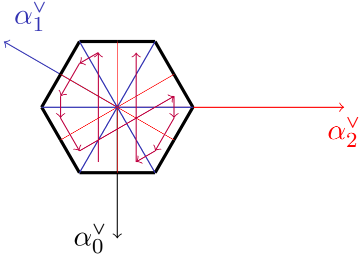
Some Algebraic Combinatorics in Sage

Anne Schilling, UC Davis

Sage Days 60 in Chennai, India
August 15, 2014



A Story

One of my passions are

crystal bases which provide a combinatorial tool to study algebraic/geometric structures such as

- quantum groups
- affine Schubert calculus
- symmetric functions
- representation theory

Combinatorics lends itself to computational analysis!

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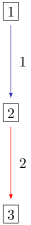
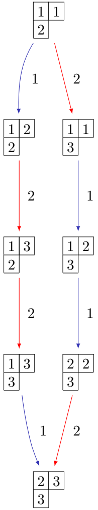
Crystals

$B(\Lambda_1)$



$B(\Lambda_1 + \Lambda_2)$



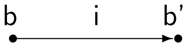
Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt} : B \to P$$
$$e_i, f_i : B \to B \cup \{\varnothing\} \quad \text{for all } i \in I$$

satisfying

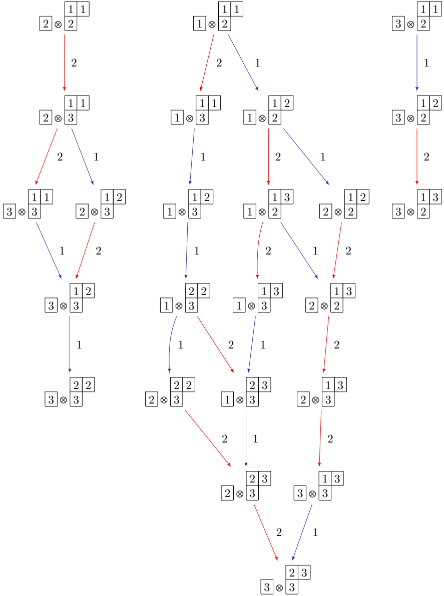
$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$
$$\langle h_i , \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$$

Write 

for $b' = f_i(b)$

Tensor products of crystals

$$B(\Lambda_1) \otimes B(\Lambda_1 + \Lambda_2)$$



Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\begin{array}{ccc} b & \otimes & b' \\ \underbrace{- - -}_{\varphi_i(b)} & & \underbrace{+ + ++}_{\varepsilon_i(b')} \end{array}$$

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Littlewood-Richardson rule in terms of crystals

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu} c_{\lambda\mu}^{\nu} V(\nu)$$

$c_{\lambda\mu}^{\nu}$ = LR coefficient

Theorem (Kashiwara-Nakashima)

$c_{\lambda\mu}^{\nu}$ is the number of highest weight vectors in $B(\lambda) \otimes B(\mu)$ of weight ν.

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Demazure crystals

Littelmann conjectured/ Kashiwara proved that there is a subset $B_w(\Lambda)$ (Demazure crystal) of $B(\Lambda)$ s.t.

$$\sum_{b \in B_w(\Lambda)} b = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_\ell} u_\Lambda$$

where

1. $i_1 \dots i_\ell$ is a reduced word of w
2. $$\mathcal{D}_i b = \begin{cases} \sum_{0 \le k \le \langle h_i, \mathrm{wt}(b) \rangle} f_i^k b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle \ge 0 \\ -\sum_{1 \le k < -\langle h_i, \mathrm{wt}(b) \rangle} e_i^k b & \text{if } \langle h_i, \mathrm{wt}(b) \rangle < 0. \end{cases}$$

Corollary

$\chi(B_w(\Lambda))$ is the Demazure character.

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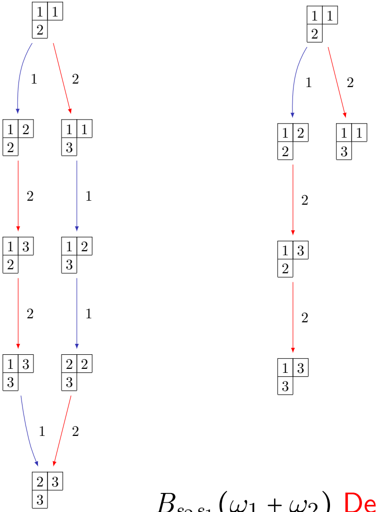
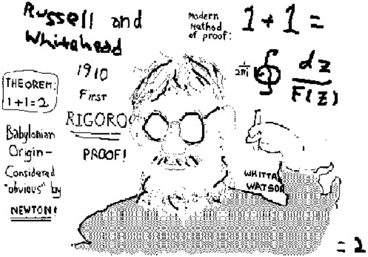


$B_{s_2 s_1}(\omega_1 + \omega_2)$ Demazure crystal
vertices given by $\{f_2^a f_1^b(u_{\omega_1+\omega_2}) \mid a, b \geq 0\}$

Sage Days 7 at IPAM in 2008

with Nicolas Thiéry
started porting crystal code to Sage



Dan Bump
uses crystals in number theory

Moral of the Story ...

End/beginning of the Story ...

Semester long program at ICERM on
`Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series`, Spring 2013

`Thematic Tutorial`

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Thematic Tutorial

Rogers–Ramanujan identitites

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

Polynomial version

$$M(L) = \sum_{n=0}^{\infty} q^{n^2} \begin{bmatrix} L-n \\ n \end{bmatrix}$$

Path interpretation

$$X(L) = \sum_{p \text{ path of length } L} q^{E(p)}$$

$$X(L) = M(L)$$

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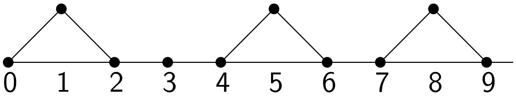

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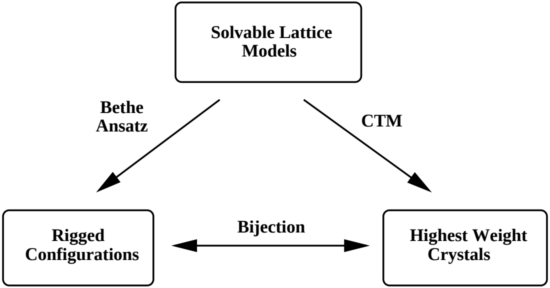
Generalized Rogers–Ramanujan identities and crystals



1988 Identity for Kostka polynomials Kerov, Kirillov, Reshetikhin
2001 $X = M$ conjecture of HKOTTY

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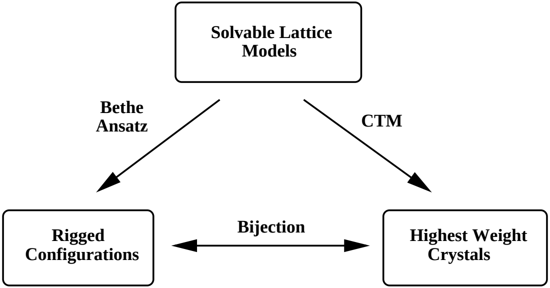

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- anyone can contribute!

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- Sage project began by William Stein in 2005
  SAGE="Software for Arithmetic Geometry Experimentation"
- Quickly expanded beyond number theory; attracted more users, developers, funding
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Sage-combinat: **"To improve the open source mathematical system** Sage **as an extensible toolbox for computer exploration in (algebraic) combinatorics, and foster code sharing between researchers in this area."**

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