



On the computation of p -adic Height Pairings

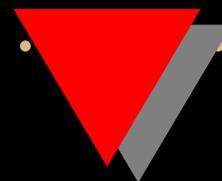
Amnon Besser



p - prime

F - number field

C/F - smooth complete curve with good
reduction at places above p



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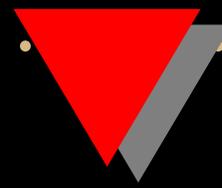
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J - Jacobian of C .

(additional data) \Rightarrow p -adic height pairing

$$h : J(F) \times J(F) \rightarrow \mathbb{Q}_p$$





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Mazur-Stein-Tate (2004) computation in the case of $g(C) = 1$.

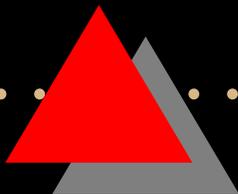
Goal: compute in general.





p -adic height pairings on curves - Coleman Gross (1985)

$\chi = \prod \chi_v : I_F/F^\times \rightarrow \mathbb{Q}_p$ a character

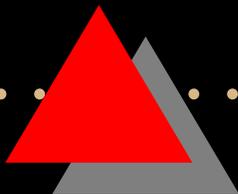




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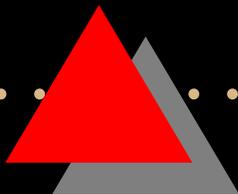
Height pairing $h = \sum h_v : J(C) \times J(C) \rightarrow \mathbb{Q}_p$





h_v for v not dividing p

v - non archimedean, C/K , $K = F_v$,
 $y, z \in \text{Div}_0(C)$ with disjoint support.





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$$h_v(y, z) = \langle y, z \rangle_v \cdot \chi_v(\pi_v).$$

$$\langle y, z \rangle_v = \tilde{y} \cdot \tilde{z}.$$

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To define h_v we need

- The universal vectorial extension of J and its logarithm.
- Coleman's integration theory.



Universal vectorial extension of J

$$T(L) = \{\omega \text{ on } C_L \text{ of third kind}\} \subset \Omega^1(L(C))$$

Third kind means simple poles and residues in \mathbb{Z}

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quotienting by $T_\ell(L)$

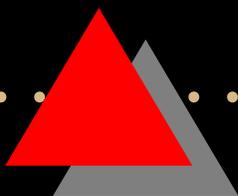
$$0 \rightarrow \Omega^1(C_L) \rightarrow T(L)/T_\ell(L) := G(L) \rightarrow J(L) \rightarrow 0$$



Universal vectorial extension of *J*

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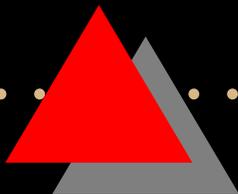
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Taking tangent spaces at 0 we get

$$0 \rightarrow H^{1,0}(C) \rightarrow H_{\text{dr}}^1(C/K) \rightarrow H^{0,1}(C) \rightarrow 0$$





The logarithm

Logarithm for a commutative group scheme over a p -adic field

$$\log_G : G(K) \rightarrow H_{\text{dr}}^1(C/K)$$

which is the identity on $H^{1,0}(C)$.

Branch of log and trace

$$\begin{array}{ccc} \times & & \\ F_v & \xrightarrow{\chi_v} & \mathbb{Q}_p \\ & \searrow \log_v \quad \swarrow \text{tr}_v & \\ & F_v & \end{array}$$

Coleman integration and the height

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For all v we have $h_v((f), z) = \chi_v(f(z))$ hence h factors via J .

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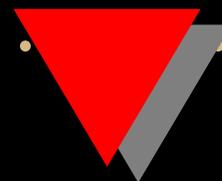
$$F_\omega = a_0 \log(z) + \dots, \text{Res}_0 \log(z) dz/z = ?.$$

However, there is no problem if either ω or η has no residue (in second case define as $-\text{Res}_0 F_\eta \omega$)



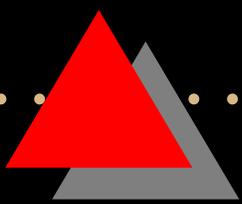
Solution: double index $\langle F_\omega, F_\eta \rangle$ depending on both F_ω and F_η , bilinear, antisymmetric and equal to above when defined.

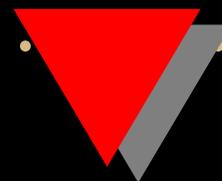




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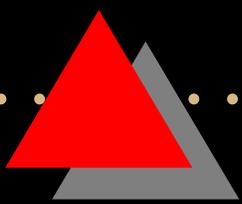


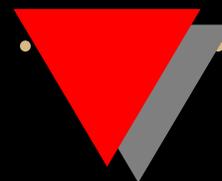


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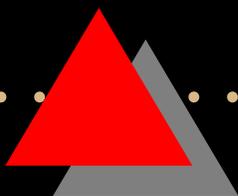
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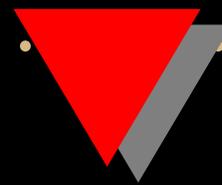
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Global index:

$\sum_{x \in C} \langle F_\omega, F_\eta \rangle_x := \langle F_\omega, F_\eta \rangle_{gl} = \langle \omega, \eta \rangle_{gl}$ where F_η, F_ω are Coleman integrals.





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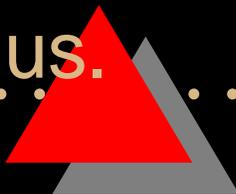
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Can replace indexes on points by an index on an annulus.





The projection formula

Theorem (B.) Define for a meromorphic form ω , $\Psi(\omega) \in H_{\text{dr}}^1(C)$ by

$$\langle \omega, \alpha \rangle_{gl} = \Psi(\omega) \cup [\alpha]$$

for α of the second kind. Then

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- For another form η

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Remarks

1. A similar projections exists in the rigid context:
 ω is a rigid form on a wide open $U \subset C$.
The projection is the unique Frobenius equivariant splitting of

$$H_{\text{dr}}^1(C) \rightarrow H_{\text{dr}}^1(U)$$

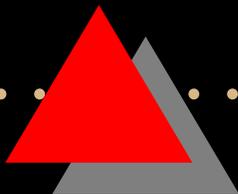


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2. The projection formula reduces the computation of the log to the computation of Coleman integrals.





Remarks

3. When computing $\langle \omega, \alpha \rangle_{gl}$ for α of the second kind only the Coleman integral of α needs to be computed.



Computation of Coleman integrals

Computation of Coleman integrals has been done by Gutnik and by Kedlaya Bradshaw for forms with residually Weierstrass singularities.

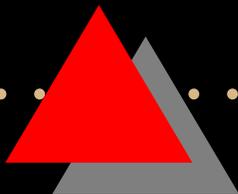




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More general forms require either more general reduction or some tricks using double indices again.

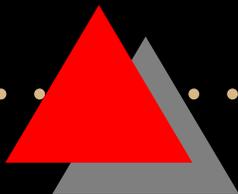




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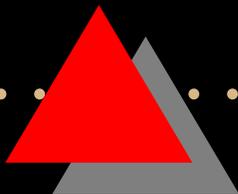
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\Rightarrow can compute Ψ .

Note: no further Coleman integration required.





Second step

“Compute” ω_y - pick any ω with residue divisor y
and compute its log = Ψ .

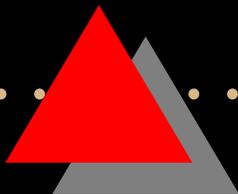




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Since its integral is known it suffices to compute $\int_z \omega$.

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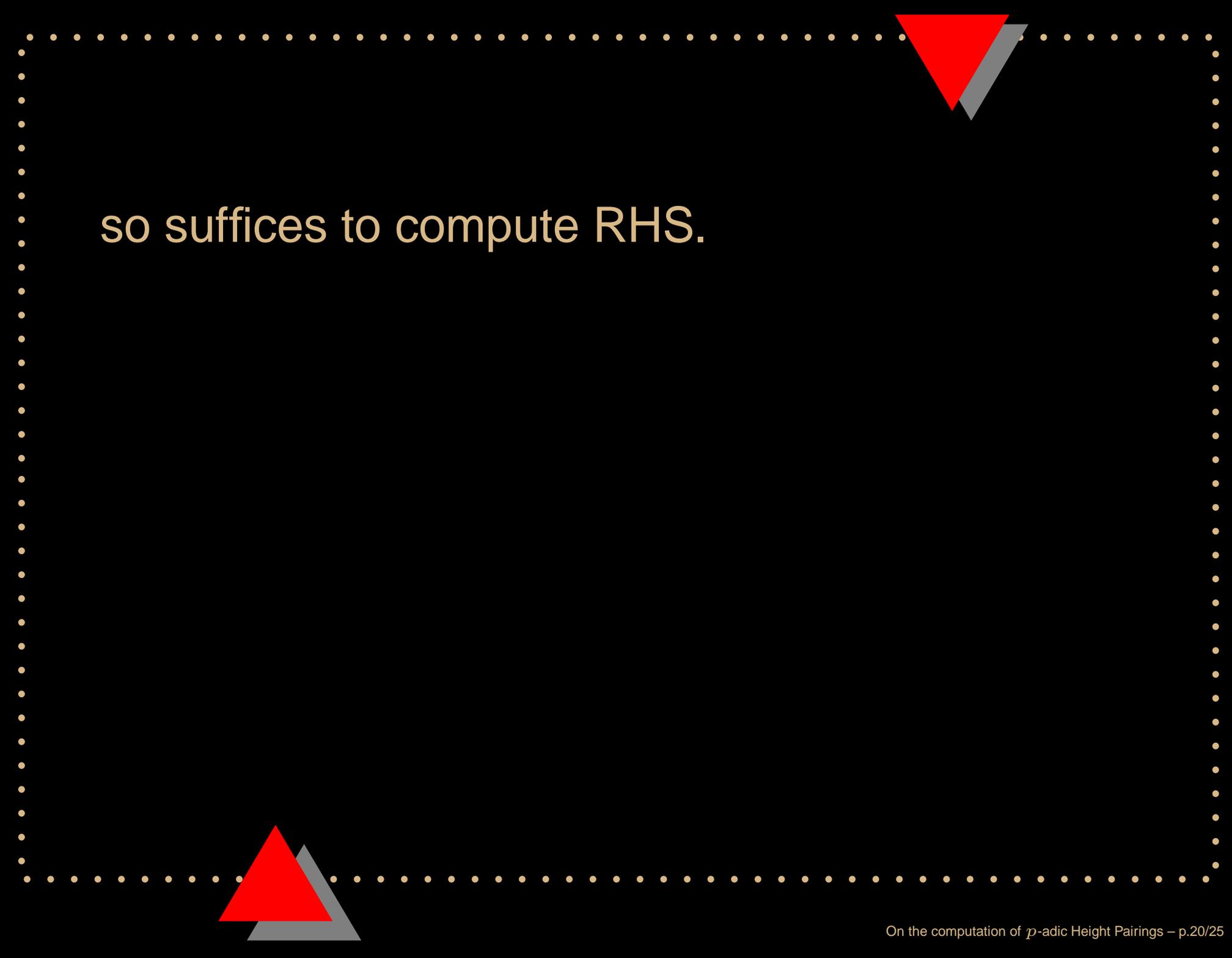
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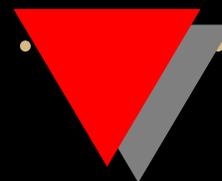
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By the assumptions on F we have

$$\sigma(F(\phi(\sigma^{-1}(P)))) - pF(P) = \frac{1}{2} \int_{-P}^P \eta$$

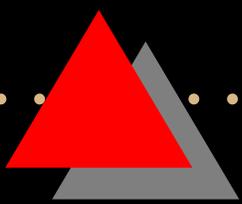


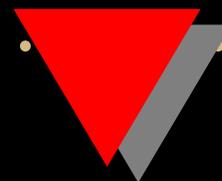
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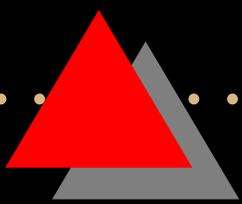


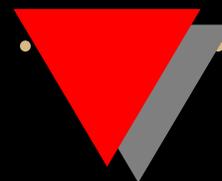
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$$\text{RHS} = \sum \text{Res } \mu \int \eta$$

sum over singular points of μ .

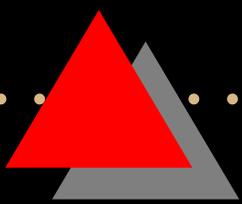


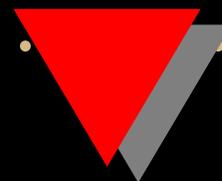


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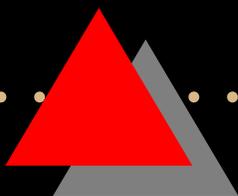


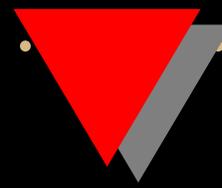
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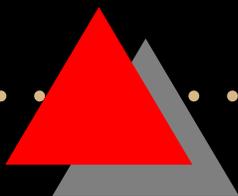
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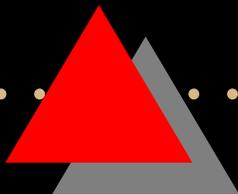
Note: η has an essential singularity on the Weierstrass discs.





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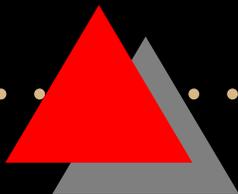


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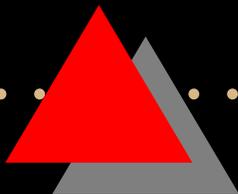
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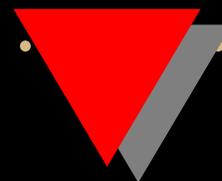
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error depends on degrees of y, z and on the reduction type of C .





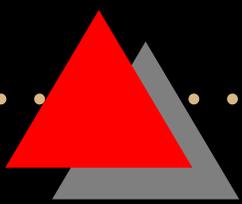
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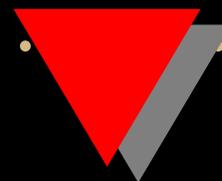


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So:

$$2^n \langle y, z \rangle_v = \langle y', z \rangle_v + v(f(z)) \sim \{y', z\}_v + v(f(z))$$





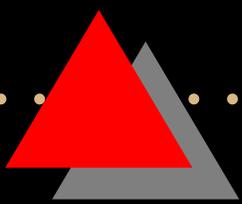
Theory of hyperelliptic curves: effective
 $2^n y = y' + (f)$, with y' of small degree.

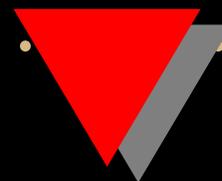
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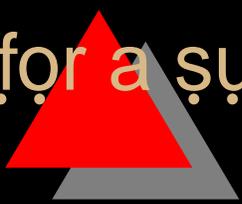
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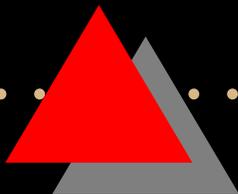
$\langle y, z \rangle_v$ has bounded denominators hence get answer for a sufficiently large n .




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$$4y = 2y_1 + (f_1^2) = y_2 + (f_2) + (f_1^2)$$

$$2^n y = y_n + (f_n) + (f_{n-1}^2) + (f_1^{2^{n-1}})$$



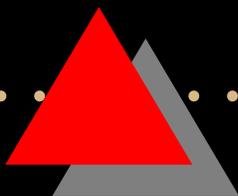

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SO

$$\langle y, z \rangle_v \sim 2^{-n} \{y', z\}_v + \sum v(f_i(z))/2^i$$





Problem

Problem: Hard to determine what's the required n .



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Solution?: Use work of Kausz to bounded number of components in reduction in terms of valuation of discriminant: To $y^2 = f$ associate the discriminant $D = \Delta(f)^g V^{8g+4}$ where V is the co-volume inside $H^0(\tilde{C}, \Omega^1)$ of the module generated by $x^i dx/y$. Then $v(D)$ bounds a weighted sum of the number of singular points of the minimal regular model.