Chromatic quasisymmetric functions and regular semisimple Hessenberg varieties

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Chromatic Symmetric Functions

$G = ([n], E)$ a finite, loopless graph.

$$\mathrm{col}(G) := \{f : [n] \to \mathbb{N} | f(i) \neq f(j) \text{ whenever } ij \in E\}$$

R. Stanley's *chromatic symmetric function*:

$$X_G(\mathbf{x}) := X_G(x_1, x_2, \ldots) := \sum_{f \in \mathrm{col}(G)} \prod_{i=1}^{n} x_{f(i)}$$

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$$G = 1 - -2 - -3$$

$$X_G(\mathbf{x}) = 6 \sum_{i<j<k} x_i x_j x_k + \sum_{i \neq j} x_i^2 x_j$$

Symmetric functions

Let R be a commutative ring (for us, $\mathbb{Q}$ or $\mathbb{Q}[t]$). Λ_R is the ring of symmetric functions with coefficients in R.

Λ_R consists of all $f \in R[[x_1, x_2, \ldots]]$ such that

- f has bounded degree, and
- $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$ for all $\sigma \in Sym(\mathbb{N})$.

Decomposition into homogeneous pieces:

$$\Lambda_R = \bigoplus_{k \geq 0} \Lambda_R^k$$

Some homogeneous symmetric functions:

Complete:

$$h_n := \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} \prod_{j=1}^{n} x_{i_j}$$

Elementary:

$$e_n := \sum_{i_1 < i_2 < \ldots < i_n} \prod_{j=1}^{n} x_{i_j}$$

Power sum:

$$p_n := \sum_{j=1}^{\infty} x_j^n$$

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition, $b \in \{h, e, p\}$.

$$b_\lambda := \prod_{j=1}^{l} b_{\lambda_j}$$

Fact: $\{b_\lambda | \lambda \in Par(k)\}$ is a basis for $\Lambda_{\mathbb{Q}}^k$.

Examples:

$$h_{(2,2,1)} = (x_1^2 + x_1 x_2 + x_2^2 + \ldots)^2 (x_1 + x_2 + \ldots)$$

$$e_{(2,2,1)} = (x_1 x_2 + x_1 x_3 + x_2 x_3 + \ldots)^2 (x_1 + x_2 + \ldots)$$

$$p_{(2,2,1)} = (x_1^2 + x_2^2 + \ldots)^2 (x_1 + x_2 + \ldots)$$

Incomparability graphs

Let P be a poset on $[n]$. The *incomparability graph* $Inc(P)$ has vertex set $[n]$ and edge set

$$\{ij | i \text{ and } j \text{ are incomparable in } P\}.$$

For $a, b \in \mathbb{N}$, P is $(a+b)$-*free* if there do not exist $x_1, \ldots, x_a, y_1, \ldots, y_b \in P$ such that

- $x_1 < \ldots < x_a$,
- $y_1 < \ldots < y_b$, and
- x_i and y_j are incomparable for all i, j.

Conjecture (Stanley-Stembridge, 1993): If P is a $3+1$-free poset on $[n]$ then $X_{Inc(P)}$ is e-positive, that is,

$$X_{Inc(P)} \in \mathbb{N}_0[\{e_\lambda : \lambda \in Par(n)\}].$$

Frobenius characteristic

$$\mathrm{Class}(S_n) := \{f : S_n \to \mathbb{Q} | f \text{ is constant on conjugacy classes}\}$$

$$\dim \mathrm{Class}(S_n) = \dim \Lambda^n_{\mathbb{Q}} = |Par(n)|$$

For $\lambda \in Par(n)$, set

$$C_\lambda := \{\sigma \in S_n : \sigma \text{ has cycle shape } \lambda\},$$

and

$$z_\lambda := n!/|C_\lambda|.$$

The *Frobenius characteristic* is the unique linear map

$$ch : \bigoplus_{n \geq 0} \mathrm{Class}(S_n) \to \Lambda_{\mathbb{Q}}$$

satisfying

$$ch(\delta_{C_\lambda}) = p_\lambda / z_\lambda.$$

Schur functions

The irreducible characters χ^λ of S_n are naturally indexed by $Par(n)$ and form a basis for $\text{Class}(S_n)$. The *Schur functions* s_λ satisfy

$$s_\lambda = ch(\chi^\lambda)$$

For $\lambda = (\lambda_1, \ldots, \lambda_l) \in Par(n)$, let μ^λ be the character of the permutation representation of S_n on the cosets of $\prod_{j=1}^{l} S_{\lambda_j}$.

$$ch(\mu^\lambda) = h_\lambda$$

$$ch(\mu^\lambda \cdot sign) = e_\lambda$$

So, if $f \in \Lambda_{\mathbb{Q}}$ is h-positive or e-positive then f is s-positive.

Theorem (V. Gasharov, 1996): Let P be a $3+1$-free poset. Then $X_{Inc(P)}$ is s-positive.

Gasharov gives a formula for the coefficient of each s_λ in $X_{Inc(P)}$. We will see it later.

Goal: a conceptual explanation of Gasharov's theorem and the Stanley-Stembridge conjecture.

Chromatic quasisymmetric functions

For $G = ([n], E)$ and $f \in \mathrm{col}(G)$, define

$$asc(f) := |\{ij \in E | i < j \text{ and } f(i) < f(j)\}|.$$

The *chromatic quasisymmetric function* of G is

$$X_G(\mathbf{x}; t) := \sum_{f \in \mathrm{col}(G)} t^{asc(f)} \prod_{j=1}^{n} x_{f(j)}.$$

$$X_G(\mathbf{x};t) := \sum_{f \in \mathrm{col}(G)} t^{asc(f)} \prod_{j=1}^{n} x_{f(j)}.$$

$$G = 1 - -2 - -3$$

$$X_G(\mathbf{x};t) = (1+4t+t^2) \sum_{i<j<k} x_i x_j x_k + t \sum_{i \neq j} x_i^2 x_j$$

$$H = 1 - -3 - -2$$

$$X_H(\mathbf{x};t) = 2(1+t+t^2) \sum_{i<j<k} x_i x_j x_k + \sum_{i<j} x_i x_j^2 + t^2 \sum_{i<j} x_i^2 x_j$$

- $X_G(\mathbf{x};t) \in \Lambda_{\mathbb{Q}[t]}$ but $X_H(\mathbf{x};t) \notin \Lambda_{\mathbb{Q}[t]}$.

Our favorite graphs

A *Hessenberg vector* is any $\mathrm{h} = (\mathrm{h}_1, \dots, \mathrm{h}_{n-1}) \in \mathbb{N}^{n-1}$ satisfying

- $i \leq \mathrm{h}_i \leq n$ for all $i \in [n-1]$ and
- $\mathrm{h}_i \leq \mathrm{h}_{i+1}$ for all $i \in [n-2]$.

The *Hessenberg graph* $\Gamma(\mathrm{h})$ associated to h has vertex set $[n]$ and edge set

$$E(h) := \{ij | i < j \leq \mathrm{h}_i\}.$$

Proposition (D. Scott-P. Suppes, 1958): A poset P is both $(3+1)$-free and $(2+2)$-free if and only if there is some Hessenberg vector h such that $Inc(P)$ is isomorphic to $\Gamma(\mathrm{h})$. If h is a Hessenberg vector then there is a $3+1$-free and $2+2$-free poset P such that $Inc(P) = \Gamma(\mathrm{h})$.

Proposition: If h is a Hessenberg vector then $X_{\Gamma(\mathrm{h})}(\mathbf{x}; t) \in \Lambda_{\mathbb{Q}[t]}$.

Schur decomposition

Let h be a Hessenberg vector and let P be the poset on $[n]$ with $Inc(P) = \Gamma(\mathrm{h})$. Let $\lambda \in Par(n)$.

A P-tableau T of shape λ is a filling of the Young diagram of shape λ with all of the elements of $[n]$ such that

- if j appears immediately to the right of j in T then $i <_P j$, and
- if j appears immediately below i in T then $j \not<_P i$.

Let $\mathcal{T}_\lambda$ be the set of all P-tableau of shape λ. For $T \in \mathcal{T}_\lambda$, set

$$inv_P(T) := |\{ij \in E(\mathrm{h}) | i < j \text{ and } row_T(i) > row_T(j)\}|.$$

Theorem: With h, P as above,

$$X_{Inc(P)}(\mathbf{x}; t) = \sum_{\lambda \in Par(n)} \left(\sum_{T \in \mathcal{T}_\lambda} t^{inv_P(T)} \right) s_\lambda.$$

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When $t = 1$, this is Gasharov's formula.

Example: $\mathrm{h} = (2,3)$, $\Gamma(\mathrm{h}) = 1 - -2 - -3$

T	1	1	2	3	13
	2	3	1	2	2
	3	2	1	1	
$inv_P(T)$	0	1	1	2	1

$$X_{\Gamma(\mathrm{h})}(\mathbf{x};t) = (1 + 2t + t^2)s_{(1,1,1)} + ts_{(2,1)}$$

Quasisymmetric functions

Let $n \in \mathbb{N}_0$ and let $S \subseteq [n-1]$. Let $P(S,n)$ be the set of all weakly decreasing sequences $J = (j_1, \ldots, j_n)$ from $\mathbb{N}$ such that $j_i > j_{i+1}$ whenever $i \in S$. Set

$$F_{S,n} := \sum_{J \in P(S,n)} \prod_{i=1}^{n} x_{j_i} \in R[[\mathbf{x}]]$$

.

The ring $\mathcal{Q}_R$ of quasisymmetric functions is the R-submodule of $R[[x]]$ generated by all $F_{S,n}$.

Note $F_{\emptyset,n} = h_n$. So, $\Lambda_R \subseteq \mathcal{Q}_R$.

(Easy) **Proposition**: For every graph $G = ([n], E)$, $X_G(\mathbf{x}; t) \in \mathcal{Q}_{\mathbb{Q}[t]}$.

Let P be a poset on $[n]$ and let $Inc(P) = ([n], E)$. For $\sigma \in S_n$, set

$$INV_P(\sigma) := \{ab \in E | a > b, \sigma^{-1}(a) < \sigma^{-1}(b)\}$$

and

$$DES_P(\sigma) := \{i \in [n-1] | \sigma(i) >_P \sigma(i+1)\}.$$

Theorem: For any poset P on $[n]$,

$$X_G(\mathbf{x}; t) = \sum_{\sigma \in S_n} t^{|INV_G(\sigma)|} F_{[n-1] \setminus DES_P(\sigma), n}.$$

When $t = 1$, this is a theorem of T. Chow.

$$X_G(\mathbf{x};t) = \sum_{\sigma \in S_n} t^{|INV_G(\sigma)|} F_{[n-1] \setminus DES_P(\sigma), n}$$

Corollary: Let $\mathrm{h} = (\mathrm{h}_1, \ldots, \mathrm{h}_{n-1})$ be a Hessenberg vector. Write

$$ch^{-1}(X_{\Gamma(\mathrm{h})}(\mathbf{x};t)) = \sum_{j \geq 0} \theta_j t^j.$$

Then, for each j such that θ_j is not the zero function, θ_j is a character of S_n and

$$\theta_j(1) = |\{\sigma \in S_n : |INV_{\Gamma(h)}(\sigma)| = j\}|.$$

"Proof": Consider the coefficient of $\prod_{j=1}^{n} x_j$ in $X_G(\mathbf{x};t)$.

The flag variety

Let $n \in \mathbb{N}$, let $G = GL_n(\mathbb{C})$ and let B be the subgroup of G consisting of those $g \in G$ that are upper triangular.

The *flag variety* is the quotient space $\mathrm{Flag}_n := G/B$.

A *flag* in $\mathbb{C}^n$ is any chain

$$\mathcal{F} : 0 = V_0 < V_1 < \ldots < V_n = \mathbb{C}^n$$

of subspaces of $\mathbb{C}^n$.

The group G acts transitively on the set of all flags in $\mathbb{C}^n$ and B is the stabilizer of a particular flag. So, the elements of Flag_n are in bijection with the set of flags in $\mathbb{C}^n$.

Hessenberg varieties of type A

Let $\mathrm{h} = (\mathrm{h}_1, \ldots, \mathrm{h}_{n-1})$ be a Hessenberg vector and let $s \in G = GL_n(\mathbb{C})$.

First definition: The *Hessenberg variety* $\mathrm{Hess}(\mathrm{h}, s)$ consists of those

$$\mathcal{F} : 0 = V_0 < V_1 < \ldots < V_n = \mathbb{C}^n$$

in Flag_n satisfying

$$sV_i \leq V_{\mathrm{h}_i}$$

for all $i \in [n-1]$.

Let $\mathsf{h} = (\mathsf{h}_1, \ldots, \mathsf{h}_{n-1})$ be a Hessenberg vector and let $s \in G = GL_n(\mathbb{C})$.

Define $M_n^{\mathsf{h}}(\mathbb{C})$ to be the set of all matrices $A = (a_{ij}) \in M_n(\mathbb{C})$ such that $a_{ij} = 0$ whenever $i > \mathsf{h}_j$.

Example:

$$M_4^{(2,3,4)}(\mathbb{C}) = \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \right\}$$

Second definition: The *Hessenberg variety* $\mathrm{Hess}(\mathsf{h}, s)$ consists of those $gB \in G/B$ such that $g^{-1}sg \in M_n^{\mathsf{h}}(\mathbb{C})$.

If $s \in G$ is diagonalizable with n pairwise distinct eigenvalues, then $\mathrm{Hess}(\mathsf{h}, n)$ is a *regular semisimple Hessenberg variety of type A*.

Theorem (De Mari-Shayman 1988, De Mari-Procesi-Shayman 1992): Let $\mathrm{Hess}(\mathsf{h}, s) \subseteq \mathrm{Flag}_n$ be a regular semisimple Hessenberg variety. Then, for all $j \in \mathbb{N}_0$,

- $H^{2j+1}(\mathrm{Hess}(\mathsf{h}, s)) = 0$, and
- $\dim H^{2j}(\mathrm{Hess}(\mathsf{h}, s), \mathbb{Q}) = |\{\sigma \in S_n : |INV_{\Gamma(\mathsf{h})}(\sigma)| = j\}|$

Note $\dim H^{2j}(\mathrm{Hess}(\mathsf{h}, s), \mathbb{Q}) = \theta_j(1)$.

Let $T = C_G(s)$. For $g \in T$ and

$$\mathcal{F} : 0 = V_0 < V_1 < \ldots < V_n = \mathbb{C}^n \in \mathrm{Hess}(\mathsf{h}, s)$$

and $i \in [n-1]$,

$$sgV_i = gsV_i \leq gV_{\mathsf{h}_i}.$$

Therefore, $g\mathcal{F} \in \mathrm{Hess}(\mathsf{h}, s)$.

So, we have an action of T on $\mathrm{Hess}(\mathsf{h}, s)$.

Note T is a torus, that is, $T \cong (\mathbb{C}^*)^n$.

The theory of Goresky-Kottwitz-MacPherson

This theory applies to the action of a torus S on a variety X when certain technical conditions are satisfied. Such conditions are satisfied by the action of T on $\mathrm{Hess}(\mathsf{h}, s)$ described above.

Given $S = (\mathbb{C}^*)^n$ and X, let F be the set of fixed points of S on X and let O be the set of 1-dimensional orbits of S on X. The technical conditions force that

- F and O are finite, and
- each orbit in F has in its closure exactly two points in O.

The *moment graph* M associated to the action of S on X has vertex set (indexed by) F and edge set (indexed by) O with $f \in F$ an endpoint of $o \in O$ if and only if $f \subseteq \overline{o}$.

Let $P = \mathbb{C}[t_1, \ldots, t_n]$ and let $R = P^F$, the direct sum of $|F|$ copies of R. Write an element of R as $(p_f)_{f \in F}$.

The G-K-M theory says that there is a collection of ideals $\{I_o : o \in O\}$ in P (determined by the action of S on X) such that

- the equivariant cohomology ring $H_S^*(X, \mathbb{C})$ is isomorphic to the subring of R consisting of those (p_f) satisfying

$$p_f - p_g \in I_o \text{ whenever } o = \{f, g\} \text{ is an edge of } M.$$

Moreover, $H_S^*(X, \mathbb{C})$ is a P-submodule of R, and

- $H^*(X, \mathbb{C}) \cong H_S^*(X, \mathbb{C})/(t_1, \ldots, t_n)H_S^*(X, \mathbb{C})$.

Applying the G-K-M theory to $\text{Hess}(\mathrm{h}, s)$

We may assume that s is diagonal. Then T consists of all diagonal matrices. A flag

$$\mathcal{F} : 0 = V_0 < V_1 < \ldots < V_n = \mathbb{C}^n$$

is fixed by T if and only if each V_i is spanned by i standard basis vectors. Every such flag lies in every $\text{Hess}(\mathrm{h}, s)$.

It follows that the elements of F are indexed by the permutations in S_n (consider the order in which standard basis vectors are added as we move up the flag).

Given $v, w \in S_n$, it turns out that $\{v, w\}$ is an edge in M if and only if there is a transposition $(ij) \in S_n$ such that

- $wv^{-1} = (ij)$, and
- ij is an edge in $\Gamma(h)$.

Given an edge $\{v, w\}$ of M, $v^{-1}w$ is a transposition (kl), and

$$I_{\{v,w\}} = (t_k - t_l).$$

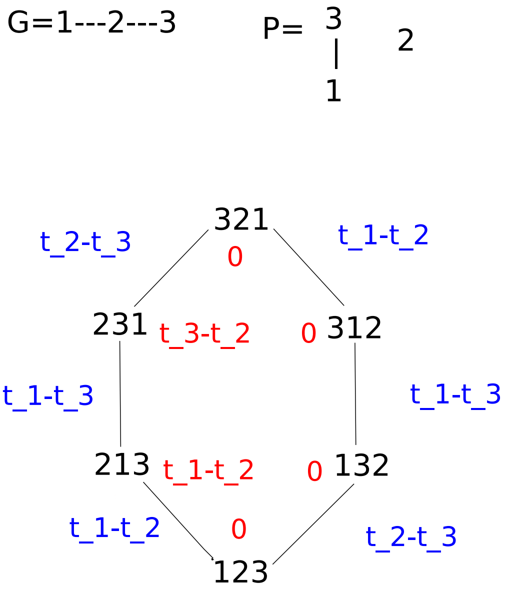

Figure: A cohomology class

The action of S_n on itself (from the right) gives an action of S_n on the moment graph M.

The natural action of S_n on indices determines and action on the polynomial ring P.

If $u \in S_n$ maps the edge $\{v, w\}$ to the edge $\{y, z\}$ then u maps $I_{\{v,w\}}$ to $I_{\{y,z\}}$.

Combining these two actions, we get a representation of S_n on $H^*(\mathrm{Hess}(\mathrm{h}, s), \mathbb{C})$.

This representation has been studied by J. Tymoczko and collaborators.

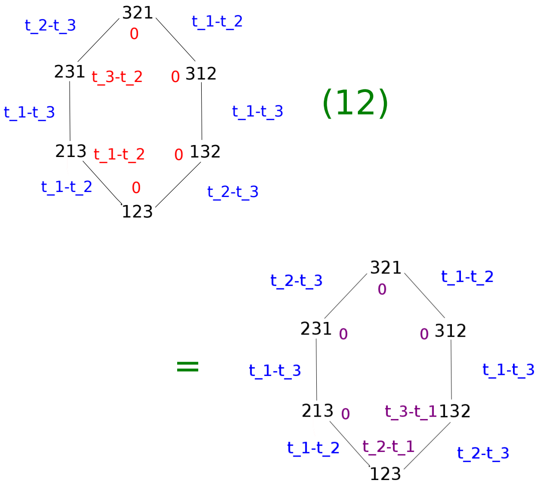

Figure: The action of a transposition

The main conjecture

Conjecture: Let $\rho_j(\mathrm{h}, s)$ be the character of S_n obtained by multiplying the character of the representation on $H^{2j}(\mathrm{Hess}(h, s), \mathbb{C})$ by the sign character. Then $ch(\rho_j(\mathrm{h}, s))$ is the coefficient of t^j in $X_{\Gamma(\mathrm{h})}(\mathbf{x}; t)$.

The conjecture holds in the following cases.

- When $\mathrm{h} = (n, \ldots, n)$. In this case $\Gamma(\mathrm{h})$ is the complete graph and $\mathrm{Hess}(\mathrm{h}, s)$ is the flag variety.
- When $\mathrm{h} = (n-1, n, \ldots, n)$.
- When $\mathrm{h} = (2, 3, \ldots, n)$. In this case $\Gamma(h)$ is the toric variety associated to the Coxeter complex of type A and the given representation was studied by C. Procesi, R. Stanley, J. Stembridge, and I. Dolgachev-V. Lunts.
- When $n \leq 4$.