}$, where $S = (s_{ij})$, $s_{ij} \in k$ is a $(g \times g)$ non-singular matrix and $S^{(p)} = (s_{ij}^p)$, independently of the choice of the basis of $\mathfrak{D}_0(K)$.*

Proof. Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_g)$ be any system of the first kind of differential forms of degree 1 on K. Then there exists a $(g \times g)$ nonsingular matrix $S = (s_{ij})$ with elements in k such that

$$\theta_i = \sum_{j=1}^{g} s_{ij}\omega_j \qquad (i = 1, \ldots, g),$$

and there is a commutative diagram

$$\begin{array}{ccc} \boldsymbol{\omega} = (\omega_1, \ldots, \omega_g) & \xrightarrow{\mathscr{C}} & H(C, \boldsymbol{\omega})\boldsymbol{\omega}^p \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle S^{(p)}} \\ \boldsymbol{\theta} = (\theta_1, \ldots, \theta_g) & \xrightarrow{\mathscr{C}} & H(C, \boldsymbol{\theta})\boldsymbol{\theta}^p \end{array}$$

Hence A is transformed to $S^{(p)} AS^{-1}$. This shows that A is determined up to transformation of the form $S^{(p)} AS^{-1}$ independently of the choice of the basis of $\mathfrak{D}_0(K)$. Q.E.D.

THEOREM 2.1. *Assume that k is algebraically closed. Let $A = H(C, \boldsymbol{\omega})$ be the Cartier–Manin matrix of the hyperelliptic curve C over k of genus g, with respect to the canonical basis $\boldsymbol{\omega}$ of $\mathfrak{D}_0(K)$ given as* (3). *Denote by $\mathbf{a} = {}^t(a_1, \ldots, a_g)$ a g-column vector with elements in k and let us put*

$$H = \{\mathbf{a}\boldsymbol{\omega} \in \mathfrak{D}_0(K) \mid A\, A^{(p)} \cdots A^{(p^{g-1})}\mathbf{a}^{p^g} = \mathbf{0}\}$$

and

$$G = \{\mathbf{a}\boldsymbol{\omega} \in \mathfrak{D}_0(K) \mid A\, \mathbf{a}^p = \mathbf{a}\}.$$

Suppose that the matrix $A\ A^{(p)} \cdots A^{(p^{g-1})}$ has rank r. *Then H is a k-vector subspace of $\mathfrak{D}_0(K)$ of dimension $g - r$ and G generates a k-vector subspace $[G]$ of dimension r. Moreover, $\mathfrak{D}_0(K)$ is isomorphic to a direct sum of H and $[G]$.*

Proof. Let us denote by

$$M = \{\mathbf{a} = {}^t(a_1, \ldots, a_g),\ a_i \in k \text{ for every } i\}$$

the set of all g-column vectors with elements in k. Then M becomes a $k[\mathscr{C}]$-module of rank g over k by defining the operation $\mathscr{C}\mathbf{a} = A\, a^p$ and $\mathscr{C}\alpha = \alpha^p\mathscr{C}$ for $\alpha \in k$. Put

$$M_1 = \{\mathbf{a} \in M \mid \mathscr{C}^g\mathbf{a} = A\, A^{(p)} \cdots A^{(p^{g-1})}\mathbf{a}^{p^g} = \mathbf{0}\}$$

and

$$M_2 = \{\mathbf{a} \in M \mid \mathscr{C}\mathbf{a} = A\, a^p = \mathbf{a}\}.$$

Suppose now that the matrix $A\ A^{(p)} \cdots A^{(p^{g-1})}$ has rank r. Then it is easy to see that M_1 is a $k[\mathscr{C}]$-submodule of M of rank $g - r$ over k. While M_2 itself is not a k-module (because $\mathscr{C}(\alpha\mathbf{a}) = A(\alpha\mathbf{a})^p = \alpha^p A\ a^p = \alpha^p\ \mathbf{a} \neq \alpha\mathbf{a}$ for $\alpha \in k$), but it generates a $k[\mathscr{C}]$-submodule $[M_2]$ of M of rank t, say over k. So there exists a system of k-basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_t\}$ of $[M_2]$ which consists of the solutions of the equation $\mathscr{C}\mathbf{a} = A\ \mathbf{a}^p = \mathbf{a}$. Now an element $\sum_{i=1}^t \alpha_i\mathbf{a}_i \in [M_2]$ is the solution of the equation $\mathscr{C}\mathbf{a} = \mathbf{a}$, if and only if $\mathscr{C}(\sum_{i=1}^t \alpha_i\mathbf{a}_i) = \sum_{i=1}^t \alpha_i{}^p\mathbf{a}_i = \sum_{i=1}^t \alpha_i\mathbf{a}_i$, if and only if α_i are elements of the prime field $\mathbb{F}_p$ of characteristic $p > 0$. Therefore there are p^t solutions for $A\ \mathbf{a}^p = \mathbf{a}$ in M and we have

$$[M_2] = \left\langle \sum_{i=1}^{t} \alpha_i\mathbf{a}_i \mid \alpha_i \in \mathbb{F}_p \text{ and } \mathbf{a}_i \in M_2 \right\rangle.$$

It is easy to see that $M_1 \cap [M_2] = \{0\}$ and $M \supseteq M_1 \oplus [M_2]$.

Now we claim that $t = r =$ the rank of the matrix $A\ A^{(p)} \cdots A^{(p^{g-1})}$, whence $M = M_1 \oplus [M_2]$. For this take an arbitrary element $\mathbf{a}_0$ of M and let $k[\mathscr{C}]\mathbf{a}_0$ be the principal module generated over k by $\mathbf{a}_0, \mathscr{C}\mathbf{a}_0, \mathscr{C}^2\mathbf{a}_0, \ldots$. $k[\mathscr{C}]\mathbf{a}_0$ is finite dimensional over k, say of rank g_0, where g_0 is the degree of the minimal polynomial $P(X)$ of $\mathscr{C}$ over k:

$$P(\mathscr{C}) = \beta_{g_0}\mathscr{C}^{g_0} + \cdots + \beta_i\mathscr{C}^i + \cdots + \beta_0 = 0, \qquad \beta_i \in k.$$

Then $\mathbf{a}_0, \mathscr{C}\mathbf{a}_0, \ldots, \mathscr{C}^{g_0-1}\mathbf{a}_0$ constitute a system of k-basis for $k[\mathscr{C}]\mathbf{a}_0$, with $g_0 \leqslant g$.

Now we put $M_1{}^0 = \{\mathbf{b} \in k[\mathscr{C}]\mathbf{a}_0 \mid \mathscr{C}^{g_0}\mathbf{b} = \mathbf{0}\}$. Then $M_1{}^0$ is a $k[\mathscr{C}]$-submodule of $k[\mathscr{C}]\mathbf{a}_0$ of finite rank, say t_0 over k. Denote by $[M_2{}^0]$ the $k[\mathscr{C}]$-submodule of $k[\mathscr{C}]\mathbf{a}_0$ generated by the solutions of the equations $\mathscr{C}\mathbf{b} = \mathbf{b}$ in $k[\mathscr{C}]\mathbf{a}_0$, with finite rank, say s_0 over k. Then we have $g_0 \geqslant t_0 + s_0$.

Suppose now that β_{n_0} is the coefficient of $P(X)$ such that $\beta_{n_0} \neq 0$ for n_0 the smallest index with this property. Put

$$\phi_i(\lambda) = \beta_i\lambda + \beta_{i-1}^p\lambda^p + \cdots + \beta_0^{p^i}\lambda^{p^i}, \qquad i = 0, \ldots, g_0.$$

Then, k being algebraically closed, we see that

$$\phi_{g_0}(\lambda) = \beta_{g_0}\lambda + \beta_{g_0-1}^p\lambda^p + \cdots + \beta_{n_0}^{p^{g_0-n_0}}\lambda^{p^{g_0-n_0}}$$

has $p^{g_0-n_0}$ solutions in k. While, by noting that $\mathscr{C}\beta_i = \beta_i{}^p\mathscr{C}$, we have

$$(1 - \mathscr{C})\left(\sum_{i=0}^{g_0-1} \phi_i(\lambda)\,\mathscr{C}^i\right) + \phi_{g_0}(\lambda)\,\mathscr{C}^{g_0} = \lambda P(\mathscr{C}) = 0.$$

Hence we see that $(1 - \mathscr{C})\mathbf{b} = \mathbf{0}$, i.e., $\mathscr{C}\mathbf{b} = \mathbf{b}$ has $p^{g_0-n_0}$ solutions in $k[\mathscr{C}]\mathbf{a}_0$.

This implies that $s_0 \geqslant g_0 - n_0$. This together with the inequality $g_0 \geqslant s_0 + t_0$ gives $n_0 \geqslant t_0$. On the other hand, we have

$$0 = P(\mathscr{C}) = \mathscr{C}^{n_0} Q(\mathscr{C}) \qquad \text{with} \quad Q(\mathscr{C}) = \sum_{i=0}^{g_0 - n_0} \beta_{n_0+i}^{1/p^{n_0}} \mathscr{C}^i.$$

Then $Q(\mathscr{C})\mathbf{a}_0$, $\mathscr{C}Q(\mathscr{C})\mathbf{a}_0, \ldots, \mathscr{C}^{n_0-1}Q(\mathscr{C})\mathbf{a}_0$ are linearly independent elements of $M_1{}^0$. So $t_0 \geqslant n_0$. Therefore $t_0 = n_0$ and $k[\mathscr{C}]\mathbf{a}_0 = M_1{}^0 \oplus [M_2{}^0]$.

$\mathbf{a}_0$ being an arbitrary element of M and M_1 and $[M_2]$ being $k[\mathscr{C}]$-modules, the assertion $t = r$ follows from

$$\mathbf{a}_0 \in k[\mathscr{C}]\mathbf{a}_0 = M_1{}^0 \oplus [M_2{}^0] \subseteq M_1 \oplus [M_2].$$

The assertions of the theorem are immediately derived from the above discussion. In fact, H (resp. G) is canonically isomorphic as a group to M_1 (resp. M_2) and H becomes a k-vector subspace of $\mathfrak{D}_0(K)$ of dimension $g - r$, while G generates a k-vector subspace $[G]$ of $\mathfrak{D}_0(K)$ of dimension r. Q.E.D.

THEOREM 2.2. *Assume that k is algebraically closed. Let G and r be as in Theorem 2.1. Then G is canonically isomorphic to the group of classes of divisors of order p of K. In other words, the number of divisor classes of order p of K is precisely p^r.*

Proof. By Artin–Schreier theory, a cyclic extension of K of degree p can be obtained by adjoining a root $\mathscr{P}^{-1}z$ of the polynomial $\mathscr{P}X - z = 0$, $z \in K$ and $\mathscr{P}X = X^p - X$. Put $Z = K(\mathscr{P}^{-1}z)$. Then Z is unramified over K, if and only if Z is unramified at every place P of K, if and only if $z \in \mathscr{P} K_P$ for every P, where K_P denotes the completion of K at P, if and only if $z \in \mathscr{P}k((u_P))$ for every P, where $k((u_P))$ is the power series field over K in a local parameter u_P, if and only if $z \in U/\mathscr{P}K$ where $U = \bigcap_P (\mathscr{P}K_P \cap K)$ (note that $z \in \mathscr{P}K \Leftrightarrow Z = K$). Furthermore, we have the following lemmas.

LEMMA A. *Let $z \in U/\mathscr{P}K$ be as above. Then*

$$z \in \prod_{j=1}^{g} \left(\mathscr{L}\left(p \sum_{i=1}^{g} P_i\right) \cap \mathscr{P}K_{P_j}\right)\Big/k,$$

where $\{P_1, \ldots, P_g\}$ is a set of distinct k-rational points on C such that the divisor $\sum_{i=1}^g P_i$ is nonspecial and $\mathscr{L}(p \sum_{i=1}^g P_i)$ is the k-vector space of functions $0 \neq \xi \in K$ such that the divisor $(\xi) \geqslant -p \sum_{i=1}^g P_i$.

Proof of Lemma A. There exists a nonspecial system of points P_i, $i = 1, \ldots, g$, on C, corresponding to the first kind differentials ω_i, $i = 1, \ldots, g$, in K in the following way. Let $0 \neq \omega_1 \in \mathfrak{D}_0(K)$ and P_1 be a point which is not a zero of ω_1. Now the Riemann–Roch theorem says that the space of the first kind differentials having zero at P_1 has dimension $g - 1$. Let $0 \neq \omega_2 \in \mathfrak{D}_0(K)$ be in it and let P_2 be a point which is not a zero of ω_2. Continuing this process g times to get g points $P_1, \ldots, P_g$ with the index of speciality $i(\sum_{i=1}^{g} P_i) = l(\sum_{i=1}^{g} P_i) - d(\sum_{i=1}^{g} P_i) - g + 1 = 1 - g + g - 1 = 0$ where $l =$ dimension and $d =$ degree of $\mathscr{L}\left(\sum_{i=1}^{g}\right) P_i)$.

Now if an element $z \in K$ is integral at $P \neq P_i$, $i = 1, \ldots, g$, then a root α of the polynomial $f(X) = X^p - X - z = 0$ is integral at the place P' over P in $Z = K(\alpha)$ (because $\nu_{P'}(\alpha) \geqslant 0$ if and only if $\nu_P(\mathrm{Norm}_{Z/K}(\alpha)) = \nu_P(-z) \geqslant 0)$. So $\{1, \alpha, \ldots, \alpha^{p-1}\}$ is an integral basis of Z at P. Moreover, α is unramified at P, since the differential exponent is $\nu_{P'}(f'(\alpha)) = \nu_{P'}(-1) = 0$. This shows that

$$\mathscr{L}\left(p \sum_{i=1}^{g} P_i\right) \cap \mathscr{P}K_{P_j} \subseteq U \qquad \text{for} \quad i = 1, \ldots, g.$$

If $z \in \mathscr{L}(p \sum_{i=1}^{g} P_i) \cap \mathscr{P}K$, then there is $X \in K$ such that $z = X^p - X$. Hence X is integral for all $P \neq P_i$, $i = 1, \ldots, g$ and at P_i, X has a pole of order at most 1. Thus X is constant and so is z. So we have the injection

$$\prod_{j=1}^{g} \left(\mathscr{L}\left(p \sum_{i=1}^{g} P_i\right) \cap \mathscr{P}K_{P_j}\right) \Big/ k \rightarrowtail U/\mathscr{P}K.$$

Finally, we want to show that for a given $z \in U$, there exist $(z_1, \ldots, z_g)$, $z_j \in \mathscr{L}(p \sum_{i=1}^{g} P_i) \cap \mathscr{P}K_{P_j}$ such that $z \equiv (z_1, \ldots, z_g) \pmod{\mathscr{P}K}$. Let $z \in U = \bigcap_P (\mathscr{P}K_P \cap K)$. Suppose that z is not integral at $P \neq P_i$, $i = 1, \ldots, g$, then z has a pole at P of order pm with some positive integer $m \geqslant 1$ and z has a power series expansion by a local parameter u_P as $z \equiv (a/u_P^{pm}) \pmod{1/u_P^{pm-1}}$ with $a \in k$. Now by applying the Riemann–Roch theorem, there exists $w_1 \in \mathscr{L}(mP + \sum_{i=1}^{g} P_i)$ (whose dimension is $1 + m$) such that $w_1 \equiv (a^{1/p}/u_P{}^m) \pmod{1/u_P^{m-1}}$. Hence we see that $z \equiv \mathscr{P}w_1 \pmod{1/u_P^{pm-1}}$ and $z - \mathscr{P}w_1$ has a pole of smaller order than that of z at P. Repeating this procedure, we may assume, without loss of generality, that z has poles only at P_i, $i = 1, \ldots, g$. Hence $z \in \mathscr{P}K_{P_j} \cap K$, $j = 1, \ldots, g$. Now we must show that $z \in \mathscr{L}\ (p \sum_{i=1}^{g} P_i)$. Since $z \in \mathscr{P}K_{P_j} \cap K$, z has an expansion of the form by the local parameter u_{P_j} at $P_j : z \equiv (a_j/u_{P_j}^{pm_j}) \pmod{1/u_{P_j}^{pm_j-1}}$ with some integer $m_j \geqslant 1$ and $a_j \in k$. If $m_j = 1$ for every $j = 1, \ldots, g$, then $z \in \mathscr{L}(p \sum_{i=1}^{g} P_i)$. If $m_j > 1$, again by the Riemann–Roch theorem, there exists $w_j \in \mathscr{L}(m_j P_j)$ such that $w_j \equiv (a_j^{1/p}/u_{P_j}^{m_j}) \pmod{1/u_{P_j}^{m_j-1}}$. Hence $z - \mathscr{P}w_j$ has a pole of smaller order than that of z at P_j. Continuing this process, we finally get $z_j \in \mathscr{L}\ (p \sum_{i=1}^{g} P_i) \cap \mathscr{P}K_{P_j}$ for each $j = 1, \ldots, g$ with the required property and hence $z \in \mathscr{L}\ (p \sum_{i=1}^{g} P_i)$. Q.E.D.

An immediate consequence of Lemma A is that we have

$$z \equiv \frac{b_i{}^p}{u_{P_i}^p} - \frac{b_i}{u_{P_i}} \pmod{u_{P_i}^0} \qquad \text{with} \quad b_i \in k \quad \text{for} \quad i = 1, \ldots, g.$$

If we write $\mathbf{u} = (u_{P_1}, \ldots, u_{P_g})$ and $\mathbf{b} = (b_1, \ldots, b_g)$, we have

$$z \equiv \frac{\mathbf{b}^p}{\mathbf{u}^p} - \frac{\mathbf{b}}{\mathbf{u}} \pmod{u^0}.$$

LEMMA B. [2, Staz 4]. *Let P_i, $i = 1, \ldots, g$ be a nonspecial system of points on C and u_{P_i} a local parameter at P_i (taken as same as in Lemma* A). *Then there exist functions $v_j \in \mathscr{L}(p \sum_{i=1}^{g} P_i)$, $j = 1, \ldots, g$ such that*

$$v_j \equiv \frac{e_{ij}}{u_{P_i}^p} - \frac{d_{ij}}{u_{P_i}} \pmod{u_{P_i}^0},$$

where $e_{ij} = 1$ if $i = j$ and 0 otherwise and $d_{ij} \in k$. If we write $\mathfrak{v} = (v_1, \ldots, v_g)$, $I = (e_{ij})$, and $D = (d_{ij})$, we have

$$\mathfrak{v} \equiv \frac{I}{\mathbf{u}^p} - \frac{D}{\mathbf{u}} \pmod{u^0}.$$

DEFINITION 2.3. The matrix D obtained in Lemma B is called *the Hasse–Witt matrix* of the hyperelliptic curve C (cf. [2]).

LEMMA C [2, Hauptstaz]. *Let z be as in Lemma* A *and $\mathfrak{v}$ as in Lemma* B. *Then z* (mod k) *is in one-to-one correspondence with the vectors $\mathbf{b} = (b_1, \ldots, b_g)$, $b_i \in k$ for all i, satisfying $D\,\mathbf{b}^p = \mathbf{b}$ modulo multiplication by elements in the prime field of characteristic $p > 0$.*

LEMMA D. *The Hasse–Witt matrix D obtained in Lemma B is identified with the Cartier–Manin matrix A. Moreover, the group*

$$\{\mathbf{b} = (b_1, \ldots, b_g),\ b_i \in k \text{ for all } i \mid D\mathbf{b}^p = \mathbf{b}\}$$

is canonically isomorphic to G in Theorem 2.1.

Proof of Lemma D. Let $\mathfrak{A}$ be the space of adeles $\xi = (\cdots \xi_P \cdots)$ in K. For a divisor X in K, we denote by $\mathfrak{A}(X)$ the k-vector space $\{\xi \in \mathfrak{A} \mid \nu_P(\xi) \geqslant -\nu_P(X)$ for all $P\}$. Then we see that $\dim_k(\mathfrak{A}/(\mathfrak{A}(X) + K)) =$ the index of speciality of X. In particular, take $X = \sum_{i=1}^{k} P_i$: the nonspecial divisor. Then $\mathfrak{A} = \mathfrak{A}(\sum_{i=1}^{k} P_i) + K$ and the factor space $\mathfrak{A}/(\mathfrak{A}(0) + K)$ is generated by the adeles $\xi_i = (\cdots 1/u_{P_i} \cdots)$ and $(\xi_1, \ldots, \xi_g)$ is the canonical basis for $\mathfrak{A}/(\mathfrak{A}(0) + K)$. The k-vector spaces $\mathfrak{D}_0(K)$ and $\mathfrak{A}/(\mathfrak{A}(0) + K)$ are dual and there is a pairing between them given in the following fashion. Let W be the canonical divisor. Then there is a sequence of k-vector spaces:

$$\mathscr{L}\left(W - \sum_{i=1}^{g} P_i\right) \subseteq \mathscr{L}\left(W - \sum_{i=1}^{g-1} P_i\right) \subseteq \cdots \subseteq \mathscr{L}\left(W - \sum_{i=1}^{j} P_i\right)$$

$$\subseteq \mathscr{L}\left(W - \sum_{i=1}^{j-1} P_i\right) \subseteq \cdots \subseteq \mathscr{L}(W - P_1) \subseteq \mathscr{L}(W)$$

$$\text{with} \quad l\left(W - \sum_{i=1}^{j-1} P_i\right) - l\left(W - \sum_{i=1}^{j} P_i\right) = 1.$$

Hence it follows from the choice of P_i and from the Riemann–Roch Theorem that

$$\omega_j \in \Omega\left(\sum_{i=1}^{j-1} P_i\right) \setminus \Omega\left(\sum_{i=1}^{j} P_i\right) \qquad \text{for each} \quad 1 \leqslant j \leqslant g,$$

where $\Omega(X) = \{\omega \in \Omega^1(K) | (\omega) \geqslant X\}$ and that $(\omega_1, ..., \omega_g)$ is a dual basis to $(\xi_1, ..., \xi_g)$.

Now let S be the matrix of scalars $(\omega_i, \xi_j) =:$ the residue of $\omega_i \xi_j$ at P_j. We may take S to be the $(g \times g)$ identity matrix by identifying the local parameters u_{P_i} with x^i/y for $i = 1, ..., g$ (note that x^i/y, $i = 1, ..., g$ can be local parameters, since $\omega_i = (x^{i-1}/y)\, dx$, $i = 1, ..., g$ are linearly independent). Hence we get

$$((\omega_i, \xi_j^p)) = ((\mathscr{C}\omega_i, \xi_j)^p) = A.$$

While we have for the functions v_j, $j = 1, ..., g$ in Lemma B,

$$(\mathbf{0}) = ((\omega_i, v_j)) = ((\omega_i, \xi_j^p)) - D((\omega_i, \xi_j)) = A - D.$$

Hence $A = D$ and the group $\{\mathbf{b} = (b_1, ..., b_g),\ b_i \in k$ for all i satisfying $D\, \mathbf{b}^p = \mathbf{b}\}$ is canonically isomorphic to G. Q.E.D.

LEMMA E. *The number of classes of divisors of order p of K is precisely p^r where r is the rank of the matrix $A\, A^{(p)} \cdots A^{(p^{g-1})}$.*

Proof of Lemma E. As an immediate consequence of Lemma D and of Theorem 2.1, we know that there are p^r solutions for the system of equations $D\, \mathbf{b}^p = \mathbf{b}$ in k. Hence there are p^r divisor classes of order p of K. Q.E.D.

This completes the proof of Theorem 2.2. Q.E.D.

COROLLARY 2.3. *The notations and the hypothesis being as in Theorems* 2.1 *and* 2.2, *we have*

(a) *The following statements are equivalent*:

(ai) $r = g$.

(aii) $|A\, A^{(p)} \cdots A^{(p^{g-1})}| \neq 0$.

(aiii) *A has rank g.*

(aiv) $\mathfrak{D}_0(K)$ *does not posess any exact differentials.*

(b) *All differentials of* $\mathfrak{D}_0(K)$ *are exact, if and only if* $A = (0)$. *When this is the case,* $A\ A^{(p)} \cdots A^{(p^{g-1})}$ *has rank 0 and there are no classes of divisors of order p of K*

Proof. (a) (ai) $\Leftrightarrow$ (aii) $\Leftrightarrow$ (aiii) are clear, since determinant is multiplicative. (ai) $\Leftrightarrow$ (aiv). Suppose (ai), then $\mathfrak{D}_0(K) = [G]$ and $\mathscr{C}\boldsymbol{\theta} = \boldsymbol{\theta}$ for every $\boldsymbol{\theta} \in \mathfrak{D}_0(K)$, whence (aiv). The converse is clear.

(b) The equivalence follows from the definition of A and from Theorem 2.1. The last assertion is a trivial consequence of Theorem 2.2. Q.E.D.

3. Ordinary Jacobian Variety $J(C)$ of C

From here on, let k be a finite field of characteristic $p > 2$ *with* p^a $(a \geq 1)$ *elements and* $\bar{k}$ *its algebraic closure.*

Let C be the hyperelliptic curve defined over k by the equation (1) and $J(C)$ its Jacobian variety. We may assume that $J(C)$ and its canonical embedding $C \to J(C)$ are also defined over k. Let π be the Frobenius endomorphism of $J(C)$ relative to k with the characteristic polynomial $P_\pi(\lambda) \in \mathbb{Z}[\lambda]$ of degree $2g$. $P_\pi(\lambda) = \sum_{i=0}^{2g} a_i\lambda^i$, $a_0 = p^{ag}$, $a_{2g} = 1$. $P_\pi(\lambda)$ is the characteristic polynomial of the l-adic and also of the p-adic representation of the Frobenius endomorphism π and it is of special interest, because (1) it determines the isogeny class of $J(C)$ [13] and (2) the p-adic values of its characteristic roots determine the formal structure of $J(C)$ up to isogeny [6]. Thus $P_\pi(\lambda)$ determines the formal and algebraic structure of $J(C)$ up to isogeny.

Henceforth, there remains the main task of determining $P_\pi(\lambda)$ explicitly. Its dependence on the Cartier–Manin matrix A of C has been illuminated by Manin [7]. That is, $P_\pi(\lambda)$ is linked to the Cartier–Manin matrix through the congruence

$$P_\pi(\lambda) \equiv (-1)^g \lambda^g \mid A_\pi - \lambda I_g \mid \qquad (\mathrm{mod}\ p), \tag{4}$$

where $\mid A_\pi - \lambda I_g \mid$ is the characteristic polynomial of the matrix $A_\pi = A\ A^{(p)} \cdots A^{(p^{a-1})}$ and I_g is the $(g \times g)$ identity matrix.

Theorem 3.1. *Let C be the hyperelliptic curve of genus g defined by* (1) *over k: a finite field of* p^a $(a \geqslant 1)$ *elements,* $p > 2$ *and* $J(C)$ *its Jacobian variety defined over k. Let* π *be the Frobenius endomorphism of* $J(C)$ *relative to k and* $P_\pi(\lambda)$ *its characteristic polynomial. Then the following statements are equivalent:*

(i) $\mid A_\pi \mid \neq 0$.

(ii) *A has rank g, i.e.,* $\mid A \mid \neq 0$.

(iii) *$A\,A^{(p)} \cdots A^{(p^{g-1})}$ has rank g.*

(iv) *The p-rank of $J(C)$ is g, that is, there are p^g points on $J(C)$ killed by p in $\bar{k}$.*

(v) *$P_\pi(\lambda)$ has g p-adic unit roots in the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.*

(vi) *The Newton polygon $\mathfrak{N}(P_\pi)$ has the segments S_1, S_2 with slopes 0 and $-a$, respectively, and looks like Fig. 1.*

(vii) *The p-divisible group $J(p)$ of $J(C)$ is isogenous to $gG_{1,0}$.*

(viii) *The formal group Γ of $J(C)$ has height g and is isogenous to $G_m(p)^g$ where $G_m(p)$ denotes the multiplicative group of height 1 and of dimension 1.*

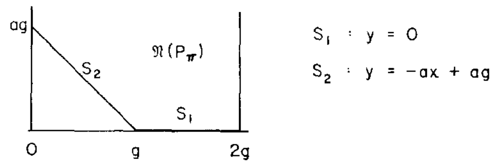


FIGURE 1

DEFINITION 3.1. When $J(C)$ satisfies any one of the conditions in Theorem 3.1, $J(C)$ is called *ordinary*.

Remarks. (1) By the Newton polygon $\mathfrak{N}(P_\pi)$ of $P_\pi(\lambda) = \sum_{i=0}^{2g} a_i\lambda^i \in \mathbb{Z}[\lambda]$, we mean the lower convex envelope of the set of points $\{(i, v_p(a_i)) \mid i = 0,\ldots, 2g\} \subset \mathbb{R} \times \mathbb{R}$ where v_p is the p-adic valuation of $\mathbb{Q}_p$. (2) We denote by ν_p the unique extension of the p-adic valuation v_p to the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, normalized so that $\nu_p(p) = 1$. (3) The formal group Γ of $J(C)$ is the connected component of the p-divisible group $J(p)$ of $J(C)$.

Proof of Theorem 3.1. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (v) $\Leftrightarrow$ (vi) are obvious.

(iii) $\Leftrightarrow$ (iv). Since the classes of divisors of order p of K correspond to the points on $J(C)(\bar{k})$ of order p, (iii) $\Rightarrow$ (iv) follows from Corollary 2.3a and (iv) $\Rightarrow$ (iii) from Theorem 2.1 and 2.2.

(i) $\Leftrightarrow$ (v). By the Manin congruence (4), $a_g \equiv (-1)^g \mid A_\pi \mid \pmod{p}$. Now assume (i). Then $v_p(a_g) = 0$. Noting also that $v_p(a_{2g}) = 0$, the Newton polygon $\mathfrak{N}(P_\pi)$ has a segment S_1 of length g and with slope 0. Therefore $P_\pi(\lambda)$ has exactly g p-adic unit roots in $\bar{\mathbb{Q}}_p$, whence the assertion (v). Conversely, assume (v) and let $\tau_1,\ldots, \tau_g$ be the p-adic unit roots of $P_\pi(\lambda)$. As $P_\pi(\lambda)$ has always together with roots τ_i, the roots p^a/τ_i, we have

$$P_\pi(\lambda) = \prod_{i=1}^{g} (\lambda - \tau_i)(\lambda - p^a/\tau_i), \quad \nu_p(\tau_i) = 0 \qquad \text{for all} \quad i = 1,\ldots, g.$$

So $v_p(a_g) = \sum_{i=1}^{g} \nu_p(\tau_i) = 0$. Hence again by the congruence (4), we get $\mid A_\pi \mid \not\equiv 0 \pmod{p}$. This proves (v) $\Rightarrow$ (i).

(vii) $\Leftrightarrow$ (viii). Assume (vii). The p-divisible group $G_{1,0}$ is isogenous to $G_m(p) \times (\mathbb{Q}_p/\mathbb{Z}_p)_k$ where $G_m(p)$ is the multiplicative group of height 1 and $(\mathbb{Q}_p/\mathbb{Z}_p)_k$ is the étale group of height 1. Hence $J(p) \sim gG_{1,0} = G_m(p)^g \times (\mathbb{Q}_p/\mathbb{Z}_p)_k^g$. The assertion (viii) follows from the facts that the connected component of $J(p)$ is the formal group of $J(C)$ and $G_m(p)^g$ is connected of height g. The converse (viii) $\Rightarrow$ (vii) is easy, because if $J(p)$ has the component $G_m(p)^g$, $J(p)$ also has its dual $(\mathbb{Q}_p/\mathbb{Z}_p)_k^g$ as its component.

(v) $\Leftrightarrow$ (vii). First (v) $\Rightarrow$ (vii) is the Manin fundamental theorem 4.1 in [6]. To show the converse, we consider the Dieudonné module $T_p(J) = T_p(G_m(p)^g) \oplus T_p((\mathbb{Q}_p/\mathbb{Z}_p)_k^g)$ corresponding to the p-divisible group $J(p)$. Since $P_\pi(\lambda)$ is the characteristic polynomial of the p-adic representation $T_p(\pi)$ of the Frobenius endomorphism π of $J(C)$ on $T_p(J)$, we may write $P_\pi(\lambda) = P_a(\lambda)\, P_0(\lambda)$ where $P_a(\lambda)$ (resp. $P_0(\lambda)$) is the characteristic polynomial of the restriction of $T_p(\pi)$ to $T_p(G_m(p)^g)$ (resp. to $T_p((\mathbb{Q}_p/\mathbb{Z}_p)_k^g)$). Both $P_a(\lambda)$ and $P_0(\lambda)$ have the same degree g. Moreover, we have

$$P_0(\lambda) = \prod_{i=1}^{g} (\lambda - \tau_i), \quad \nu_p(\tau_i) = 0 \qquad \text{for all} \quad i = 1,\ldots, g.$$

In fact, $(\mathbb{Q}_p/\mathbb{Z}_p)_k^g$ being étale, $T_p(\pi)$ induces an automorphism of $T_p((\mathbb{Q}_p/\mathbb{Z}_p))_k^g)$ and hence all the characteristic roots of $P_0(\lambda)$ must have the p-adic value 0.
Q.E.D.

THEOREM 3.2. *With the notation as in Theorem* 3.1, *suppose that $J(C)$ is elementary and ordinary. Then we have*

(a) *$P_\pi(\lambda)$ is $\mathbb{Q}$-irreducible.*

(b) *The endomorphism algebra $\mathscr{A} = \mathrm{End}_k(J(C)) \otimes Q$ is commutative and coincides with its center $\Phi = \mathbb{Q}(\pi)$.*

(c) *$\Phi = \mathbb{Q}(\pi)$ is a CM-field of degree $2g$. Let $\beta = \pi + \bar{\pi}$ where $\bar{\pi}$ denotes the complex conjugate of π. Then β is totally real and $[\mathbb{Q}(\pi) : \mathbb{Q}] = g$ and $|\beta| < 2p^{a/2}$, $(\beta, p) = 1$, and $P_\pi(\lambda) = \lambda^2 - \beta\lambda + p^a \in \mathbb{Q}(\beta)[\lambda]$.*

(d) *$J(C)$ is k-simple.*

Proof. It is well known that if $J(C)$ is elementary, $P_\pi(\lambda) = P(\lambda)^e$ for some integer e with $P(\lambda)$ $\mathbb{Q}$-irreducible and $P(\pi) = 0$ and that $\mathscr{A}$ is a division algebra of dimension e^2 over its center $\Phi = \mathbb{Q}(\pi)$.

Now suppose that $J(C)$ is elementary and ordinary. Then by Theorem 3.1, $P_\pi(\lambda)$ has the p-adic decomposition

$$P_\pi(\lambda) = \prod_{i=1}^{g} (\lambda - \tau_i)(\lambda - p^a/\tau_i), \qquad \nu_p(\tau_i) = 0 \qquad \text{for every} \quad 1 \leqslant i \leqslant g.$$

Hence at every prime ν over p in Φ, $\operatorname{ord}_\nu(\pi) = 0$ or a. Thus the local invariant i_ν of $\mathscr{A}$ at ν (defined by Tate [13] as $i_\nu = \operatorname{ord}_\nu(\pi) \cdot f_\nu/a$ where f_ν is the residue degree at ν) is an integer for every ν over p. Noting that there are no real primes, (because if π is real, $\pi = \pm p^{a/2}$ and $\operatorname{ord}_\nu(\pi) = a/2$), we see that the least common denominator of all the i_ν is 1. Since e is the period of $\mathscr{A}$ in the Brauer group of Φ and so is the least common denominator of all the i_ν, we get $e = 1$, whence the assertions (a), (b), and (d).

Now we shall prove (c). Since π is imaginary with $\deg(\pi) = 2g$, $\mathbb{Q}(\pi)$ is a CM-field of degree $2g$. Put $\beta = \pi + \bar{\pi}$. In every embedding $\Phi = \mathbb{Q}(\pi)$ into $\mathbb{C}$, $|\pi| = p^{a/2}$ by the Riemann hypothesis, so $\beta = \pi + p^a/\pi$ is real and $\mathbb{Q}(\beta)$ becomes totally real with $[\mathbb{Q}(\beta) : \mathbb{Q}] = g$ and $\mathbb{Q}(\pi)$ becomes imaginary over it (i.e., π satisfies the equation $P_\pi(\pi) = \pi^2 - \beta\pi + p^a = 0$ over $\mathbb{Q}(\beta)$). As $J(C)$ is ordinary by the hypothesis, $P_\pi(\lambda)$ must split. Hence at every prime ν over p, we have $\operatorname{ord}_\nu(\beta) = 0$, whence $(\beta, p) = 1$. Q.E.D.

EXAMPLE 3.3. Consider the curve $C: y^2 = 1 - x^5$ defined over the prime field $\mathbb{F}_p$ where p is a prime of the form $10n + 1$, $n \in \mathbb{N}$. C has genus 2 and the Cartier–Manin matrix A of C is given by

$$A = \begin{pmatrix} \binom{(p-1)/2}{(p-1)/5} & 0 \\ 0 & \binom{(p-1)/2}{2(p-1)/5} \end{pmatrix} \quad \text{with } (:) \text{ binomial coefficient.}$$

It is easy to see that $|A| \neq 0$ in $\mathbb{F}_p$. So $J(C)$ is ordinary by Theorem 3.1. We have

$$P_\pi(\lambda) \equiv \lambda^4 - \left\{ \binom{(p-1)/2}{(p-1)/5} + \binom{(p-1)/2}{2(p-1)/5} \right\} \lambda^3 + |A| \lambda^2 \pmod{p}.$$

So $P_\pi(\lambda)$ must split with roots of orders 0 and 1. Hence half the places have $\operatorname{ord}_\nu(\pi) = 0$ and the other half have $\operatorname{ord}_\nu(\pi) = 1$. So i_ν is an integer for every prime ν over p, and hence $J(C)$ is simple over $\mathbb{F}_p$.

This is a rather special example (cf. Honda [3]). Let ζ be the endomorphism of $J(C)$ corresponding to the birational automorphism $(x, y) \to (\zeta x, y)$ of C. Put $L = \mathbb{Q}(\zeta)$. Then L is the decomposition field of $p = 10n + 1$ with $[L : \mathbb{Q}] = 4$ and moreover $L = \mathbb{Q}(\pi)$. Since $\mathscr{A}$ contains a field L of degree 4, $J(C)$ is isogenous to a product of a simple abelian variety. But p splits in Φ and the local invariants of $\mathscr{A}$ are all integers. Hence $\mathscr{A} = \Phi = L = \mathbb{Q}(\zeta)$. This shows that for all primes p of the form $10n + 1$, $n \in \mathbb{N}$, $J(C)$ are of the same CM-type (L) and hence are isogenous to each other.

4. The Jacobian Variety $J(C)$ of C with $|A| = 0$

Theorem 4.1. *With the notation as in Section* 3, *suppose that the Cartier–Manin matrix A of C has the determinant $|A| = 0$ in k. Then we have* (a) *If the matrix $A\,A^{(p)} \cdots A^{(p^{a-1})}$ has rank 0, then the matrix $A\,A^{(p)} \cdots A^{(p^{g-1})}$ also has rank 0.*

(b) *When* (a) *is the case, the following statements are equivalent*:

(bi) *The p-rank of $J(C)$ is 0, that is, there are no points on $J(C)$ defined over $\bar{k}$, killed by p.*

(bii) *The characteristic polynomial $P_\pi(\lambda)$ has the p-adic decomposition $P_\pi(\lambda) = \prod_{i=1}^{2g} (\lambda - \tau_i)$ with $0 < \nu_p(\tau_i) < a$.*

(biii) *The formal group Γ of $J(C)$ has height $2g$ and coincides with the p-divisible group $J(p)$ of $J(C)$.*

Proof. (a) Let $l \geqslant 1$ be an integer and let us denote by ρ_l the rank of the matrix $A_l = A\,A^{(p)} \cdots A^{(p^{l-1})}$, and $A_0 = I_g$.

Suppose now that $A_\pi = A\,A^{(p)} \cdots A^{(p^{a-1})}$ has rank 0. If $a \leqslant g$, there is nothing to prove. So we assume now that $a > g$. Let R_l be the k-vector space of the roots of the system of equations $\mathscr{C}^l\mathbf{x} = \mathbf{0}$ in $\bar{k}$, i.e., $R_l = \{\mathbf{x} \mid \mathscr{C}^l\mathbf{x} = A_l\mathbf{x}^{p^l} = \mathbf{0}\}$, $R_0 = \{0\}$ and $R_g = H$ (in Theorem 2.1). We know that the rank of R_l is $g - \rho_l$. First we shall prove the following lemma.

Lemma. *Put $\delta_l = \rho_{l-1} - \rho_l$. Then δ_l is the rank of the k-vector space R_l/R_{l-1} and*

$$\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_g \geqslant \delta_{g+1} = \cdots = \delta_n = 0 \qquad \textit{for any} \quad n \geqslant g + 1.$$

Proof of Lemma. It is easily seen that $R_l \supset R_{l-1}$ and $\delta_l = (g - \rho_l) - (g - \rho_{l-1})$ is the rank of the space R_l/R_{l-1}. Let $\mathbf{u}_1^{(g)},\ldots, \mathbf{u}_{\delta_g}^{(g)}$ be a basis of R_g/R_{g-1}. Applying the Cartier operator $\mathscr{C}$, we get

$$\mathscr{C}\mathbf{u}_1^{(g)},\ldots, \mathscr{C}\mathbf{u}_{\delta_g}^{(g)} \in R_{g-1},$$

and modulo R_{g-2}, they are linearly independent. Hence we get the inequality $\delta_g + g - \rho_{g-2} \leqslant g - \rho_{g-1}$, whence $\delta_{g-1} \geqslant \delta_g$. Continuing the same discussion, we have $\delta_1 \geqslant \delta_2 \geqslant \cdots \geqslant \delta_g$. It remains to show that $\delta_g \geqslant \delta_{g+1} = \cdots = \delta_n = 0$. But this is an immediate consequence of Theorem 2.1, because $R_g = R_n$ for every $n \geqslant g + 1$. Q.E.D.

Now we shall prove the theorem. The assertion (a) follows immediately from the lemma. In fact, take $n = a$, then $\rho_a = 0$ by the hypothesis and $\rho_g = \rho_{g+1} = \cdots = \rho_a = 0$.

(b) We shall prove (a) $\Rightarrow$ (bi) $\Rightarrow$ (bii) $\Rightarrow$ (biii) $\Rightarrow$ (bi).

(a) $\Rightarrow$ (bi). See Corollary 2.3 b.

(bi) $\Rightarrow$ (bii). We first note that the p-rank of $J(C)$ coincides with the rank of the toroidal component $G_{1,0}$ of $J(p)$. As we have seen in the proof of Theorem 3.1(v) $\Leftrightarrow$ (vii), the characteristic roots of $P_\pi(\lambda)$ corresponding to the toroidal component have the p-adic values 0 and a. Now assume (bi). Then (bii) follows from the above fact and from the Riemann hypothesis that all the characteristic roots must have the absolute value $p^{a/2}$.

(bii) $\Rightarrow$ (biii). Assume (bii). Then by the Manin theorem 4.1 in [6], the p-divisible group $J(p)$ of $J(C)$ has no toroidal component. So $J(p)$ is connected. Hence the formal group Γ of $J(C)$ has height $2g$ and coincides with the p-divisible group $J(p)$.

(biii) $\Rightarrow$ (bi). This is a trivial consequence of the fact that the p-rank of $J(C)$ is equal to the rank of the toroidal component of $J(p)$. Q.E.D.

Remarks 4.2. (1) The Cartier–Manin matrix A of C in Theorem 4.1 provides us merely a connected p-divisible group of height $2g$. So in order to determine the local structure of $J(C)$ up to isogeny, we must classify the connected p-divisible groups of height $2g$ into isogeny classes. Manin [6] is the first to observe that the local decomposition of $J(C)$ parallels the p-adic factorization of the characteristic polynomial $P_\pi(\lambda)$ of π.

(2) Let $2s$ (resp. r) be the number of the p-adic roots τ_i of $P_\pi(\lambda)$ with $\nu_p(\tau_i) = a/2$ (resp. 0). Then we can factor $P_\pi(\lambda)$ into the form

$$P_\pi(\lambda)$$

$$= \prod_{\substack{i=1\\ \nu_p(\tau_i)=a/2}}^{2s} (\lambda - \tau_i) \cdot \prod_{\substack{i=1\\ \nu_p(\tau_i)=0}}^{r} (\lambda - \tau_i)(\lambda - p^a/\tau_i) \cdot \prod_{\substack{i=1\\ 0<\nu_p(\tau_i)<a/2}}^{g-s-r} (\lambda - \tau_i)(\lambda - p^a/\tau_i).$$

(Note that $J(p)$ is connected, if and only if $r = 0$.)

In the forthcoming sections, we shall determine, up to isogeny, the type of the formal group Γ, of the p-divisible group $J(p)$ and then the algebraic structure of $J(C)$ up to isogeny, in the cases, $[s = g, r = 0]$, $[s = 0, r = 0]$, and $[0 < s < g, 0 < r < g]$, respectively.

(3) In principle, the characteristic polynomial $P_\pi(\lambda)$ can be explicitly determined by making use of the well-known Lefsechtz formulas for the hyperelliptic curve C over k (cf. [5]).

5. Supersingular Jacobian Variety $J(C)$ of C

Theorem 5.1. *Suppose that the Cartier–Manin matrix A of C has the determinant $|A| = 0$ in k and that the matrix A_π has rank 0. Then we have*

(a) *The following statements are equivalent:*

(ai) *$s = g$, i.e, all the characteristic roots of $P_\pi(\lambda)$ have the p-adic value $a/2$.*

(aii) *The Newton polygon $\mathfrak{N}(P_\pi)$ has only one nonvertical segment with slope $-a/2$ and looks like Fig. 2.*

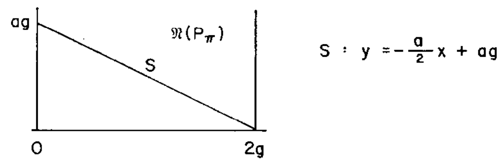


Figure 2

(b) *When* (a) *is the case, the p-divisible group $J(p)$ of $J(C)$ is isogenous to $gG_{1,1}$ and so is the formal group Γ of $J(C)$.*

(c) *The following statements are equivalent:*

(ci) *The p-divisible group $J(p)$ of $J(C)$ has the isogeny type $gG_{1,1}$.*

(cii) *The Newton polygon of the characteristic polynomial of π^n for some integer $n \geqslant 1$ has only one nonvertical segment with slope $-an/2$.*

Definition 5.1. When $J(C)$ has the p-divisible group $J(p)$ isogenous to $gG_{1,1}$, $J(C)$ is called *supersingular*.

Proof of Theorem 5.1. (a) (ai) $\Rightarrow$ (aii). By the hypothesis,

$$P_\pi(\lambda) = \prod_{\substack{i=1 \\ v_p(\tau_i)=a/2}}^{2g} (\lambda - \tau_i) = \sum_{i=0}^{2g} a_i\lambda^i.$$

So we have $v_p(a_i) = (2g - i)\, a/2$ for every $0 \leqslant i \leqslant 2g$. Hence the equation for the nonvertical segment of $\mathfrak{N}(P_\pi)$ is given by $y = -(a/2)x + ag$.

(aii) $\Rightarrow$ (ai). Clear.

(b) This follows from the Manin theorem 4.1 in [6], and Theorem 4.1b.

(c) First note that over any finite extension k_n of k of degree $n \geqslant 1$, there exists an Abelian variety B_n of dimension g whose all the characteristic roots of the Frobenius endomorphism relative to k_n have the p-adic value $an/2$. (For example, $B_n = E^g$ where E is an elliptic curve with vanishing Hasse invariant.) Then by Manin's Theorem 4.1 in [6], B_n has the p-divisible group

$B_n(p)$ isogenous to $gG_{1,1}$. There is a one-to-one correspondence due to Tate (see Waterhouse [15]) and to Manin:

$$\mathrm{Hom}_{k_n}(J(C), B_n) \otimes \mathbb{Z}_p \leftrightarrow \mathrm{Hom}_{k_n}(J(p), B_n(p)).$$

(ci) $\Rightarrow$ (cii) Now suppose (ci). Then there exists an element $\phi(p) \in \mathrm{Hom}_{k_n}(J(p), B_n(p)) \subseteq \mathrm{Hom}_{\bar{k}}(gG_{1,1}, gG_{1,1}) = \mathrm{End}_{\bar{k}}(gG_{1,1}) \simeq M_g(\mathrm{End}_{\bar{k}}(G_{1,1}))$ (M_g denotes the $(g \times g)$ matrix algebra.) By the above correspondence, we get the element $\phi \in \mathrm{Hom}_{k_n}(J(C), B_n)$. Hence the characteristic polynomial of $\pi^n_{J(C)} = \pi^n$ coincides with that of the Frobenius endomorphism π_{B_n} of B_n relative to k_n. Therefore the p-adic exponents of the eigenvalues of π^n are $an/2$. Thus (cii) follows from by applying the argument (ai) $\Rightarrow$ (aii) with π^n for π.

(cii) $\Rightarrow$ (ci). Suppose (cii). Then there are $2g$ characteristic roots with p-adic value $an/2$. Hence by applying the Manin Theorem 4.1 in [6] with k_n for k and π^n for π, the p-divisible group $J(p)$ of $J(C)$ is isogenous to $gG_{1,1}$.

Q.E.D.

THEOREM 5.2. *A supersingular Jacobian variety $J(C)$ of C over k is isogenous over some finite extension of k to a product $E \times \cdots \times E$ (g copies) of a supersingular elliptic curve E (cf. Oort* [9]).

Proof. Recall that an elliptic curve is called supersingular if its endomorphism algebra is noncommutative. We employ the same notation as in Theorem 3.2: $\mathscr{A}$ the endomorphism algebra of $J(C)$ and $\Phi = \mathbb{Q}(\pi)$ the center of $\mathscr{A}$. The algebraic integer π satisfying the Riemann hypothesis $|\pi| = p^{a/2}$ in all embedding of Φ into $\mathbb{C}$, are called the Weil numbers. As the notation suggests, we may identify the Frobenius endomorphism with a Weil number.

Now by the assumption, all the characteristic roots of $P_\pi(\lambda)$ have the p-adic value $a/2$.

I. Suppose that there are real primes in Φ.

Case Ia. If a is even, $\pi = \pm p^{a/2}$ is rational. Hence $\Phi = \mathbb{Q}$, $P_\pi(\lambda) = (\lambda \pm p^{a/2})^{2g}$, $[\mathscr{A} : \mathbb{Q}] = (2g)^2$, and $\mathscr{A} = M_g(Q_{p,\infty}) : a$ $(g \times g)$ matrix algebra over the quaternion algebra $Q_{p,\infty}$ over $\mathbb{Q}$ which is ramified only at p and ∞. Then by Tate [13], $J(C)$ is isogenous over k to g copies of a supersingular elliptic curve over k, all of whose endomorphisms are defined over k and whose characteristic polynomial is $(\lambda \pm p^{a/2})^2$.

Case Ib. If a is odd, $\pi = \pm p^{a/2} \notin \mathbb{Q}$, but π^2 becomes rational. We have $\Phi = \mathbb{Q}(p^{1/2})$, $[\Phi : \mathbb{Q}] = 2$. So there are two infinite primes with local invariants $\frac{1}{2}$, and only one prime over p with local invariant 0. Thus the least common denominator of all the local invariants is 2. Hence we obtain a k-simple constituent X of $J(C)$ with $\dim X = \frac{1}{2} \cdot 2 \cdot \deg(\pi) = 2$. Passing to the quadratic extension k_2 of k, we have $\mathbb{Q}(\pi^2) = \mathbb{Q}$ and X becomes isogenous to the product of a supersingular elliptic curve. Hence by applying the same argument as in Case Ia, the algebra $\mathscr{A}^{(2)}$ attached to $J(C)$ relative to k_2 becomes a matrix algebra over $Q_{p,\infty}$ and the characteristic polynomial of π^2 is given by $P_{\pi^2}(\lambda) = (\lambda - p^a)^{2g}$. Hence $J(C)$ is isogenous over k_2 to g copies of a supersingular elliptic curve over k_2.

II. Suppose now that there are no real primes in Φ. So $\mathbb{Q}(\pi)$ is totally imaginary. Put $\beta = \pi + p^a/\pi$. Then β is real and $\mathbb{Q}(\beta)$ becomes totally real and $\mathbb{Q}(\pi)$ is imaginary quadratic over it. We can write $P_\pi(\lambda) = \lambda^2 - \beta\lambda + p^a \in \mathbb{Q}(\beta)[\lambda]$ with $|\beta| < 2p^{a/2}$. The solution of $P_\pi(\lambda) = 0$ is a Weil number. Now the hypothesis that all the characteristic roots of $P_\pi(\lambda) = 0$ have the p-adic value $a/2$ implies that $(\beta, p) \neq 1$ and hence p ramifies or stays prime in $\mathbb{Q}(\beta)$. Write $\beta = \pm p^b\alpha$ with $b \in \mathbb{Q}$ and $\alpha = 0$ or an algebraic integer satisfying $(\mathrm{Norm}(\alpha), p) = 1$.

Case IIa. If $\alpha = 0$, then $\beta = 0$ and $\mathbb{Q}(\beta) = \mathbb{Q}$, $\mathbb{Q}(\pi) = \mathbb{Q}((-p^a)^{1/2})$ with $[\mathbb{Q}(\pi) : \mathbb{Q}] = 2$. Hence we get Weil numbers $\pi = \pm p^{a/2} \cdot \sqrt{-1}$, whose second powers become rational. So if a is odd or a is even and $p \not\equiv 1 \pmod 4$, they give supersingular elliptic curves whose all endomorphisms are not defined over k, but are defined over k_2. Hence the characteristic polynomial of π^2 is given by $P_{\pi^2}(\lambda) = (\lambda + p^a)^{2g}$, and hence $J(C)$ is isogenous over k_2 to g copies of a supersingular elliptic curve over k_2.

Case IIb. If $\alpha \neq 0$ and $2b < a$, then we have $\pi = \pm(p^b\alpha \pm p^b(\alpha^2 - 4p^{a-2b})^{1/2})/2$. Since $\alpha^2 - 4p^{a-2b} \equiv \alpha^2 \pmod{4p}$, we have $\nu_p(\pi) = b < a/2$. But this contradicts to our hypothesis. So we can suppose that $2b \geqslant a$. As $\beta^2 - 4p^a = p^a(p^{2b-a}\alpha^2 - 4) < 0$ and $p \neq 2$, we must have $|p^{b-a/2}| < 2$. So it follows that $\pi = \pm p^{a/2}(p^{b-a/2}\ \alpha \pm i\,|p^{2b-a}\alpha^2 - 4|^{1/2})/2$ with $\mathrm{Norm}((p^{b-a/2}\alpha \pm i\,|p^{2b-a}\alpha^2 - 4|^{1/2})/2) = 1$. Hence $\nu_p(\pi) = a/2$. Since $|p^{b-a/2}\alpha| < 2$, we have $|p^{b-a/2}\alpha/2| < 1$ and $|p^{2b-a}\alpha^2 - 4|^{1/2}/2 < 1$. Hence $(p^{b-a/2}\alpha \pm i|p^{2b-a}\alpha^2 - 4|^{1/2})/2$ is a root of unity. Therefore some powers of π becomes rational, say $\pi^t = \pm p^{ta/2} \in \mathbb{Q}$. So if a is even (resp. odd), the characteristic polynomial of π^t (resp. π^{2t}) is given by $P_{\pi^t}(\lambda) = (\lambda \pm p^{ta/2})^{2g}$ (resp. $P_{\pi^{2t}}(\lambda) = (\lambda \pm p^{at})^{2g}$), whence $J(C)$ is isogenous over the extension k_t of degree t (resp. k_{2t} of degree $2t$) of k to g copies of a supersingular elliptic curve over k_t (resp. k_{2t}).

A typical example of Case IIb is when the characteristic polynomial $P_\pi(\lambda)$ of π of $J(C)$ relative to k is given by $P_\pi(\lambda) = \lambda^{2g} + p^{ag}$. Q.E.D.

It is a classical result that an elliptic curve E over k is supersingular if and only if the Hasse invariant of E is zero. In the following, we shall give a generalization of this fact to higher-dimensional cases.

THEOREM 5.3. *Suppose that the Cartier–Manin matrix A of C is* (0) *in k. Then $J(C)$ is supersingular and is isogenous over some finite extension of k to g copies of a supersingular elliptic curve.*

Proof. $A = (0)$ certainly satisfies the hypothesis of Theorem 4.1(a), so that $J(p)$ has no toroidal components $\Leftrightarrow$ $J(C)$ has no p-torsion points $\Leftrightarrow$ The Tate group of the dual of $J(C)$ is 0. First we shall prove the following two lemmas.

LEMMA A. *Let F be the Frobenius morphism of $K = k(x, y)$ onto $K^p = k^p(x^p, y^p)$, $J(C)$ onto $J(C)^{(p)}$ and $J(p)$ onto $J(p)^{(p)}$ induced by the pth power map $a \to a^p$ of k and $F' = V$ its dual morphism. Then for the canonical basis $\boldsymbol{\omega} = (\omega_1, \dots, \omega_g)$ of $\mathfrak{D}_0(K)$ $(\simeq \mathfrak{D}_0(J(C)))$ given as* (3), *we have*

$$\mathscr{C}'\boldsymbol{\omega} = \boldsymbol{\omega} \circ V = A^{(1/p)}\boldsymbol{\omega}, \qquad \mathscr{C}\boldsymbol{\omega} = \boldsymbol{\omega} \circ F = A\boldsymbol{\omega}^p.$$

Proof of Lemma A. Let $\mathfrak{O}$ be the ring of integers in the absolutely unramified extension L of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_{p^a}$. So p generates the maximal ideal of $\mathfrak{O}$. We can lift the equation for C to L, which we write $\tilde{C} : y^2 = \tilde{f}(x)$ where $\tilde{f}$ is a polynomial over $\mathfrak{O}$ without multiple roots of degree $2g + 1$ such that $\tilde{C} \bmod p = C$. Let $t_i = x^i/y$, $i = 1, \dots, g$ and $\mathbf{t} = (t_1, \dots, t_g)$. As we have seen in Section 2, $\mathbf{t}$ is a system of local parameters of C at the origin and the canonical basis ω_i, $i = 1, \dots, g$ of $\mathfrak{D}_0(K)$ can be written as

$$\omega_i = d\phi_i + \sum_{l=1}^{g} c_{lp-i} t_l{}^p \frac{dx}{x}, \qquad \phi_i \in K.$$

Now the differential forms of degree 1 and of the first kind on the algebraic function field of $\tilde{C}$ can have the form

$$\tilde{\omega}_i = d\tilde{\phi}_i + \sum_{l=1}^{g} \tilde{c}_{lp-i} t_l{}^p \frac{dx}{x} \qquad \text{with} \quad \tilde{\omega}_i \bmod p = \omega_i \quad \text{for} \quad i = 1, \dots, g.$$

Let $\tilde{\mathbf{\Gamma}} = (\tilde{\Gamma}_i)$, $i = 1, \dots, g$ be the formal group of $J(\tilde{C})$ with respect to the local parameters $\mathbf{t}$, so that $\tilde{\mathbf{\Gamma}} \bmod p = \mathbf{\Gamma}$. We consider the isogeny of $\mathbf{\Gamma}$ (resp. $\tilde{\mathbf{\Gamma}}$) of multiplication by p. On $\tilde{\mathbf{\Gamma}} = (\tilde{\Gamma}_i)$ over $\mathfrak{O}$, there exist systems of power series $\tilde{\mathbf{U}}(\mathbf{t}) = (\tilde{U}_i(\mathbf{t}))$, $\tilde{\mathbf{W}}(\mathbf{t}) = (W_i(\mathbf{t}))$ in $\mathfrak{O}[[\mathbf{t}]]$ such that

$$\tilde{\mathbf{W}}(\mathbf{t}) = \mathbf{1} + \cdots,$$
$$\mathbf{t} \circ (p1_{\tilde{\Gamma}}) = p\tilde{\mathbf{W}}(\mathbf{t}) + \tilde{\mathbf{U}}(\mathbf{t}^p).$$

So by reducing modulo p, we get

$$\mathbf{t} \circ (p1_{\Gamma}) = \mathbf{U}(\mathbf{t}^p) = (U_i(\mathbf{t}^p)), \qquad \text{where} \quad \mathbf{U} = \tilde{\mathbf{U}} \bmod p.$$

Now we know that in characteristic $p > 0$, the multiplication by p can be expressed as the product of F and V taken in either order: $p1_{\Gamma} = FV = VF$ (cf. Manin [6, Proposition 1.4]). So we have

$$\mathbf{t} \circ F = \mathbf{t}^p, \qquad \mathbf{t} \circ V = \mathbf{t} \circ (p1_{\Gamma}/F) = (\mathbf{b}_1\mathbf{t}, \dots, \mathbf{b}_g\mathbf{t}),$$

where $\mathbf{b}_i\mathbf{t} = \sum_{l=1}^{g} b_{li}t_l$ with b_{li} the coefficient of $t_l{}^p$ in $U_i(\mathbf{t}^p)$. Expanding $\tilde{\omega}_i$ into power series of $\mathbf{t} = (t_1,\ldots,t_g)$, we have

$$\tilde{\omega}_i = \sum_{l=1}^{g} dt_l(\tilde{a}_{li} + \cdots + \tilde{c}_{lp-i}t_l^{p-1} + \cdots) = \sum_{l=1}^{g} \tilde{h}_{li}(t_l)\, dt_l\,,$$

where $\tilde{a}_{li} \equiv 1 \pmod p$ for all i, l. So it follows that

$$\tilde{\omega}_i \circ (p1_{\tilde{\Gamma}}) = p\tilde{\omega}_i = \sum_{l=1}^{g} p\tilde{h}_{li}(t_l)\, dt_l\,.$$

On the other hand, we also have

$$\tilde{\omega}_i \circ (p1_{\tilde{\Gamma}}) = \sum_{l=1}^{g} \tilde{h}_{li}(\tilde{U}_i(t_l{}^p) + p\tilde{W}_i(t_l)) \cdot (\tilde{U}_i'(t_l{}^p)\, pt_l^{p-1} + p\tilde{W}_i'(t_l))\, dt_l\,.$$

Hence we get the equality

$$\sum_{l=1}^{g} \tilde{h}_{li}(t_l)\, dt_l = \sum_{l=1}^{g} \tilde{h}_{li}(\tilde{U}_i(t_l{}^p) + p\tilde{W}_i(t_l)) \cdot (\tilde{U}_i'(t_l{}^p)\, t_l^{p-1} + \tilde{W}_i(t_l))\, dt_l\,.$$

Read it modulo p and compare the coefficients of t_l^{p-1} of both sides for each $l = 1,\ldots, g$. Since

$$\tilde{h}_{li}(\tilde{U}_i(t_l{}^p) + p\tilde{W}_i(t_l)) \equiv \tilde{h}_{li}(\tilde{U}_i(t_l{}^p)) \equiv 1 \qquad \pmod p,$$

and

$$U_i'(t_l{}^p) = b_{li}\,,$$

we get $c_{lp-i} = b_{li}$ for i, $l = 1,\ldots, g$. This proves that

$$\mathscr{C}'\omega_i = \sum_{l=1}^{g} c_{lp-i}^{1/p}\omega_l = \omega_i \circ V.$$

By duality, we also get

$$\mathscr{C}\omega_i = \sum_{l=1}^{g} c_{lp-i}\omega_l{}^p = \omega_i \circ F. \qquad \text{Q.E.D.}$$

LEMMA B. *The hypothesis and the notation are as in Theorem* 5.3 *and Lemma* A. *Then* $p1_{J(C)} = p1_\Gamma$, F *and* V *are purely inseparable and moreover, we have*

$$F^2 = V^2 = -p1_{J(C)}\,.$$

Proof of Lemma B. Since $J(C)$ has no points of order p in $\bar{k}$, $p1_{J(C)}$ is purely inseparable of degree p^{2g} (cf. [12, Chap. I, Proposition 7]). According to Serre [11], every purely inseparable isogeny is the product of elementary isogenies of height 1, of one of two types i_1, i_2 defined as follows. Let $\mathfrak{N}$ be the p-Lie algebra of differentiations of $J(C)$. The isogeny of type i_1 corresponds to the subspace $\{\partial \in \mathfrak{N} \mid \partial^p = 0\}$ of $\mathfrak{N}$ and that of type i_2 to the subspace $\{\partial \in \mathfrak{N} \mid \partial^p = \partial\}$ of $\mathfrak{N}$. The dual (or the transpose) of type i_1 is again of type i_1 and has kernel 0, while that of type i_2 becomes separable and has kernel of order p. Since the Cartier–Manin matrix A of C is the matrix of the map $\partial \to \partial^p$ in $\mathfrak{N}$, $A = (0)$ implies that V is the g product of the isogenies of type i_1. So it is purely inseparable of degree p^g. It follows that F is also the g product of the dual of the isogenies of type i_1. Hence F is also purely inseparable of degree p^g. Therefore, F^2, V^2, and $p1_{J(C)}$ are purely inseparable of degree p^{2g} and they differ only by an automorphism. Let σ be an automorphism of K (modulo translation automorphism). σ has the form: $x^\sigma = \epsilon x$, $y^\sigma = \eta y$ where ϵ, η roots of unity (cf. [10]). It has the matrix representation $M(\sigma)$ of degree g with respect to the canonical basis ω_i, $i = 1,\dots, g$ of $\mathfrak{D}_0(K)$:

$$(\omega_i^\sigma,\dots, \omega_g^\sigma) = M(\sigma)(\omega_1,\dots, \omega_g).$$

$M(\sigma)$ can be put into the form

$$\begin{pmatrix} \epsilon_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon_g \end{pmatrix},$$

where ϵ_i roots of unity. In particular, the hyperelliptic automorphism is represented by the matrix

$$\begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}.$$

Now if $A = (0)$, then ω_i, $i = 1,\dots, g$ are given by

$$\omega_i = d\left(y^{-p} \sum_{j+i \not\equiv 0 \pmod p} c_j \frac{x^{j+i}}{j+i}\right), \qquad 0 \leqslant j \leqslant \frac{p-1}{2}(2g+1).$$

Under the automorphism σ, ω_i is transformed to

$$\omega_i^\sigma = d\left(y^{-p}\eta^{-p} \sum_{j+i \not\equiv 0 \pmod p} \epsilon^{j+i} c_j \frac{x^{j+i}}{j+i}\right).$$

But the identity $\omega_i^\sigma = \epsilon_i \omega_i$ for $i = 1,\dots, g$ must hold. Thus the only possibility is when $\eta = \pm 1$ and $\epsilon = 1$, whence $\epsilon_i = \pm 1$ for every i. Thus all the nontrivial automorphisms have order 2. Hence we have $F^2 = -p1_{J(C)}$ and $V^2 = p^2/F^2 = -p1_{J(C)}$. Q.E.D.

The end of the proof of Theorem 5.3. $\pi = F^a$, $\pi' = p^a/F^a$ are purely inseparable isogenies of $J(C)$. The characteristic polynomial of π^2 is given by $P_{\pi^2}(\lambda) = (\lambda + p^a)^{2g}$. Hence $\nu_p(\pi) = a/2$ and $J(p) \sim gG_{1,1}$. Thus $J(C)$ is supersingular by Theorem 5.1. Q.E.D.

EXAMPLE 5.4. $A = (0)$ is a sufficient condition for $J(C)$ to be supersingular, but it is not a necessary one. We shall illustrate some examples that $J(C)$ with $A \neq (0)$ becomes supersingular.

Let C be the hyperelliptic curve of genus 3 with the equation $y^2 = 1 - x^7$ defined over the prime field $\mathbb{F}_p$ of characteristic $p > 2$. The Cartier–Manin matrix A of C is given by $A = (c_{m,n})_{m,n=1,2,3}$, where

$$c_{m,n} = \binom{(p-1)/2}{(mp-n)/7} \cdot (-1)^{(mp-n)/7} \qquad \text{with} \quad c_{m,n} = 0 \quad \text{if} \quad 7 \nmid mp - n.$$

Let ζ be a primitive seventh root of unity and put $L = \mathbb{Q}(\zeta)$. So $[L : \mathbb{Q}] = 6$. Now for any prime $p \neq 7$, there exists the smallest positive integer f such that $p^f \equiv 1 \pmod 7$ and $fr' = 6$ where r' is the degree (over $\mathbb{Q}$) of the decomposition field K_0 of p.

Case I. If $p \equiv 3$ or $5 \pmod 7$, then $p^6 \equiv 1 \pmod 7$, so $f = 6$, $r' = 1$. Hence p stays prime in L. For primes $p \equiv 3 \pmod 7$, the Cartier–Manin matrix A of C has the form

$$c_{1,3} = \binom{(p-1)/2}{(p-3)/7} \cdot (-1)^{(p-3)/7},$$

$$c_{3,2} = \binom{(p-1)/2}{(3p-2)/7} \cdot (-1)^{(3p-2)/7}, \qquad \text{and} \qquad c_{m,n} = 0, \text{ otherwise.}$$

For primes $p \equiv 5 \pmod 7$, the Cartier–Manin matrix A of C has the form

$$c_{2,3} = \binom{(p-1)/2}{(2p-3)/7} \cdot (-1)^{(2p-3)/7},$$

$$c_{3,1} = \binom{(p-1)/2}{(3p-1)/7} \cdot (-1)^{(3p-1)/7}, \qquad \text{and} \qquad c_{m,n} = 0, \text{ otherwise.}$$

In both cases, $|A| = 0$ and $A \neq (0)$, $A\,A^{(p)} \neq (0)$, but $A\,A^{(p)}A^{(p^2)} = (0)$.

Case II. If $p \equiv 6 \pmod 7$, we have $p^2 \equiv 1 \pmod 7$, so $f = 2$, $r' = 3$. Hence p decomposes in the real cubic field $K_0 = Q(\zeta + \zeta^{-1})$. In this case, the Cartier–Manin matrix A of C is $A = (0)$.

Now let π be the pth power endomorphism of $J(C)$ relative to $\mathbb{F}_p$. Then $\pi^f \in K_0$ and the characteristic polynomial of π^f is given as follows:

$$\begin{aligned} P_{\pi^f}(\lambda) &= (\lambda + p^3)^6 \qquad \text{if} \quad \text{Case I}, \\ &= (\lambda + p)^6 \qquad \text{if} \quad \text{Case II}. \end{aligned}$$

(Cf. Honda [3].) Hence $J(p)$ is isogenous to $3G_{1,1}$ in both cases. In Case I (resp. Case II), $J(C)$ is isogenous over the extension of $\mathbb{F}_p$ of degree 6 (resp. over the quadratic extension of $\mathbb{F}_p$) to 3 copies of a supersingular elliptic curve.

6. The Jacobian Variety $J(C)$ of C with the Symmetric Formal Group

Theorem 6.1. *Suppose that the Cartier–Manin matrix A of C has the determinant $|A| = 0$ in k and that the matrix $A_\pi = A\, A^{(p)} \cdots A^{(p^{a-1})}$ has rank 0. Then we have*

(a) *The following statements are equivalent:*

(ai) $s = 0$ *and* $P_\pi(\lambda) = \prod_{i=1}^{g} (\lambda - \tau_i)(\lambda - p^a/\tau_i)$ *with* τ_i *simple roots, and* $v_p(\tau_i) = ac$, $0 < c < \frac{1}{2}$ *for every* $1 \leqslant i \leqslant g$.

(aii) $P_\pi(\lambda) = \sum_{i=0}^{2g} a_i \lambda^i$ *is a distinguished polynomial over* $\mathbb{Z}_p$ *and the coefficients* a_i *satisfy the conditions:*

$$\operatorname*{Min}_{0 \leqslant i \leqslant 2g} \frac{v_p(a_i)}{a(2g - i)} = \frac{v_p(a_g)}{ag} = c = \frac{n_c}{n_c + m_c},$$

where n_c, m_c *are positive integers such that* $1 \leqslant n_c < m_c$, $(n_c, m_c) = 1$, *and* $n_c + m_c = g$.

(aiii) *The p-divisible group* $J(p)$ *of* $J(C)$ *is isogenous to* $G_{n_c, m_c} + G_{m_c, n_c}$ *where* n_c, m_c *are integers such that* $1 \leqslant n_c < m_c$, $(n_c, m_c) = 1$, *and* $n_c + m_c = g$ *and so is the formal group* Γ *of* $J(C)$.

(b) *When* (a) *is the case, the Newton polygon* $\mathfrak{N}(P_\pi)$ *of* $P_\pi(\lambda)$ *has two segments* S_1, S_2 *indexed from the right with slopes* $-ac$, $-a(1 - c)$, *respectively. The vertices of* $\mathfrak{N}(P_\pi)$ *are* $(2g, 0)$, $(g, v_p(a_g))$, *and* $(0, ag)$ *and it looks like Fig. 3.*

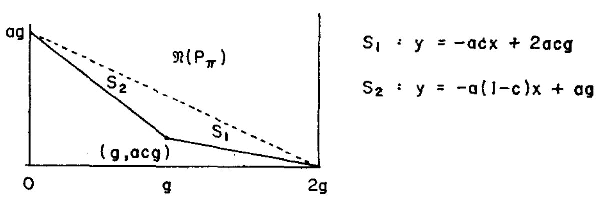


Figure 3

(c) *If the Newton polygon $\mathfrak{N}(P_\pi)$ has the shape as (Fig. 3), then the p-divisible group $J(p)$ of $J(C)$ is isogenous to $t(G_{n_c,m_c} + G_{m_c,n_c})$ where n_c, m_c are positive integers such that $1 \leqslant n_c < m_c$, $(n_c, m_c) = 1$, and $n_c + m_c = d =$: the number of distinct characteristic roots τ_i of $P_\pi(\lambda)$ with $\nu_p(\tau_i) = ac$, $0 < c < \frac{1}{2}$, and $td = g$. In other words, $\mathfrak{N}(P_\pi)$ determines the isotypic components of $J(p)$ (rather than its simple components).*

DEFINITION 6.1. The formal group of the type $G_{n,m} + G_{m,n}$ where n, m are positive integers such that $1 \leqslant n < m$, $(n, m) = 1$, and $n + m = g$ is called *the symmetric formal group of dimension g.*

Proof of Theorem 6.1. (a) (ai) $\Rightarrow$ (aii). Put $p^a/\tau_i = \tau_{g+i}$ for $i = 1,\ldots, g$. Then $\nu_p(\tau_i) = ac$, $\nu_p(\tau_{g+i}) = a(1-c)$ for every $1 \leqslant i \leqslant g$, from which we have immediately that $v_p(a_{2g}) = 0$, $v_p(a_{2g-i}) \geqslant aci$ for every $1 \leqslant i \leqslant g$, $v_p(a_g) = acg$, and $v_p(a_{g-i}) \geqslant acg + ia(1-c)$ for every $1 \leqslant i \leqslant g$. Hence it follows that

$$\frac{v_p(a_{2g-i})}{ai} \geqslant c, \qquad \frac{v_p(a_g)}{ag} = c, \qquad \text{and} \qquad \frac{v_p(a_{g-i})}{a(g+i)} \geqslant c.$$

Therefore, we get

$$\operatorname*{Min}_{0 \leqslant i \leqslant 2g} \frac{v_p(a_i)}{a(2g-i)} = \frac{v_p(a_g)}{ag} = c.$$

Now put $n_c = cg$ and $m_c = g - n_c = (1-c)g$. Then n_c, m_c are positive integers satisfying $1 \leqslant n_c < m_c$, $n_c + m_c = g$, $(n_c, m_c) = 1$, and $c = n_c/(n_c + m_c)$. (In fact, if $(n_c, m_c) \neq 1$, then $n_c = dn_c'$ with $n_c' = cg/d$. This implies that $P_\pi(\lambda)$ has g/d distinct roots with $\nu_p(\tau_i) = ac$, which contradicts to the hypothesis of (ai).)

(aii) $\Rightarrow$ (aiii). See Manin [6, Theorem 4.1′].

(aiii) $\Rightarrow$ (ai). Suppose that $P_\pi(\lambda)$ has no such decomposition as (ai). Then we have either

$$\operatorname{Min} \frac{v_p(a_i)}{a(2g-i)} = \frac{t_1}{t_2} \neq \frac{n_c}{n_c + m_c}$$

or

$$\operatorname{Min} \frac{v_p(a_i)}{a(2g-i)} = \frac{v_p(a_l)}{a(2g-l)} = \frac{n_c}{n_c + m_c} \qquad \text{for} \quad l > g.$$

In the first case, $J(p)$ is isogenous to the formal group of the type $G_{t_1,t_2-t_1} + G_{t_2-t_1,t_1}$ which is obviously nonisogenous to $G_{n_c,m_c} + G_{m_c,n_c}$. In the latter case, $J(p)$ is isogenous to $G_{n_c,m_c} + G_{m_c,n_c} + G'$ with dimension of $G' > 1$. But this is impossible, because $n_c + m_c + \dim G' > g$.

(b) The assertion follows immediately from the proof of (ai) $\Rightarrow$ (aii) and from the hypothesis $0 < c < \frac{1}{2}$.

(c) Corresponding to the segment S_1, we get g roots τ_i with $\nu_p(\tau_i) = ac$, $0 < c < \frac{1}{2}$ for every $1 \leqslant i \leqslant g$. If there are d distinct roots $\tau_1, \ldots, \tau_d$ among them, then $\prod_{i=1}^{d} (\lambda - \tau_i) \in \mathbb{Z}_p[\lambda]$ and $P_\pi(\lambda)$ has the p-adic decomposition as

$$P_\pi(\lambda) = \Bigg(\prod_{\substack{i=1 \\ \nu_p(\tau_i)=ac}}^{d} (\lambda - \tau_i)(\lambda - p^a/\tau_i) \Bigg)^{g/d}.$$

Hence $J(p)$ is isogenous to $t(G_{n_c', m_c'} + G_{m_c', n_c'})$ with $1 \leqslant n_c' < m_c'$, $(n_c', m_c') = 1$, and $m_c' + n_c' = d$. So $\mathfrak{N}(P_\pi)$ determine the isotypic component of $J(p)$. Q.E.D.

THEOREM 6.2. *Suppose that $J(C)$ is elementary and that the p-divisible group $J(p)$ of $J(C)$ is isogenous to the symmetric formal group of dimension g : $G_{n,m} + G_{m,n}$ $1 \leqslant n < m$, $(n, m) = 1$, and $n + m = g$. Then the following statements are equivalent:*

(i) *g divides the residue degree at every prime v in Φ lying over p.*

(ii) *$P_\pi(\lambda)$ is $\mathbb{Q}$-irreducible, but $P_\pi(\lambda) = P_{\nu_1}(\lambda)\, P_{\nu_2}(\lambda)$ where*

$$P_{\nu_1}(\lambda) = \prod_{\substack{i=1 \\ \nu_p(\tau_i)=an/g}}^{g} (\lambda - \tau_i) \text{ and } P_{\nu_2}(\lambda) = \prod_{\substack{i=1 \\ \nu_p(\tau_i)=am/g}}^{g} (\lambda - \tau_i) \text{ are } \mathbb{Q}_p\text{-irreducible.}$$

(iii) *$J(C)$ is k-simple.*

(iv) *$\mathscr{A} = \Phi = \mathbb{Q}(\pi)$ is a CM-field of degree $2g$. Φ has the imaginary quadratic field K_0 in which p splits.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). As $J(C)$ is elementary, $P_\pi(\lambda) = P(\lambda)^e$ with $P(\lambda)$ $\mathbb{Q}$-irreducible and $P(\pi) = 0$. Corresponding to the primes ν in $\Phi = \mathbb{Q}(\pi)$ over p, $P(\lambda)$ is decomposed into the product of $\mathbb{Q}_p$-irreducible factors $P_\nu(\lambda)$. Now we shall compute the local invariants of $\mathscr{A} = \mathrm{End}_k(J(C)) \otimes \mathbb{Q}$ at primes ν in $\Phi = \mathbb{Q}(\pi)$. First note that there are no real primes in Φ. Now by Manin [6], $P_\pi(\lambda)$ has the p-adic factorization in the ring $W(\bar{k})\,[p^{1/g}]$ where $W(\bar{k})$ denotes the ring of Witt vectors over $\bar{k}$, as

$$P_\pi(\lambda) = \prod_{i=1}^{g} (\lambda - p^{an/g} x_i) \cdot \prod_{i=1}^{g} (\lambda - p^{am/g} y_i),$$

where x_i, y_i are invertible elements in $W(\bar{k})\ [p^{1/g}]$. So $P_\nu(\lambda)$ splits in the ring $W(\bar{k})\,[p^{1/g}]$ into linear factors $(\lambda - p^{an/g} x_i)$, $(\lambda - p^{am/g} y_i)$. So the local invariants are

$$i_\nu = \mathrm{ord}_\nu(\pi) \cdot [\Phi_\nu : \mathbb{Q}_p]/a = \mathrm{ord}_\nu(\pi) \cdot f_\nu/a = \frac{(an/g) \cdot f_\nu}{a} \text{ or } \frac{(am/g) \cdot f_\nu}{a},$$

where f_ν is the residue degree at ν with $1 \leqslant f_\nu \leqslant g$. Hence $e = 1$, if and only if all the i_ν are integers, if and only if $P_{\nu_i}(\lambda)$, $i = 1, 2$ are $\mathbb{Q}_p$-irreducible, if and only if $f_{\nu_i} = g$ for $i = 1, 2$. This proves the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (iv). Since π is imaginary with $\deg(\pi) = 2g$, $\Phi = \mathbb{Q}(\pi)$ is a CM-field of degree $2g$. Corresponding to the p-adic decomposition (ii) of $P_\pi(\lambda)$, there are two valuations ν_1, ν_2 in Φ over p with $\operatorname{ord}_{\nu_1}(\pi) = an/g$ and $\operatorname{ord}_{\nu_2}(\pi) = am/g$. In other words, there are two prime ideals ν_1, ν_2 over p such that $(\pi^g) = \nu_1^{an}\nu_2^{am}$. Now the Riemann hypothesis $|\pi| = p^{a/2}$ implies that $(p) = \nu_1\nu_2$ and ν_1, ν_2 are complex conjugates. Since $f_{\nu_i} = g$ for $i = 1, 2$, p splits in an imaginary quadratic subfield K_0 of Φ, whence the assertion (iv).

(iv) $\Rightarrow$ (i). Suppose that the CM-field $\Phi = \mathbb{Q}(\pi)$ has an imaginary quadratic subfield K_0 in which p splits: $(p) = \nu\nu'$ where ν, ν' are complex conjugates. Take an ideal $\mathfrak{A}$ such that $\mathfrak{A}^g = \nu^{an}\nu'^{am}$ with $1 \leqslant n < m$, $(n, m) = 1$ and $n + m = g$. Then $\mathfrak{A}$ satisfies $\mathfrak{A}\mathfrak{A}' = (p^a)$ where $\mathfrak{A}'$ denotes the conjugate of $\mathfrak{A}$, and hence we can find an algebraic integer $\tau \in \Phi$ such that $(\tau) = \mathfrak{A}$ (cf. Honda [4]). Thus $(\tau^g) = \nu^{an}\nu'^{am}$ and $\operatorname{ord}_\nu(\pi) = an/g$, $\operatorname{ord}_{\nu'}(\pi) = am/g$, and we see that $P_\pi(\lambda)$ has g p-adic roots τ_i with order an/g together with g p-adic roots τ_i with order am/g. Hence the local invariants are $i_\nu \equiv (n/g) \cdot f_\nu$ and $(m/g) \cdot f_\nu \pmod{\mathbb{Z}}$. But the commutativity hypothesis of $\mathscr{A}$ implies that $i_\nu \equiv 0 \pmod{\mathbb{Z}}$. This holds true if and only if f_ν and $f_{\nu'}$ are divisible by g. Q.E.D.

Example 6.3. We again consider the curve $y^2 = 1 - x^7$ defined over the prime field $\mathbb{F}_p$ where p is a prime such that $p \equiv 2$ or $4 \pmod 7$. The Cartier–Manin matrix A of C is given by

$A = (c_{m,n})_{m,n=1,2,3}$ where for $p \equiv 2 \pmod 7$,

$$c_{1,2} = \binom{(p-1)/2}{(p-2)/7} \cdot (-1)^{(p-2)/7}, \; c_{m,n} = 0 \text{ otherwise},$$

and for $p \equiv 4 \pmod 7$,

$$c_{2,1} = \binom{(p-1)/2}{(2p-1)/7} \cdot (-1)^{(2p-1)/7}, \; c_{m,n} = 0 \text{ otherwise}.$$

So $|A| = 0$ and $A\,A^{(p)} = (0)$ in both cases.

Now it is easy to see that the primes $p \equiv 2$ or $4 \pmod 7$ satisfy $p^3 \equiv 1 \pmod 7$. So in the notations of Example 5.4, we have $f = 3$ and $r' = 2$. Hence p splits in the unique subfield $K_0 = \mathbb{Q}((-7)^{1/2})$ of $L = \mathbb{Q}(\zeta)$. Moreover, Honda [3] has shown that for any $s \geqslant 1$, $\mathbb{Q}(\pi^{3s}) = K_0$. Hence $2 \leqslant [\Phi : \mathbb{Q}] \leqslant 6$ and $[\mathscr{A} : \mathbb{Q}] \leqslant 3^2 \cdot 2$. As $\mathscr{A}$ contains the subfield $L = \mathbb{Q}(\zeta)$ of degree $2 \cdot 3$, $\mathscr{A}$ is a simple algebra over K_0. Now note that $K_0 = \mathbb{Q}((-7)^{1/2})$ has the basis $\{1, (1 + (-7)^{1/2})/2\}$.

So we have

$$\pi^3 = a_1 + a_2\left(\frac{1+(-7)^{1/2}}{2}\right), \qquad a_1, a_2 \in \mathbb{Z} \quad \text{with} \quad N(\pi^3) = p^3.$$

Hence the characteristic polynomial of π^3 is given by

$$P_{\pi^3}(\lambda) = (\lambda^2 - (2a_1 + a_2)\lambda + p^3)^3 =: Q(\lambda)^3,$$

where $Q(\lambda)$ is $\mathbb{Q}$-irreducible and $(2a_1 + a_2)^2 - 4p^3 = -7a_2^2 < 0$. Since p splits in K_0, the polynomial $Q(\lambda)$ must factor p-adicly, giving two primes ν_1, ν_2 with $\mathrm{ord}_{\nu_1}(\pi^3) = 1$ and $\mathrm{ord}_{\nu_2}(\pi^3) = 2$. Hence $\mathrm{ord}_{\nu_1}(\pi) = \frac{1}{3}$ and $\mathrm{ord}_{\nu_2}(\pi) = \frac{2}{3}$. Hence over some finite extension of $\mathbb{Q}_p$, $P_\pi(\lambda)$ has three roots τ_i with the order $\frac{1}{3}$ and hence we get $n_{1/3} = 1$, $m_{1/3} = 3 - 1 = 2$. So $J(p)$ is isogenous to $G_{1,2} + G_{2,1}$.

What is the algebraic structure of $J(C)$? First we know that there are no real primes in Φ. The local invariants are $i_\nu = 1, 2$ and hence $\mathscr{A}$ is commutative with $[\mathscr{A} : \mathbb{Q}] = 6$. Thus $J(C)$ is simple over $\mathbb{F}_p$.

7. The Jacobian Variety $J(C)$ of C with the Formal Structure of Mixed Types

THEOREM 7.1. *Suppose that the Cartier–Manin matrix A of C is such that $A \neq (0)$, but $|A| = 0$ in k. Let $|A_\pi - \lambda I_g| = \sum_{i=0}^{g} b_i\lambda^i$, $b_g = 1$ be the characteristic polynomial of $A_\pi = A\,A^{(p)} \cdots A^{(p^{a-1})}$. Then we have*

(a) *The following statements are equivalent*:

(ai) *There is an integer $1 < t < g$ such that $(b_t, p) = 1$ and $b_j \equiv 0 \pmod p$ for all $j = 0, \ldots, t-1$.*

(aii) *There exist the polynomials $P_0(\lambda)$, $P_a(\lambda)$, and $g(\lambda)$ over $\mathbb{Z}_p$ such that $P_0(\lambda) = \prod_{i=1}^{g-t}(\lambda - \tau_i)$, $P_a(\lambda) = \prod_{i=1}^{g-t}(\lambda - p^a/\tau_i)$ with $\nu_p(\tau_i) = 0$ for every $1 \leqslant i \leqslant g - t$, $g(\lambda) \equiv \lambda^{2t} \pmod p$ and that $P_\pi(\lambda) = P_0(\lambda)\,P_a(\lambda)\,g(\lambda)$.*

(aiii) *The p-divisible group $J(p)$ of $J(C)$ has the component $(g-t)G_{1,0}$. The formal group Γ of $J(C)$ has height $g + t$.*

(aiv) *The p-rank of $J(C)$ is $g - t$.*

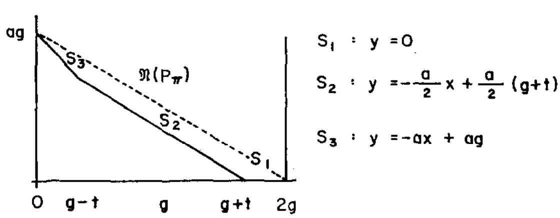


FIGURE 4

(b) *Assume that* (a) *is true*; *then the following statements are equivalent*:

(bi) $g(\lambda) = \prod_{i=1}^{2t} (\lambda - \tau_i)$ *with* $v_p(\tau_i) = a/2$ *for every* $1 \leqslant i \leqslant 2t$.

(bii) *The Newton polygon* $\mathfrak{N}(P_\pi)$ *of* $P_\pi(\lambda)$ *has the shape of Fig.* 4.

When the above is true, the p-divisible group $J(p)$ *of* $J(C)$ *is isogenous to* $(g - t)G_{1,0} + tG_{1,1}$ *and* $\mathbf{\Gamma}$ *to* $G_m(p)^{g-t} + tG_{1,1}$.

(c) *Assume that* (a) *is true*; *then the following statements are equivalent*:

(ci) $g(\lambda) = \prod_{i=1}^{t} (\lambda - \tau_i)(\lambda - p^a/\tau_i)$ *with* τ_i *simple roots but* $v_p(\tau_i) = ac$, $0 < c < \frac{1}{2}$ *for every* $1 \leqslant i \leqslant t$.

(cii) *Write* $g(\lambda) = \sum_{i=0}^{2t} d_i\lambda^i$. *Then* $g(\lambda)$ *is a distinguished polynomial over* $\mathbb{Z}_p$ *and the coefficients* d_i *satisfy the conditions*:

$$\underset{0 \leqslant i \leqslant 2t}{\mathrm{Min}} \frac{v_p(d_i)}{a(2t - i)} = \frac{v_p(d_t)}{at} = \frac{n}{n + m},$$

where n, m are positive integers satisfying $1 \leqslant n < m$, $(n, m) = 1$, *and* $n + m = t$.

(ciii) *The p-divisible group* $J(p)$ *of* $J(C)$ *is isogenous to* $(g - t)G_{1,0} + G_{n,m} + G_{m,n}$ *where n, m are positive integers satisfying* $1 \leqslant n < m$, $(n, m) = 1$ *and* $n + m = t$, *and* $\mathbf{\Gamma}$ *to* $G_m(p)^{g-t} + G_{n,m} + G_{m,n}$.

When the above is true, the Newton polygon $\mathfrak{N}(P_\pi)$ *of* $P_\pi(\lambda)$ *has the shape of Fig.* 5.

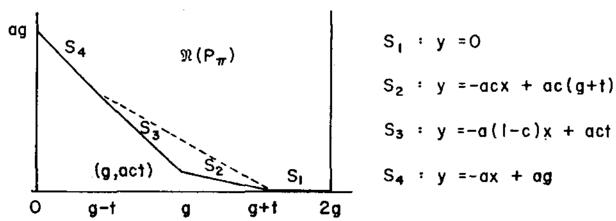


FIGURE 5

Proof. (a) (ai) $\Rightarrow$ (aii). Assume (ai). Then

$$P_\pi(\lambda) \equiv (-1)^{g+t}\lambda^{g+t} \{(-1)^{g-t}\lambda^{g-t} + \cdots + b_t\} \pmod{p},$$

where λ^{g+t} and $(-1)^{g-t}\lambda^{g-t} + \cdots + b_t$ are relatively prime. So by Hensel's lemma, there exist polynomials $P_0(\lambda)$, $h(\lambda)$ over $\mathbb{Z}_p$ such that

$$\begin{aligned} P_0(\lambda) &\equiv (-1)^{g-t}\lambda^{g-t} + \cdots + b_t \quad \pmod{p}, \quad \deg P_0(\lambda) = g - t, \\ h(\lambda) &\equiv (-1)^{g+t}\lambda^{g+t} \quad \pmod{p}. \end{aligned}$$

Moreover, in the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, $P_0(\lambda) = \prod_{i=1}^{g-t} (\lambda - \tau_i)$ with $v_p(\tau_i) = 0$ for every $1 \leqslant i \leqslant g - t$, because b_t is a p-adic unit. Since $P_\pi(\lambda)$ has always together with a root τ_i, the root p^a/τ_i, $h(\lambda)$ contains the factor

$P_a(\lambda) = \prod_{i=1}^{g-t} (\lambda - p^a/\tau_i)$ with $\nu_p(\tau_i) = 0$ for every $1 \leqslant i \leqslant g - t$. So there exists $g(\lambda) \in \mathbb{Z}_p[\lambda]$ such that $g(\lambda) \equiv (-1)^{2t}\lambda^{2t} \pmod p$ and that $h(\lambda) = P_a(\lambda)\, g(\lambda)$.

(aii) $\Rightarrow$ (aiii). The first part follows from the Manin theorem 4.1 in [6]. The formal group $\mathbf{\Gamma}$ of $J(C)$ is the connected component $J(p)/(\mathbb{Q}_p/\mathbb{Z}_p)_k^{g-t}$ of $J(p)$, whence it has height $2g - (g - t) = g + t$.

(aiii) $\Rightarrow$ (aiv). This follows from the fact that the p-rank of $J(C)$ coincides with the rank of the component $G_{1,0}$ in $J(p)$.

(aiv) $\Rightarrow$ (ai). The Dieudonné module corresponding to the $J(p)$ contains the factors $T_p(G_m(p)^{g-t}) \oplus T_p((\mathbb{Q}_p/\mathbb{Z}_p)_k^{g-t})$. Hence we can write $P_\pi(\lambda) = P_a(\lambda)\ P_0(\lambda)\ g(\lambda)$ where $P_a(\lambda)$ (resp. $P_0(\lambda)$: resp. $g(\lambda)$) is the characteristic polynomial of the restriction of the p-adic representation $T_p(\pi)$ of the Frobenius endomorphism π to $T_p(G_m(p)^{g-t})$ (resp. $T_p((\mathbb{Q}_p/\mathbb{Z}_p)_k^{g-t})$: resp. $T_p(J(p)/(g - t)G_{1,0})$). Both $P_a(\lambda)$ and $P_0(\lambda)$ have the same degree $g - t$ and moreover, $P_0(\lambda) = \prod_{i=1}^{g-t} (\lambda - \tau_i)$ with $\nu_p(\tau_i) = 0$ for every $1 \leqslant i \leqslant g - t$, since $(\mathbb{Q}_p/\mathbb{Z}_p)_k$ is étale. As $P_\pi(\lambda)$ satisfies the congruence (4) in Section 3:

$$P_\pi(\lambda) \equiv (-1)^g\lambda^g \mid A_\pi - \lambda I_g \mid \pmod p,$$

we have $\mid A_\pi - \lambda I_g \mid \equiv \lambda^t P_0(\lambda) \pmod p$. Here take $b_t \equiv P_0(0) \pmod p$. Then $(b_t, p) = 1$ and $b_j \equiv 0 \pmod p$ for all $j = 0,..., t - 1$.

(b) (bi) $\Rightarrow$ (bii). Putting $P_\pi(\lambda) = P_0(\lambda)\, P_a(\lambda)\, g(\lambda) = \sum_{i=0}^{2g} a_i\lambda^i$, we have immediately that $v_p(a_{2g-i}) = 0$ for every $0 \leqslant i \leqslant g - t$, $v_p(a_{g+t-i}) = (a/2)i$ for every $1 \leqslant i \leqslant 2t$, $v_p(a_{g-t-i}) = a(t + i)$ for every $1 \leqslant i \leqslant g - t$. Hence the Newton polygon $\mathfrak{N}(P_\pi)$ of $P_\pi(\lambda)$ has the segments S_1, S_2, S_3 with slopes 0, $-a/2$ and $-a$, respectively, and looks like Fig. 4.

(bii) $\Rightarrow$ (bi). Any segment $(j, v_p(a_j)) \leftrightarrow (l, v_p(a_l))$ with $l > j$ of $\mathfrak{N}(P_\pi)$ with slope $-m$ gives the roots $\tau_1, ..., \tau_{l-j}$ of $P_\pi(\lambda)$ in $\bar{\mathbb{Q}}_p$ with $\nu_p(\tau_i) = m$ for every $1 \leqslant i \leqslant l - j$. Moreover, $\prod_{i=1}^{l-j} (\lambda - \tau_i)$ with $\nu_p(\tau_i) = m$, is in $\mathbb{Z}_p[\lambda]$ and divides $P_\pi(\lambda)$. Hence the segments S_1, S_2, and S_3 correspond respectively to the factors $P_0(\lambda)$, $g(\lambda)$, and $P_a(\lambda)$. Therefore, the p-divisible group $J(p)$ is isogenous to $(g - t)\, G_{1,0} + tG_{1,1}$, and $\mathbf{\Gamma}$ to $G_m(p)^{g-t} + tG_{1,1}$.

(c) Since $g(\lambda)$ is the characteristic polynomial of the restriction of $T_p(\pi)$ to the Dieudonnλ module $T_p(J(p)/(g - t)G_{1,0})$, we have $g(\lambda) = \prod_{i=1}^{t} (\lambda - \tau_i)\ (\lambda - p^a/\tau_i)$ with $0 < \nu_p(\tau_i) < a/2$. Hence the same proof as Theorem 6.1a for $g(\lambda)$ yield the equivalences (ci) $\Leftrightarrow$ (cii) $\Leftrightarrow$ (ciii).

Now the factorization $P_\pi(\lambda) = \sum_{i=0}^{2g} a_i\lambda^i = P_0(\lambda)\, P_a(\lambda)\, g(\lambda)$ gives $v_p(a_{2g-i}) = 0$ for every $0 \leqslant i \leqslant g - t$, $v_p(a_{g+t-i}) \geqslant aci$ for every $1 \leqslant i < t$, $v_p(a_g) = act$ and $v_p(a_{g-i}) \geqslant act + a(1 - c)i$ for every $1 \leqslant i < t$, $v_p(a_{g-t}) = at$ and $v_p(a_{g-t-i}) \geqslant a(t + i)$ for every $1 \leqslant i \leqslant g - t$. Hence the Newton polygon $\mathfrak{N}(P_\pi)$ has the segments S_i, $i = 1,..., 4$ with slopes 0, $-ac$, $-a(1 - c)$, and $-a$, respectively, and looks like Fig. 5. Q.E.D.

THEOREM 7.2. *Let π be a Weil number of order a and suppose that the center $\Phi = \mathbb{Q}(\pi)$ of $\mathscr{A} = \mathrm{End}_k(J(C)) \otimes \mathbb{Q}$ is a CM-field of degree 2g. Put $\beta = \pi + \bar{\pi} = \pi + p^a/\pi$. Then we have*

(a) *$J(C)$ is elementary.*

(b) $P_\pi(\lambda) = \lambda^2 - \beta\lambda + p^a \in \mathbb{Q}(\beta)[\lambda]$. Moreover, we have

(b1) *$(\beta, p) = 1 \Leftrightarrow J(C)$ is ordinary.*

(b2) *Assume that $(\beta, p) \neq 1$ and let $f(\lambda) = \sum_{i=0}^{g} d_i\lambda^i$, $d_g = 1$ be the minimal polynomial of β. Then we have*

(b2.1) *If $\beta = \pm p^{a/2}\alpha$ with α an algebraic integer satisfying $(Norm(\alpha), p) = 1$, then $J(C)$ is supersingular.*

(b2.2) *If there exists the integer t such that $(d_t, p) = 1$, but $d_j \equiv 0 \pmod p$ for every $1 \leqslant j < t$ (take the smallest t if there are more than one such integers), then the p-divisible group $J(p)$ contains the component $(g - t)G_{1,0}$. Moreover, if there is a valuation ν over p in $\mathbb{Q}(\beta)$ such that $ord_\nu(\beta) = a/2$ and that ν is unramified in Φ, then the p-divisible group $J(p)$ is isogenous to $(g - t)G_{1,0} + tG_{1,1}$, but $J(C)$ is k-simple.*

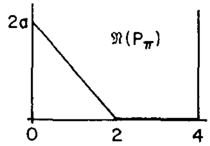
Proof. (a) This is the main theorem of Honda [4].

(b) For (b1), see Theorem 3.2 and for (b2.1), see Theorem 5.2.

(b2.2) It follows from the hypothesis that $f(\lambda) \equiv \lambda^t(\lambda^{g-t} + \cdots + d_t) \pmod p$. Hence $f(\lambda)$ gives $(g - t)$ p-adic roots with order 0. At these places ν, we have $\mathrm{ord}_\nu(\beta) = 0$ and the equation $\lambda^2 - \beta\lambda + p^a = 0$ must split, giving roots of orders 0 and a. Hence the local invariants i_ν are integers, so satisfies the commutativity condition for $\mathscr{A}$. This argument also shows that the p-divisible group $J(p)$ contains the component $(g - t)G_{1,0}$. Now we have a distinguished polynomial over $\mathbb{Z}_p$ corresponding to the factor λ^t of $f(\lambda)$ modulo p. Suppose that there is a valuation ν_2 in $\mathbb{Q}(\beta)$ over p such that $\mathrm{ord}_{\nu_2}(\beta) = a/2$. Then we may write $\beta = \pm p^{a/2}\alpha$ with α an invertible element in $\mathbb{Q}_p(\beta)$ such that $(\alpha, p) = 1$. The equation $\lambda^2 - \beta\lambda + p^a = 0$ gives $\pi = p^{a/2}Y$ where Y satisfies the equation $Y^2 - \alpha Y + 1 = 0$. In modulo ν_2 (i.e., in $\mathbb{F}_p$, since ν_2 is ramified) if $Y^2 - \alpha Y + 1 = 0$ has no solution, then it must be irreducible over $\mathbb{Q}_p(\beta)$. Hence Y generates an unramified quadratic extension over $\mathbb{Q}_p(\beta)$ and hence we get the unique extension of ord_{ν_2} to $\Phi = \mathbb{Q}(\pi)$ with residue degree 2. So π has $\mathrm{ord}_{\nu_2}(\pi) = a/2$ for the unique extension (again denoted) ord_{ν_2} in Φ over ord_{ν_2}. This shows that $P_\pi(\lambda)$ has $2t$ p-adic roots with order $a/2$ and hence $J(p)$ contains the factor $tG_{1,1}$. Thus $J(p)$ is isogenous to $(g - t)G_{1,0} + tG_{1,1}$. Now we compute the local invariant i_{ν_2}; $i_{\nu_2} = ((a/2) \cdot 2)/a \in \mathbb{Z}$. Hence $\mathscr{A} = \Phi$ with $[\mathscr{A} : \mathbb{Q}] = 2g$ and hence $J(C)$ is k-simple. Q.E.D.

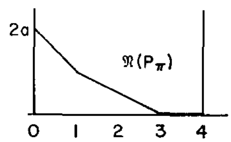
EXAMPLE 7.3. For hyperelliptic curves C of genus 2 over k, we have more complete classification theorem for the p-divisible group $J(p)$ of $J(C)$. The notation in Theorem 7.2 remains in force.

(a) $|A| \neq 0 \Leftrightarrow (\beta, p) = 1 \Leftrightarrow$



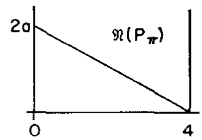
$\Leftrightarrow J(p) \sim 2G_{1,0} \Leftrightarrow J(C)$ is ordinary.

(b) $[|A| = 0$, but $A\ A^{(p)} \neq 0] \Leftrightarrow [(\beta, p) \neq 1$, but $(\mathrm{Tr}(\beta), p) = 1] \Leftrightarrow$



$\Leftrightarrow J(p) \sim G_{1,0} + G_{1,1}$.

(c) $[|A| = 0$ and $A\ A^{(p)} = (0)] \Leftrightarrow [(\beta, p) \neq 1$, $(\mathrm{Tr}(\beta), p^{a/2}) \neq 1$ and $(\mathrm{Norm}(\beta), p^a) \neq 1] \Leftrightarrow (\beta, p^{a/2}) \neq 1 \Leftrightarrow$



$\Leftrightarrow J(p) \sim 2G_{1,1} \Leftrightarrow J(C)$ is supersingular.

Proof. β being a real quadratic over $\mathbb{Q}$ and $\beta = \xi + \eta(d)^{1/2}$ with $\xi, \eta \in \mathbb{Q}$, and d square free, we have $P_\pi(\lambda) = \lambda^2 - \beta\lambda + p^a \in \mathbb{Q}(\beta)[\lambda]$ and $P_\pi(\lambda) = \lambda^4 - \mathrm{Tr}(\beta)\lambda^3 + (2p^a + \mathrm{Norm}(\beta))\lambda^2 - \mathrm{Tr}(\beta)\, p^a\lambda + p^{2a} \in \mathbb{Q}[\lambda]$ and $|A_\pi| \equiv \mathrm{Norm}(\beta) \pmod p$. Hence the assertions follow immediately. Q.E.D.

EXAMPLE 7.4. We shall give an example of k-simple Abelian variety of dimension 2 equipped with the mixed type of formal structure $G_{1,0} + G_{1,1}$. Let $k = \mathbb{F}_{7^2}$ and let $\beta = 6 + (29)^{1/2}$ in $\mathbb{Q}((29)^{1/2})$. Then $|\beta| < 2 \cdot 7$ and $\pi^2 - \beta\pi + 7^2 = 0$ gives a Weil number of order 2 and $\Phi = \mathbb{Q}(\pi)$ is a CM-field of degree 4. Since $(\beta, 7) \neq 1$, the Abelian variety X determined by π, up to isogeny, is nonordinary. Then the minimal polynomial of β over $\mathbb{Q}$ is given by $f(\lambda) = \lambda^2 - 12\lambda + 7$ and $f(\lambda) \equiv \lambda(\lambda + 2) \pmod 7$. So there are two valuations over 7 in $\mathbb{Q}(\beta)$: $\mathrm{ord}_{\nu_1}(\beta) = 0$ and $\mathrm{ord}_{\nu_2}(\beta) = 1$. At ord_{ν_2}, $\lambda^2 - \beta\lambda + 7^2 = 0$ splits, giving roots with orders 0 and 2. Hence the p-divisible group $X(p)$ of X has the component $G_{1,0}$. At ord_{ν_2}, $7 \mid \beta$ in $\mathbb{Q}_7$ and hence $\lambda^2 - \beta\lambda + 7^2 = 0$ has the solution $\pi = 7 \cdot \alpha$ where α satisfies the equation $\alpha^2 - 3\alpha + 1 = 0$. This α

generates an unramified quadratic extension over $\mathbb{Q}_7$. So there is a unique extension (again denoted) ord_{ν_2} of ord_{ν_2} to Φ with $\mathrm{ord}_{\nu_2}(\pi) = 1$. Hence $X(p)$ is isogenous to $G_{1,0} + G_{1,1}$. The characteristic polynomial of π over $\mathbb{Q}$ is $P_\pi(\lambda) = \lambda^4 - 12\lambda + 105\lambda^2 - 588\lambda + 7^4$, which is easily seen to be $\mathbb{Q}$-irreducible. Thus X is k-simple. Q.E.D.

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