On the computation of p-adic Height Pairings

Amnon Besser

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Mazur-Stein-Tate (2004) computation in the case of $g(C) = 1$.

Goal: compute in general.

p-adic height pairings on curves - Coleman Gross (1985)

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Height pairing $h = \sum h_v : J(C) \times J(C) \to \mathbb{Q}_p$

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$h_v(y, z) = \langle y, z \rangle_v \cdot \chi_v(\pi_v)$.

$\langle y, z \rangle_v = \tilde{y} \cdot \tilde{z}$.

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To define h_v we need

- The universal vectorial extension of J and its logarithm.
- Coleman's integration theory

Universal vectorial extension of J

$$T(L) = \{\omega \text{ on } C_L \text{ of third kind}\} \subset \Omega^1(L(C))$$

Third kind means simple poles and residues in $\mathbb{Z}$

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quotienting by $T_\ell(L)$

$$0 \to \Omega^1(C_L) \to T(L)/T_\ell(L) := G(L) \to J(L) \to 0$$

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Taking tangent spaces at 0 we get

$$0 \to H^{1,0}(C) \to H^1_{\mathrm{dr}}(C/K) \to H^{0,1}(C) \to 0$$

The logarithm

Logarithm for a commutative group scheme over a p-adic field

$$\log_G : G(K) \to H^1_{\mathrm{dr}}(C/K)$$

which is the identity on $H^{1,0}(C)$.

Branch of log and trace

$$\begin{array}{ccc} F_v^{\times} & \xrightarrow{\chi_v} & \mathbb{Q}_p \\ & {\scriptstyle \log_v} \searrow \quad \nearrow {\scriptstyle \mathrm{tr}_v} & \\ & F_v & \end{array}$$

Coleman integration and the height

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For all v we have $h_v((f), z) = \chi_v(f(z))$ hence h factors via J.

The double index and $\log$

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However, there is no problem if either ω or η has no residue (in second case define as $-\operatorname{Res}_0 F_\eta \omega$)

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Global index:
$\sum_{x \in C} \langle F_\omega, F_\eta \rangle_x := \langle F_\omega, F_\eta \rangle_{gl} = \langle \omega, \eta \rangle_{gl}$ where F_η, F_ω are Coleman integrals.

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Can replace indexes on points by an index on an annulus.

The projection formula

Theorem (B.) Define for a meromorphic form ω, $\Psi(\omega) \in H^1_{\text{dr}}(C)$ by

$$\langle \omega, \alpha \rangle_{gl} = \Psi(\omega) \cup [\alpha]$$

for α of the second kind. Then

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- For $\omega \in T(K)$, $\Psi(\omega) = \log(\omega)$
- For another form η

$$\langle \omega, \eta \rangle_{gl} = \Psi(\omega) \cup \Psi(\eta)$$

Remarks

1. A similar projections exists in the rigid context: ω is a rigid form on a wide open $U \subset C$. The projection is the unique Frobenius equivariant splitting of

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The projection is the unique Frobenius equivariant splitting of

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2. The projection formula reduces the computation of the $\log$ to the computation of Coleman integrals.

Remarks

3. When computing $\langle \omega, \alpha \rangle_{gl}$ for α of the second kind only the Coleman integral of α needs to be computed.

Computation of Coleman integrals

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More general forms require either more general reduction or some tricks using double indices again.

Sketch of the algorithm at p

Observation: may assume our divisors are anti-symmetric

Start with antisymmetric y and z.

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$\Rightarrow$ can compute Ψ.

Note: no further Coleman integration required.

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"Compute" ω_y - pick any ω with residue divisor y and compute its log $= \Psi$.

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Since its integral is known it suffices to compute $\int_z \omega$.

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By the assumptions on F we have

$$\sigma(F(\phi(\sigma^{-1}(P)))) - pF(P) = \frac{1}{2}\int_{-P}^{P} \eta$$

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$$\text{RHS} = \sum \operatorname{Res} \mu \int \eta$$

sum over singular points of μ.

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Note: η has an essential singularity on the Weierstrass discs.

How to compute local heights away from p

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error depends on degrees of y, z and on the reduction type of C.

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$\langle y, z\rangle_v$ has bounded denominators hence get answer for a sufficiently large n.

$$2y = y_1 + (f_1)$$

$$4y = 2y_1 + (f_1^2) = y_2 + (f_2) + (f_1^2)$$

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$$\langle y, z \rangle_v \sim 2^{-n} \{y', z\}_v + \sum v(f_i(z))/2^i$$

Problem

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Solution?: Use work of Kausz to bounded number of components in reduction in terms of valuation of discriminant: To $y^2 = f$ associate the discriminant $D = \Delta(f)^g V^{8g+4}$ where V is the co-volume inside $H^0(\tilde{C}, \Omega^1)$ of the module generated by $x^i dx/y$. Then $v(D)$ bounds a weighted sum of the number of singular points of the minimal regular model.