Zeta Functions, Point Counting, and Mirror Symmetry

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Arithmetic Mirror Symmetry?

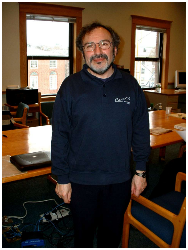

Figure: Philip Candelas

Figure: Xenia de la Ossa

Figure: Fernando Rodriguez Villegas

- Theoretical physicists have made conjectures about the number of points on certain varieties over finite fields.
- The motivation comes from mirror symmetry.

Building a Model

Locally, space-time should look like

$$M_{3,1} \times V.$$

- $M_{3,1}$ is four-dimensional space-time
- V is a d-dimensional complex manifold
- Physicists require $d = 3$ (6 real dimensions)
- V is a Calabi-Yau manifold

A-Model or B-Model?

Choosing Complex Variables

- $z = a + ib$, $w = c + id$
- $z = a + ib$, $\overline{w} = c - id$

Mirror Symmetry

Physicists say . . .

- Calabi-Yau manifolds appear in pairs (V, V°).
- The universes described by $M_{3,1} \times V$ and $M_{3,1} \times V^\circ$ have the same observable physics.

Mirror Symmetry for Mathematicians

The physicists' prediction led to mathematical discoveries!

Mathematicians say . . .

- Calabi-Yau manifolds appear in paired families $(V_\alpha, V_\alpha^\circ)$.
- The families V_α and V_α° have dual geometric properties.

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- Resolve singularities in the quotient X_t/G to obtain Y_t
- Y_t is the mirror family to smooth quartics in $\mathbb{P}^3$
- Smooth quartics in $\mathbb{P}^3$ have many complex deformation parameters; Y_t has 1

The residue map

We will use a residue map to describe the cohomology of a K3 hypersurface X:

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Let Ω_0 be a holomorphic 3-form on $\mathbb{P}^3$. We may represent elements of $H^3(\mathbb{P}^3 - X)$ by forms $\frac{m\Omega_0}{f^k}$, where m is a homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_3]$ of degree 0, 4, or 8, and $k = (\deg\, m)/4 + 1$. Then:

$$\mathrm{Res}(\frac{m\Omega_0}{f^k}) \in H^{(3-k,k-1)}(X).$$

The Griffiths-Dwork technique

Procedure

Suppose we have a pencil of K3 hypersurfaces X_t in $\mathbb{P}^3$.

1.

$$\frac{d}{dt}\int \operatorname{Res}\left(\frac{P\Omega}{f^k(t)}\right) = \int \operatorname{Res}\left(\frac{d}{dt}\left(\frac{P\Omega}{f^k(t)}\right)\right)$$
$$= -k\int \operatorname{Res}\left(\frac{f'(t)P\Omega}{f^{k+1}(t)}\right)$$

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2. Since $H^*(X_t, \mathbb{C})$ is a finite-dimensional vector space, only finitely many of the classes $\mathrm{Res}\left(\frac{d^j}{dt^j}\left(\frac{\Omega}{f^k(t)}\right)\right)$ can be linearly independent

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3. Use the reduction of pole order formula to compare classes of the form $\mathrm{Res}\left(\frac{P\Omega}{f^{k+1}(t)}\right)$ to classes of the form $\mathrm{Res}\left(\frac{Q\Omega}{f^k(t)}\right)$

Picard-Fuchs Equations for the Holomorphic Form

The Picard-Fuchs differential equation satisfied by the period of the holomorphic form is:

$$\left((t^4-1)\frac{d^3}{dt^3}+6t^3\frac{d^2}{dt^2}+7t^2\frac{d}{dt}+t\right)\int\omega=0.$$

If we set $\lambda=t^4$ and $\theta=\lambda\frac{d}{d\lambda}$, we obtain a generalized hypergeometric equation:

$$\left(\theta(\theta-1/4)(\theta-1/2)-\lambda(\theta+1/4)^3\right)\int\omega=0.$$

Hypergeometric Functions

Definition

Let $A, B \in \mathbb{N}$. A hypergeometric function is a function on $\mathbb{C}$ of the form:

$$
\begin{aligned}
{}_AF_B(\alpha;\beta|z) &= {}_AF_B(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B|z) \\
&= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_A)_k}{(\beta_1)_k\cdots(\beta_B)_k k!} z^k,
\end{aligned}
$$

where $\alpha \in \mathbb{Q}^A$ are *numerator parameters*, $\beta \in \mathbb{Q}^B$ are *denominator parameters*, and the Pochhammer notation is defined by

$$
(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.
$$

Solving the Picard-Fuchs Equation

The solution to the Picard-Fuchs equation for the holomorphic form is a generalized hypergeometric function:

$$ {}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1\,\middle|\,t^{-4}\right). $$

Fermat Monomials

Let's consider the Fermat quartic pencil X_t, as described by polynomials f_t.

- We may represent a homogeneous monomial $x^a y^b z^c w^d$ by the 4-tuple (a, b, c, d).
- We can classify a monomial m using the action of $G = (\mathbb{Z}/(4))^2$ and the Griffiths-Dwork derivative $\frac{d}{dt} \int \mathrm{Res}\left(\frac{m\Omega}{f_t^k}\right)$.

Classifying Monomials

Up to permutations of the variables, we have three types of equivalence classes:

- 1 class:

$$(0, 0, 0, 0), (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3)$$

- 3 classes:

$$(0, 0, 2, 2), (1, 1, 3, 3), (2, 2, 0, 0), (3, 3, 1, 1)$$

- 12 classes:

$$(0, 1, 1, 2), (1, 2, 2, 3), (2, 3, 3, 0), (3, 0, 0, 1)$$

More Picard-Fuchs Equations

Each type of monomial equivalence class yields a Picard-Fuchs equation of a different sort.

- 1 class: 3rd-order differential equation
- 3 classes: 2nd-order differential equation
- 12 classes: 1st-order differential equation

The Congruent Zeta Function

- Let $X/\mathbb{F}_q$ be an algebraic variety over the finite field of $q = p^s$ elements.
- Let $N_s(X) = \#X(\mathbb{F}_{q^s})$ be the number of $\mathbb{F}_{q^s}$-rational points on X.

Definition

The Zeta function of X is

$$Z(X/\mathbb{F}_q, T) := \exp\left(\sum_{s=1}^{\infty} N_s(X)\frac{T^s}{s}\right) \in \mathbb{Q}[[T]].$$

Dwork and the Weil Conjectures

- $Z(X/\mathbb{F}_q, T)$ is rational
- We can factor $Z(X/\mathbb{F}_q, T)$ using polynomials with integer coefficients:

$$Z(X/\mathbb{F}_p, T) := \frac{\prod_{j=1}^{n} P_{2j-1}(T)}{\prod_{j=0}^{n} P_{2j}(T)},$$

- $\dim_{\mathbb{C}} X = n$
- $P_0(t) = 1 - T$ and $P_{2n}(T) = 1 - p^n T$
- For $1 \leq j \leq 2n-1$, $\deg P_j(T) = b_j$, where $b_j = \dim H^j_{dR}(X)$.

The Fermat quartic pencil

Let X_t be the Fermat quartic pencil. Xenia de la Ossa and Shabnam Kadir (building on results of Dwork) showed:

$$Z(X_t/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-pT)(1-p^2T)Q_t(T)}$$

$$Q_t(T) = R_{(0,0,0,0)}(T)R^3_{(0,0,2,2)}(T)R^{12}_{(0,1,1,2)}(T)$$

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where

- $R_{(0,0,0,0)}(T) = (1 \pm pT)(1 - a_t T + p^2 T)$
- $R_{(0,0,2,2)}(T) = (1 \pm pT)(1 \pm pT)$
- $R_{(0,1,1,2)}(T) =$
  $\begin{cases} [(1-pT)(1+pT)]^{1/2} & \text{when } p \equiv 3 \bmod 4 \\ (1 \pm pT) & \text{otherwise} \end{cases}$

Mirror Quartics

Let Y_t be the mirror family to quartics in $\mathbb{P}^3$ (constructed using Greene-Plesser and the Fermat pencil). Then de la Ossa and Kadir showed:

$$Z(Y_t/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-pT)^{19}(1-p^2T)R_{(0,0,0,0)}(T)}.$$

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The factor $R_{(0,0,0,0)}(T)$ corresponds to periods of the holomorphic form and its derivatives, and is invariant under mirror symmetry.

Zeta function and monomial equivalence classes

- The 1, 3, and 12 monomial equivalence classes correspond to the factors of the zeta function.
- The orders of the corresponding Picard-Fuchs equations correspond to the degrees of the polynomials in each factor.

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Kloosterman's Work

Kloosterman ('07) gives a general explanation of the factorization of zeta functions of monomial deformations of Fermat varieties using Monsky-Washnitzer cohomology. He builds on work by Candelas, de la Ossa, & Rodriguez-Villegas and Kadir & Yui.

Counting Points

We can use group actions to count points on the Fermat pencil and the alternate pencils.

- Let $N(t)$ be the number of points on a hypersurface X_t over $\mathbb{F}_q$, where $q = p^s$
- Let $N^*(t)$ be the number of points where all coordinates are nonzero

Point Counts and Hypergeometric Functions

For the Fermat pencil, there is a relationship between the point count and the truncation of the solution to the Picard-Fuchs equation:

$$N_{\mathbb{F}_p}(t) - N_{\mathbb{F}_p}(0) \equiv \left[{}_3F_2 \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1 \middle| \, t^{-4} \right) \right]_0^{\frac{p-1}{4}-1} \mod p$$

Here, if $u(z) = \sum_{n=0}^{\infty} a_n$, $[u(z)]_i^j$ is the truncation $\sum_{n=i}^{j} a_n$.

Gauss Sums

- Let $\chi_{1/(q-1)} : \mathbb{F}_q^* \to K^*$ be a fixed generator of the character group of $\mathbb{F}_q^*$, where K is $\mathbb{C}$ or $\mathbb{C}_p$.

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- Let $\psi : \mathbb{F}_q \to K^*$ be a (fixed) additive character.
- For $s \in \frac{1}{(q-1)}\mathbb{Z}/\mathbb{Z}$ we let $g(s)$ denote the Gauss sum

$$g(s) = \sum_{x \in \mathbb{F}_q} \chi_s(x)\psi(x).$$

Finite Field Analogues

We can think of Gauss sums as the finite field analogue of the Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^x \frac{dt}{t}.$$

More generally, let $\alpha_1, \ldots, \alpha_A, \beta_1, \ldots, \beta_B \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}$. Katz defines the finite field analogue of a hypergeometric function as:

$$H(\alpha; \beta | t) = \frac{1}{q-1} \sum_{s \in \frac{1}{q-1}\mathbb{Z}/\mathbb{Z}} g(s+\alpha_1) \cdots g(s+\alpha_n)$$

$$\cdot g(-s-\beta_1) \cdots g(-s-\beta_m) \overline{\chi_s}(t)$$

Point Counting for the Fermat Quartic Pencil

$$N_{\mathbb{F}_p}(t) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^4}{g(4s)} \chi_{4s}(4t)$$

$$+ \frac{3}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s)^2 g(s+\frac{1}{2})^2}{g(4s)} \chi_{4s}(4t)$$

$$+ \frac{12}{p-1} \sum_{s \in \frac{1}{p-1}\mathbb{Z}/\mathbb{Z}} \frac{g(s) g(s+\frac{1}{4})^2 g(s+\frac{1}{2})}{g(4s)} \chi_{4s}(4t)$$

Point Counting, Monomials, and Picard-Fuchs Equations

There are three terms in the expression for $N_{\mathbb{F}_p}(t) - N_{\mathbb{F}_p}(0)$, with coefficients 1, 3, and 12, respectively.

- The terms correspond to our equivalence classes of monomials.
- Each of the sums yields an approximation to the solution of the Picard-Fuchs equation for the corresponding monomials.

The Hodge Diamond

Calabi-Yau Threefolds

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{2,1}(X) & & h^{2,1}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

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 & & & 1 & & &
\end{array}
$$

If X and X° are mirror, $h^{2,1}(X) \cong h^{1,1}(X^\circ)$ and $h^{1,1}(X) \cong h^{2,1}(X^\circ)$.

Arithmetic Mirror Symmetry for Threefolds

If X and X° are mirror Calabi-Yau threefolds, we can expect a relationship between $Z(X/\mathbb{F}_q, T)$ and $Z(Y/\mathbb{F}_q, T)$ due to the interchange of Hodge numbers.

We know:

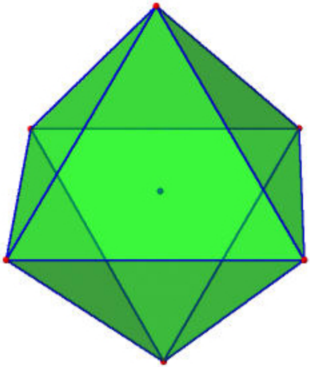
- $$Z(X, p, T) = \frac{cP_3(T)}{(1-T)(1-p^3T)P_2(T)P_4(T)}$$
- $\deg P_2(T) = \deg P_4(T)$

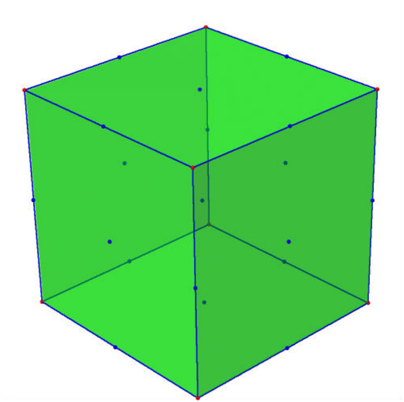
Mirror symmetry implies:

- $\deg P_2 + 2 = \deg P_3^\circ$
- $\deg P_2^\circ + 2 = \deg P_3$

Batyrev's Insight

We can describe mirror families of Calabi-Yau manifolds using objects called reflexive polytopes.





Toric Experimentation?

- For mirror pairs of Calabi-Yau threefolds in (weighted) projective spaces, the zeta functions have a common factor.
- This phenomenon can be studied using reflexive simplices.
- For other reflexive polytope pairs, we have mirror families but not necessarily mirror pairs of varieties.