On Convergence in the Sato-Tate Conjecture

William Stein (joint work with Barry Mazur)

Sage Days 5, Clay Math Institute, 2007

Purpose

Find a possible "next question to ask", now that so much is understood about the Sato-Tate conjecture due to work of Taylor, Haris, et al.

Hecke Eigenvalues

Let E be a **non-CM** elliptic curve over $\mathbb{Q}$, and

$$a_p = p + 1 - \#E(\mathbf{F}_p).$$

Theorem (Hasse): $-1 < \dfrac{a_p}{2\sqrt{p}} < 1.$

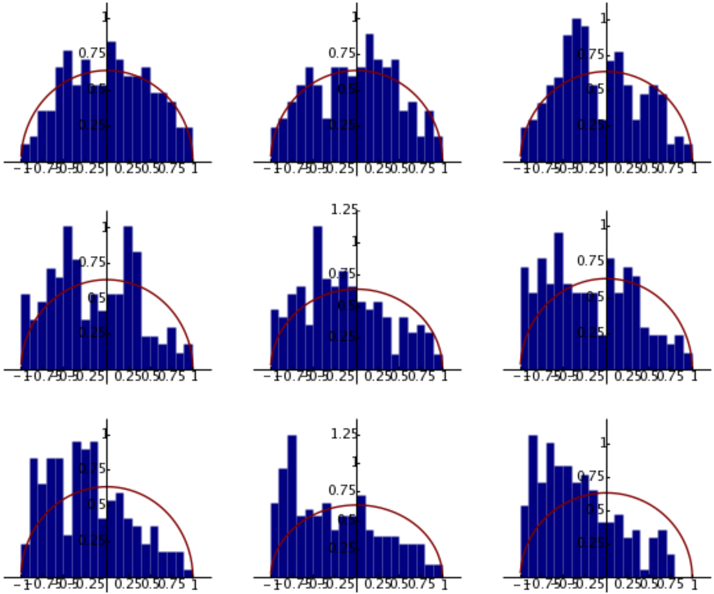
Sato and Tate: How are these numbers distributed? A conjecture...

Convergence to the semicircle distribution

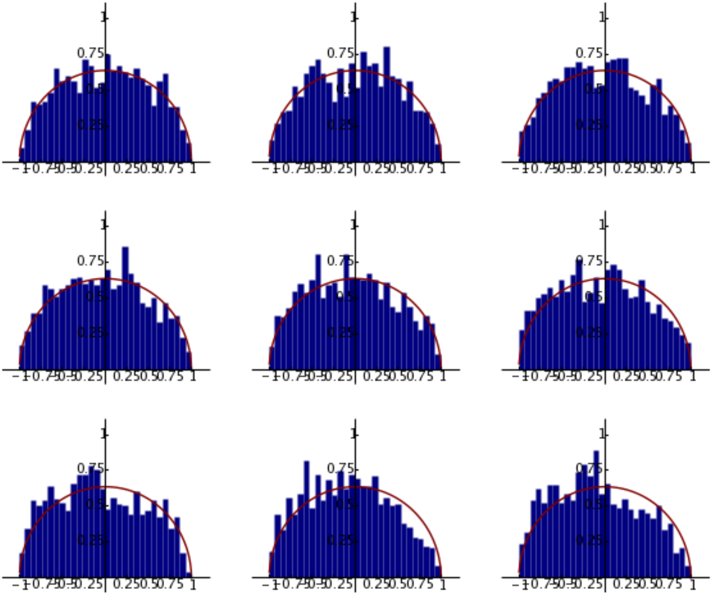
The following slides each contain 8 plots. Each plot displays the distribution of normalized a_p for the lowest conductor elliptic curves of different rank and all a_p for $p < C$, for $C = 10^3, 10^4, 10^5, 10^6$.

Rank 0	Rank 1	Rank 2
Rank 3	Rank 4	Rank 5
Rank 6	Rank 7	Rank 8

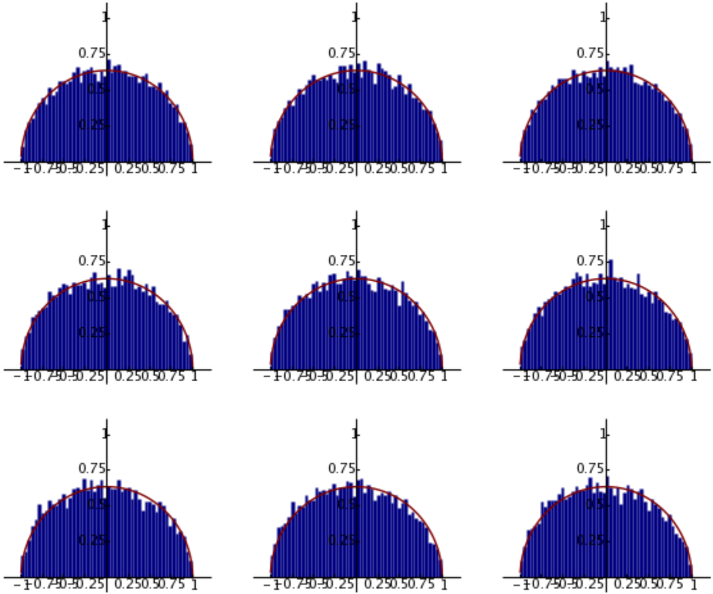
Sato-Tate Frequency Histograms: $C = 10^3$



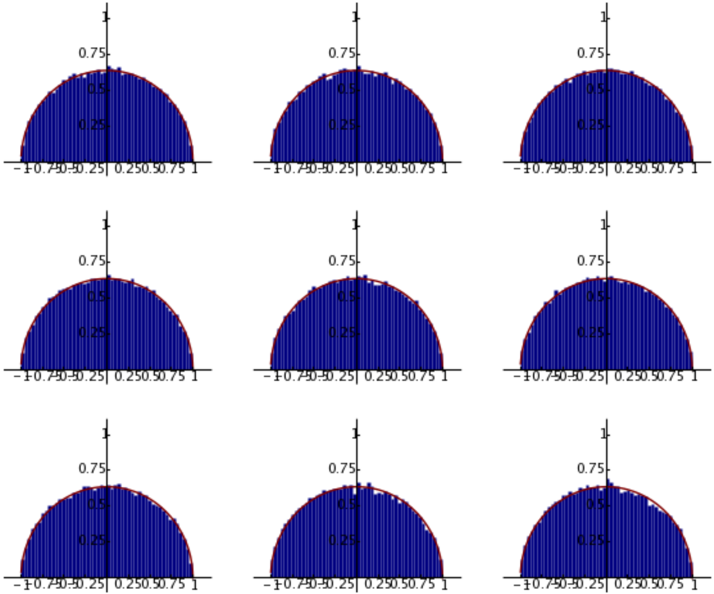
Sato-Tate Frequence Histograms: $C = 10^4$



Sato-Tate Frequence Histograms: $C = 10^5$



Sato-Tate Frequence Histograms: $C = 10^6$



Quantify the convergence?

Barry Mazur: "How can we precisely quantify the convergence of the blue data to the red semicircle theoretical distribution?"

Some Functions (copy on blackboard)

E an elliptic curve; $a_p = p + 1 - \#E(\mathbf{F}_p)$

- $X(T) = \dfrac{\int_{-1}^{T} \sqrt{1-x^2}dx}{\int_{-1}^{1} \sqrt{1-x^2}dx} =$ area under arc of semicircle

- $Y_C(T) = \dfrac{\#\left\{\text{primes } p < C : -1 < \frac{a_p}{2\sqrt{p}} < T\right\}}{\#\{\text{primes } p < C\}}.$

- $\Delta(C) = \sqrt{\int_{-1}^{1} (X(T) - Y_C(T))^2 dT} =$ the L_2-norm of the difference of $X(T)$ and $Y_C(T)$, and $\Delta(C)_\infty$ the L_∞-norm.

The Sato-Tate Conjecture

Let $\Delta(C)_\infty$ be the max of the difference between the theoretical semicircle distribution and actual data using primes up to C.

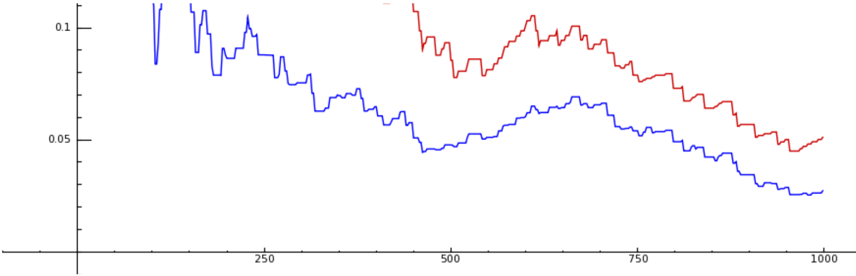
Sato-Tate Conjecture:

$$\lim_{C\to\infty} \Delta(C)_\infty = 0$$

Theorem (Taylor, M. Harris, et al.): If E has multiplicative reduction at some prime, then the Sato-Tate conjecture is true. [Key part of proof is that symmetric power L-functions (see Mark Watkins' talk) are modular.]

Plotting Δ (up to 10^3)

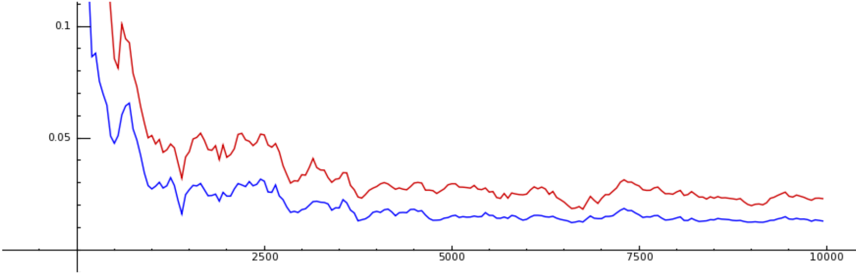
```
sage: e37a = SatoTate(EllipticCurve('37a'), 10^6)
sage: show(e37a.plot_Delta(10^3, plot_points=400,
 max_points=100), ymax=0.1, ymin=0, figsize=[10,3])
```



The red line is $\Delta(C)_\infty$ and the blue line is $\Delta(C)$. By Sato-Tate, they both go to 0 as $C \to \infty$.

Plotting Δ (up to 10^4)

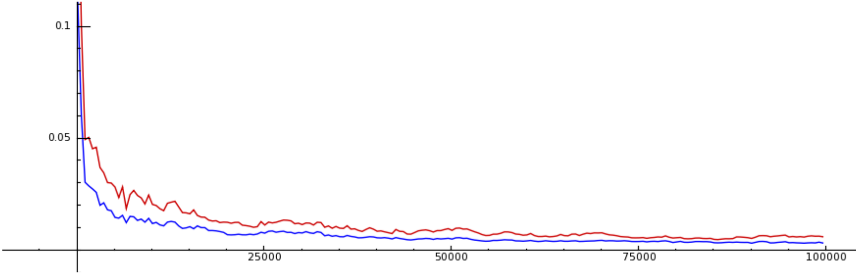
```
sage: e37a = SatoTate(EllipticCurve('37a'), 10^6)
sage: show(e37a.plot_Delta(10^4, plot_points=200,
 max_points=100), ymax=0.1, ymin=0, figsize=[10,3])
```



The red line is $\Delta(C)_\infty$ and the blue line is $\Delta(C)$. By Sato-Tate, they both go to 0 as $C \to \infty$.

Plotting Δ (up to 10^5)

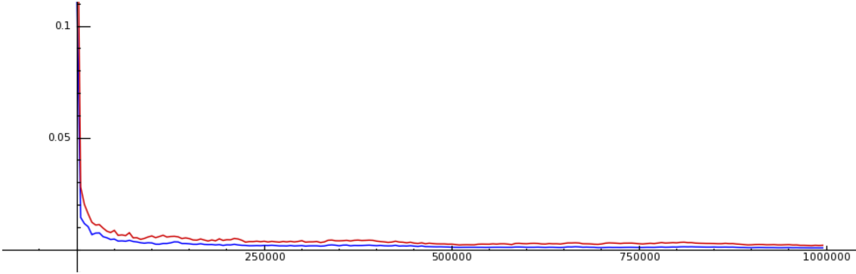
```
sage: e37a = SatoTate(EllipticCurve('37a'), 10^6)
sage: show(e37a.plot_Delta(10^5, plot_points=200,
 max_points=100), ymax=0.1, ymin=0, figsize=[10,3])
```



The red line is $\Delta(C)_\infty$ and the blue line is $\Delta(C)$. By Sato-Tate, they both go to 0 as $C \to \infty$.

Plotting Δ (up to 10^6)

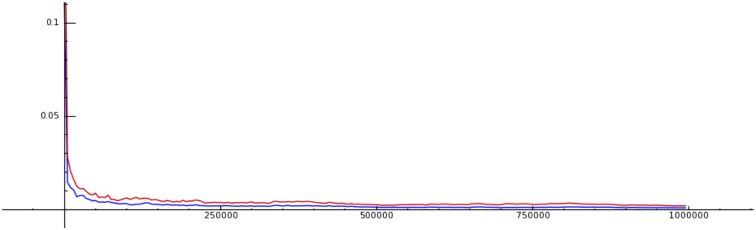
```
sage: e37a = SatoTate(EllipticCurve('37a'), 10^6)
sage: show(e37a.plot_Delta(10^6, plot_points=200,
 max_points=100), ymax=0.1, ymin=0, figsize=[10,3])
```



The red line is $\Delta(C)_\infty$ and the blue line is $\Delta(C)$. By Sato-Tate, they both go to 0 as $C \to \infty$.

“The next question to ask...”

QUESTION: What about the speed of convergence? I.e., *how* does $\Delta(C)$ or $\Delta(C)_\infty$ converge to 0?



The Akiyama-Tanigawa Conjecture

Conjecture (Akiyama-Tanigawa [Math Comp., 1999]): For every $\epsilon > 0$, for $C \gg 0$ we have

$$\Delta(C)_\infty \leqslant \frac{1}{C^{1/2-\epsilon}}.$$

Theorem (A-T): This conjecture implies the Generalized Riemann Hypothesis for $L(E, s)$.

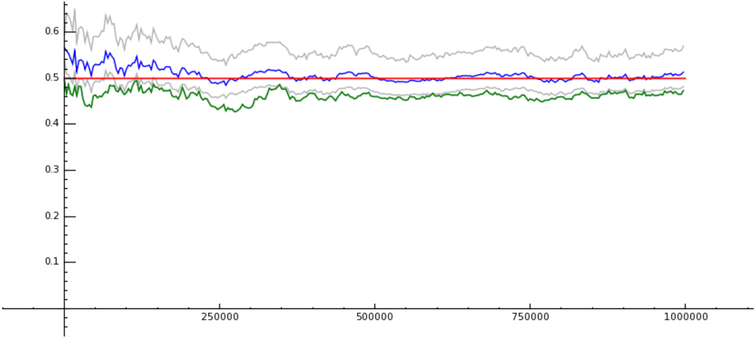
See Barry Mazur's forthcoming Notices paper for more discussion, references, and pretty pictures.

Log Plots

Let's test out Akiyama-Tanigawa, instead of plotting $\Delta(C)$ which just goes to 0 quickly, we instead plot $-\log_C(\Delta(C))$.

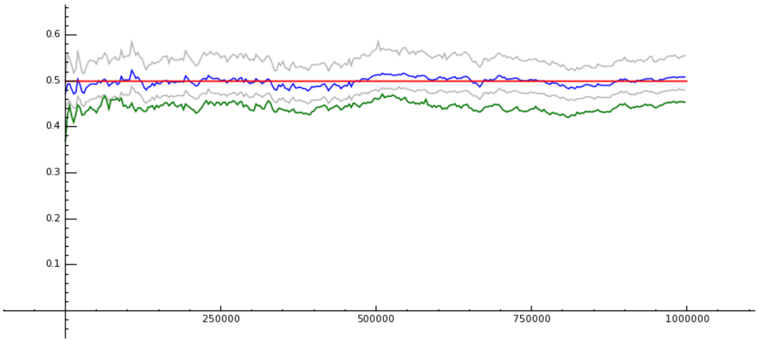
1. How does this function compare to $\dfrac{1}{2}$? I.e., does it eventually get within ϵ of $\frac{1}{2}$.
2. Can we find a simple function that conjecturally nicely approximates $-\log_C(\Delta(C))$?

Rank 0 curve 11a; $p < 10^6$; with 300 sample points



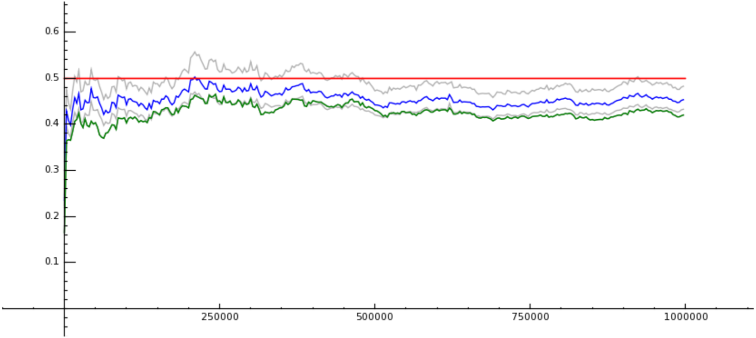
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 1 curve 37a; $p < 10^6$



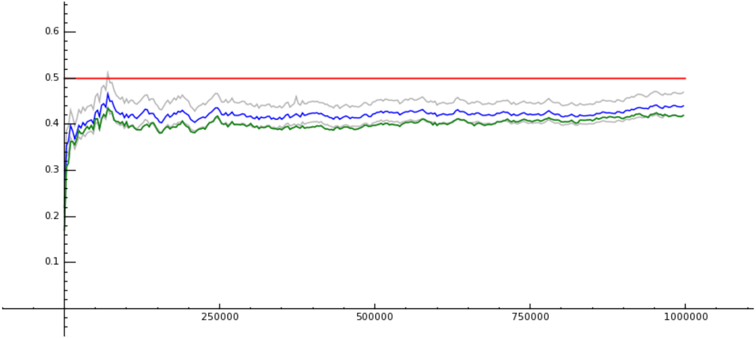
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 2 curve 389a; $p < 10^6$



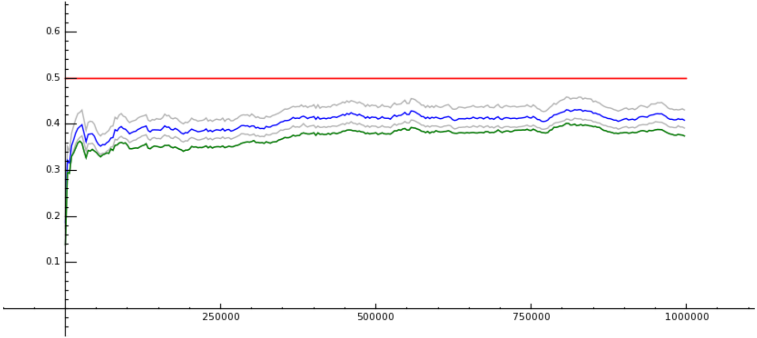
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 3 curve 5077a; $p < 10^6$



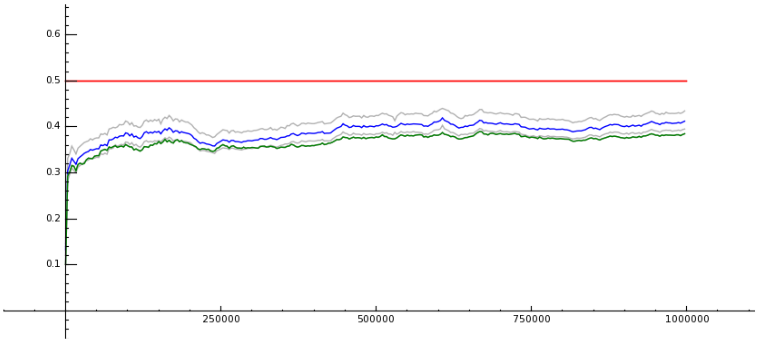
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 4 curve [1,-1,0,-79,289]; $p < 10^6$



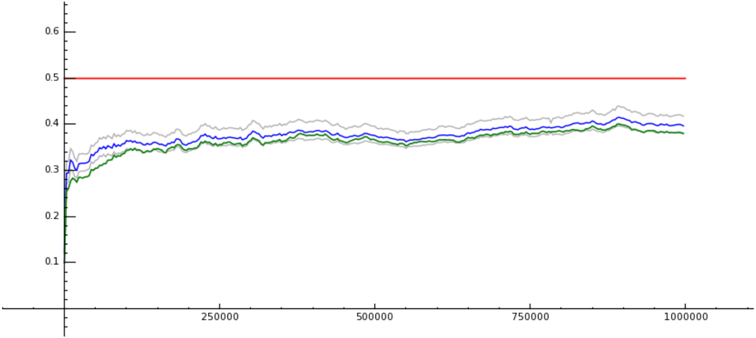
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 5 curve [0, 0, 1, -79, 342]; $p < 10^6$



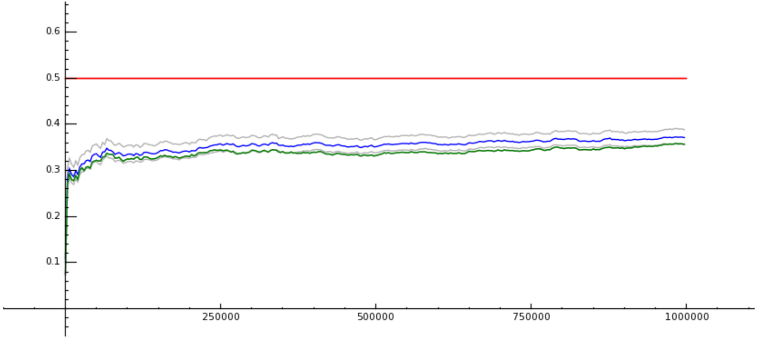
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 6 curve [1, 1, 0, -2582, 48720]; $p < 10^6$



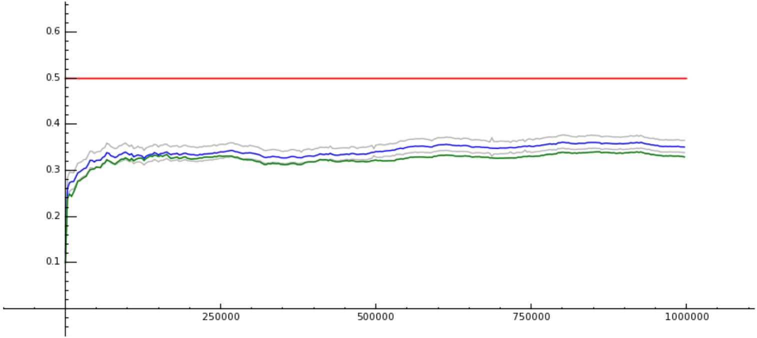
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 7 curve [0, 0, 0, -10012, 346900]; $p < 10^6$



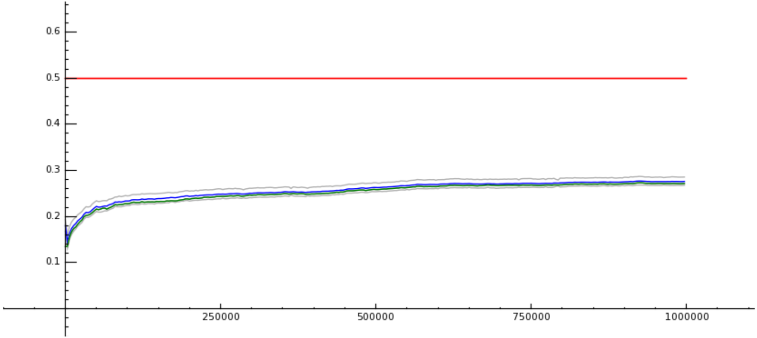
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Rank 8 curve [0, 0, 1, -23737, 960366]; $p < 10^6$



- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

Elkies rank $\geqslant 28$ curve; $p < 10^6$



- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Red line is $1/2$.

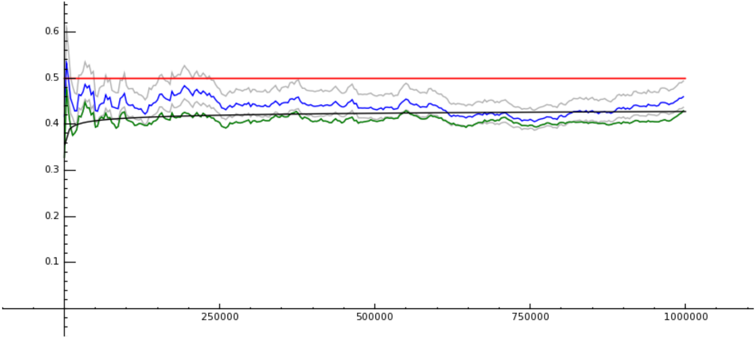
OK, are those lines really going up to $1/2$???

Understanding the Data Better?

Can one predict the asymptotic shape of the curve $\Delta(C)$, say, in terms of either arithmetic invariants of the curve or perhaps in terms of zeros of $L(E, s)$ on the critical strip?

For some curves $\Delta(C)$ is quickly very close to $1/2$, e.g., the curves of rank 0 and 1 above.

Fitting the "random" Rank 0 curve $y^2 = x^3 + 19x + 234$



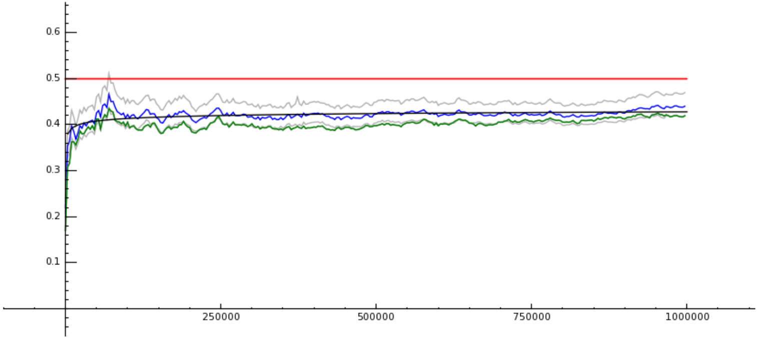
- The black curve is
$$\frac{1}{2} - \frac{1}{\log(X)}.$$
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Conductor = 24093568 $= 2^7 \cdot 41 \cdot 4591$

Low zeros?

```
sage: EllipticCurve('11a').Lseries_zeros(10)
[6.36261389, 8.60353962, 10.0355091,
 11.4512586, 13.5686391, 15.9140726,
 17.0336103, 17.9414336, 19.1857250,
 20.3792605]
```

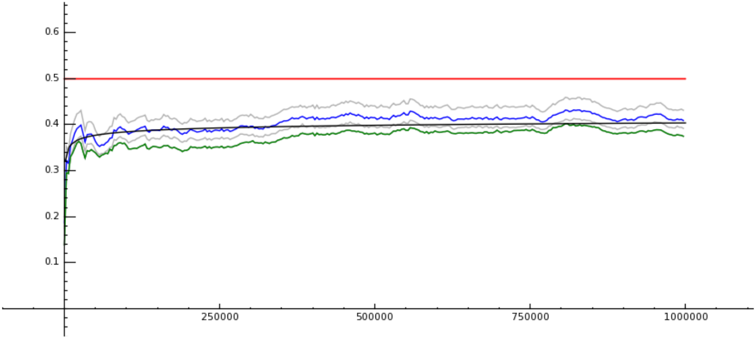
```
sage: EllipticCurve([19,234]).Lseries_zeros(10)
[0.255961213, 0.739839807, 1.03144159,
 1.78804887, 2.11227980, 2.42762599,
 3.11102036, 3.26810134, 3.68155235,
 4.13888170]
```

Fitting the Rank 3 Curve 5077a



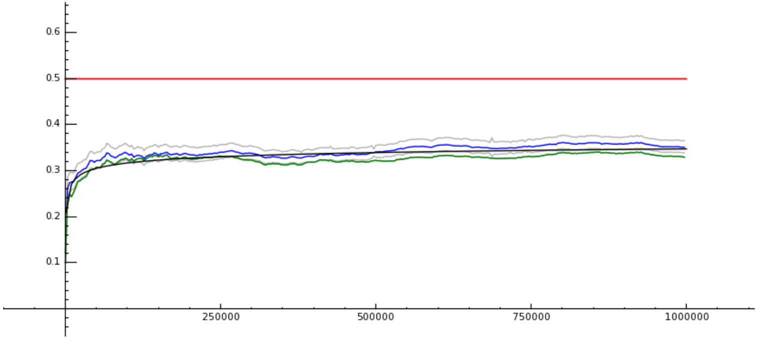
- The black curve is
$$\frac{1}{2} - \frac{3/3}{\log(X)}.$$
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.

Fitting the Rank 4 [1,-1,0,-79,289]; $p < 10^6$



- The black curve is
$$\frac{1}{2} - \frac{4/3}{\log(X)}.$$
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.

Fitting Rank 8 [0, 0, 1, -23737, 960366]; $p < 10^6$

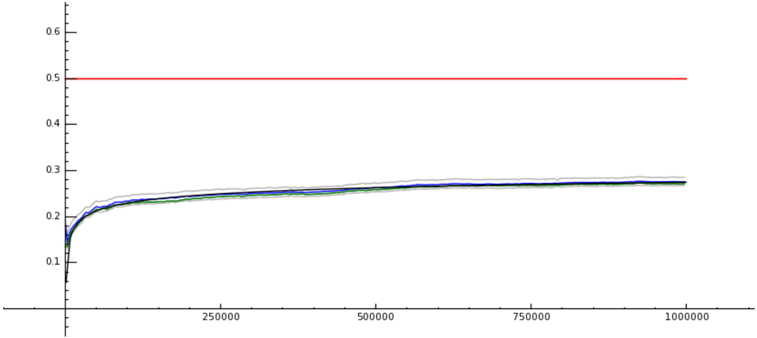


- The black curve is

$$\frac{1}{2} - \frac{19/9}{\log(X)}.$$

- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.

Fitting Rank 28 curve; $p < 10^6$



- The black curve is
$$\frac{1}{2} - \frac{28/9}{\log(X)}.$$
- Green line is $-\log_C(\Delta(C)_\infty)$.
- Blue line is $-\log_C(\Delta(C))$, with a grey tubular *numerical integration error bound*.
- Changing the 28/9 at all moves the black curve *visibly* away from the green and blue plots!

Conjectural convergence of the measure of convergence

Conjecture (Stein): For any E there is a constant α such that

$$\frac{1}{2} - \frac{\alpha}{\log(C)} \leqslant -\log_C(\Delta(C)) \leqslant \frac{1}{2}$$

for all C**.**

This further refines the Akiyama-Tanigawa conjecture about converge of the function $\Delta(C)$ (that measures convergence in the Sato-Tate conjecture). Recall that for all $\epsilon > 0$, AT conjecture that have

$$\Delta(C) \leqslant O\left(\frac{1}{C^{1/2-\epsilon}}\right)$$

$$-\log_C(\Delta(C)) \gg 1/2 - \epsilon$$

The Sato-Tate convergence parameter

For an elliptic curve E let $k(C)$ be the constant that minimizes the L_2 norm of this (i.e. the distance between the black and blue curves above!):

$$\frac{1}{2} - \frac{k(C)}{\log(C)} + \log_C(\Delta(C))$$

Thus $k(C)$ is a function of k.

(I haven't attempted to prove that $k(C)$ exists.)

Definition: The *Sate-Tate convergence parameter* of E is

$$k_E = \lim_{C\to\infty} k(C).$$

(I don't know if this exists. replace by limsup and liminf?)

Challenge: Find a conjectural formula for k_E in terms the critical zeros of $L(E, s)$?

Another future direction...

We have

$$X^{1/2-1/\log(X)} = \frac{X^{1/2}}{X^{1/\log(X)}} = e \cdot X^{1/2}.$$

We thus entertain the possibility (following the format of the people who work with random matrices etc.) that the true distribution is well approximated by something like

$$a \cdot (\log X)^b \cdot X^c$$

for appropriate constants a, b, c.
So for the rank 3 example above we might choose

$$a = e, \qquad b = 0, \qquad c = 1/2,$$

but there may be better choices?

More future direction...

1. Restrict to intervals $[a, b] \subset (-1, 1)$. (This seems to have little to know impact.)
2. Push computations much further (next slide).

Pushing Computations Further

1. Drew Sutherland (an MIT postdoct) has some amazingly fast *multithreaded* code for computing all a_p for $p < C$ quickly (and much much more – over 20,000 lines of new (pure) C code.
2. On sage.math his code computes all a_p for $p < C = 10^7$ in less than 5 seconds!
3. For comparison, $C = 10^7$ takes Sage (via PARI) 94 seconds and Magma (via M Watkins' code) 81.25 seconds (on sage.math, a 16-core opteron 246.).
4. Drew: "My guess then is that on an idle system it would take about 5 minutes to do p to 10^9."