

# number\_theory\_talk

## Analytic Number Theory with Sage

Analytic number theory begins with estimations on Prime Numbers.

The Prime Number Theorem says  $\pi(x) := |\{\text{primes} \leq x\}|$  is asymptotic to  $\frac{x}{\log(x)}$ .

In sage we don't need to write a program for values of  $\pi(x)$  as there is already this built in function available!

**File:** /usr/lib/sagemath/devel/sage/sage/functions/prime\_pi.pyx

**Type:** <type 'sage.functions.prime\_pi.PrimePi'>

**Definition:** prime\_pi(x, mem\_mult=1)

**Docstring:**

Return the number of primes  $\leq x$ .

EXAMPLES:

```
sage: prime_pi(7)
4
sage: prime_pi(100)
25
sage: prime_pi(1000)
168
sage: prime_pi(100000)
9592
sage: prime_pi(0.5)
0
sage: prime_pi(-10)
0
sage: prime_pi(500509)
41581
```

The following test is to verify that ticket #4670 has been essentially resolved:

```
sage: prime_pi(10**10)
455052511
```

The prime\_pi function allows for use of additional memory:

```
sage: prime_pi(500509, 8)
41581
```

The prime\_pi function also has a special plotting method, so it plots quickly and perfectly as a step function:

```
sage: P = plot(prime_pi, 50, 100)
```

```
prime_pi(12.5)
```

5

Primes less or equal to 12.5 are 2, 3, 5, 7 and 11. We can make a plot for this function in Sage that can interact.

```
@interact
def _(n=('Range', 300)):
    show(line([(i, prime_pi(i)) for i in srange(1, n, 0.3)], figsize=4,
color='red'))
```

We may compare the graph of  $\pi(x)$  with  $\frac{x}{\log(x)}$  by plotting it together.

```
@interact
```

```
def _(n=('Range', 100)):
    x=var('x')
    show(line([(i, prime_pi(i)) for i in srange(1, n, 0.3)], figsize=6,
color='red', legend_label='$\pi(x)$')
+ plot(x/log(x), 2, n, figsize=6, legend_label='$\frac{x}{\log(x)}$'))
```

We see, even though the graphs of  $\pi(x)$  and  $\frac{x}{\log(x)}$  grow alike, they differ a lot too. This is because  $\pi(x) = \frac{x}{\log(x)} + O\left(\frac{x}{\log^2(x)}\right)$

But there is another function that approximate  $\pi(x)$  better. It is the logarithmic integral  $Li(x) := \int_1^x \frac{1}{\log(t)} dt$ . This function is a also built in function in sage.

We will see how the plots of these functions look together.

```
@interact
def _(n=('Range', 200)):
    x=var('x')
    show(line([(i, prime_pi(i)) for i in srange(1, n, 0.3)], figsize=6,
color='red', legend_label='$\pi(x)$')
+ plot(li(x), 2, n, figsize=6, color='green', legend_label='$Li(x)$') +
plot(x/log(x), 2, n, figsize=6, legend_label='$\frac{x}{\log(x)}$'))
```

We see the function  $Li(x)$  and  $\pi(x)$  are very close to each other. But which one is greater ? Let's do some experiment to find out what is really happening. For this we may plot  $Li(x) - \pi(x)$ .

```
@interact
def _(f=('from', 2), t=('to', 100)):
    x=var('x')
    show(line([(x, li(x)-prime_pi(x)) for x in srange(f, t, 0.2)],
figsize=6))
```

The above graph suggests  $\forall x \geq 2$  the difference  $Li(x) - \pi(x) \geq 0$ . This gives an interesting problem to solve:

Can we find sufficiently large  $x$  such that  $x > 2$  such that  $Li(x) - \pi(x) < 0$  ?

What can we say about  $\lim_{x \rightarrow \infty} Li(x) - \pi(x)$  and  $\overline{\lim}_{x \rightarrow \infty} Li(x) - \pi(x)$  ?

Littlewood proved that  $\pi(x) > Li(x)$  infinitely often. Skewes proved that the first such number

is less than  $10^{10^{10^{34}}}$ . This number is called Skewes' number.

Now we will see, we can go even further in Analytic Number Theory with Sage. Because we can work in complex field and many complex valued functions appearing in Number theory are already implemented in sage. For example the Riemann Zeta Function  $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$  for  $\text{Re}(s) > 1$  and defined elsewhere in  $\mathbb{C} - \{1\}$  by analytic continuation.

```
CC(zeta(3))
```

```
1.20205690315959
```

```
zeta(1) # zeta has a pole at 1.
```

```
Infinity
```

```
CC(zeta(2+10000*I))
```

```
0.922315539900211 - 0.258362555325381*I
```

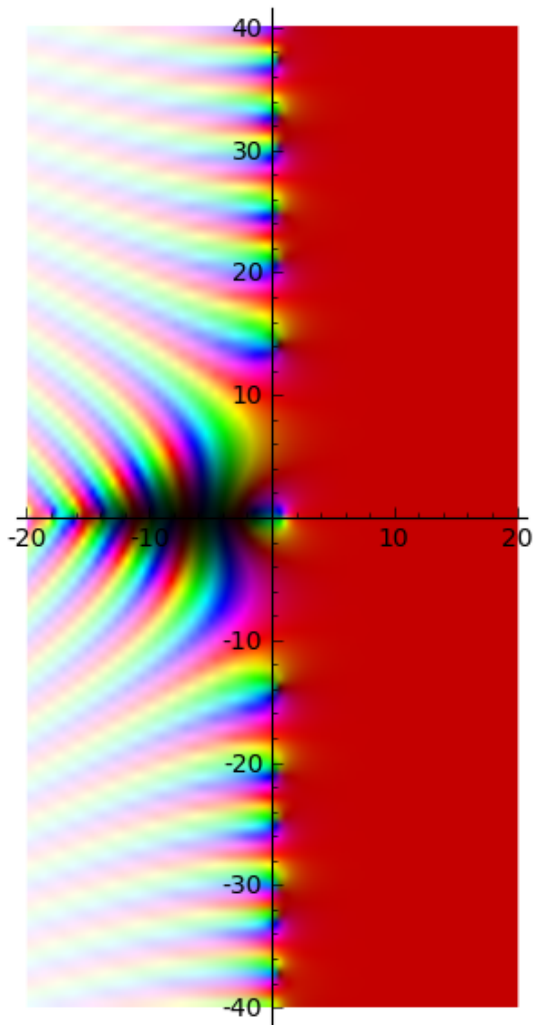
$\zeta(s)$  has trivial zeros at even negative even integers which we can see in the graph below.

```
@interact
def _(f=('from', -13), t=('to', -1), fn=('function', 0)):
    x=var('x')
    show(plot(zeta(x), f, t, figsize=6) + plot(fn, f, t, color='red',
figsize=6))
```

In the above plot we may experiment with the order of decay of the Riemann zeta function along positive real axis ( $> 1$ ). we may also experiment with the oscillation on negative real axis.

Below is a plot of zeta function  $\zeta(\sigma + it)$  when  $\sigma \in [-20, 20]$  and  $t \in [-20, 20]$ . Here brightness characterize the magnitude and hue characterize argument.

```
complex_plot(zeta, (-20, 20), (-40, 40) )
```



```
@interact
def _(a=('abscissa', 1/2), f=('from', -20), t=('to', 20)):
    z=var('z')
    show(line([(real(zeta(a+I*z)), imag(zeta(a+I*z))) for z in srange(f, t,
0.01)], figsize=6))
```

## Multiplication Table Problem

Let  $S_N := \{n : n = ab, 1 \leq a \leq N, 1 \leq b \leq N\}$ .

What can we say about  $\frac{|S_N|}{N^2}$ ? Is it  $o(1)$  as  $N \rightarrow \infty$ ?

This question was first asked by Erdos and he also gave an answer.

Below is a plot of  $\frac{|S(N)|}{N^2}$ .

```
def mult_growth(st, end):
    return [(i, len(mult_list(i))) for i in range(st, end+1)]
```

```

def mult_list(n):
    L=[i*j for i in range(1, n+1) for j in range(1, n+1)]
    return list(Set(L))
@interact
def mult_growth_graph(st=20, end=400): #f=1/x):
    L=mult_list(st, end)
    show(line([(A[0], (float(A[1]))/(float(A[0]))^2) for A in L ],
ymin=0.25)) #+ plot(f, st, end))

```

The above graph suggests that  $\frac{|S(N)|}{N^2} = o(1)$ . Erdos proved that  $\frac{|S(N)|}{N^2} = O\left(\frac{1}{(\log N)^\delta \sqrt{\log \log N}}\right)$ .

Here  $\delta$  is a constant having value 0.0806... Later in 2008 Kevin Ford proved that

$$|S(N)| \asymp \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}.$$

But there is no asymptotic formula known for this problem. This is an interesting Sage project to try ! For example can we find an upper and lower bound to this constant, i.e. find  $c_1, c_2$  such that

$$c_1 \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}} \leq |S(N)| \leq c_2 \frac{N^2}{(\log N)^\delta (\log \log N)^{3/2}}$$

Another interesting project is to approximate  $\delta$  to few more decimal places (without having the expression for delta as in Ford's paper ).

In my research I am interested in what happens if we multiply two arbitrary subsets of  $\{1, \dots, n\}$  with given density. For example define

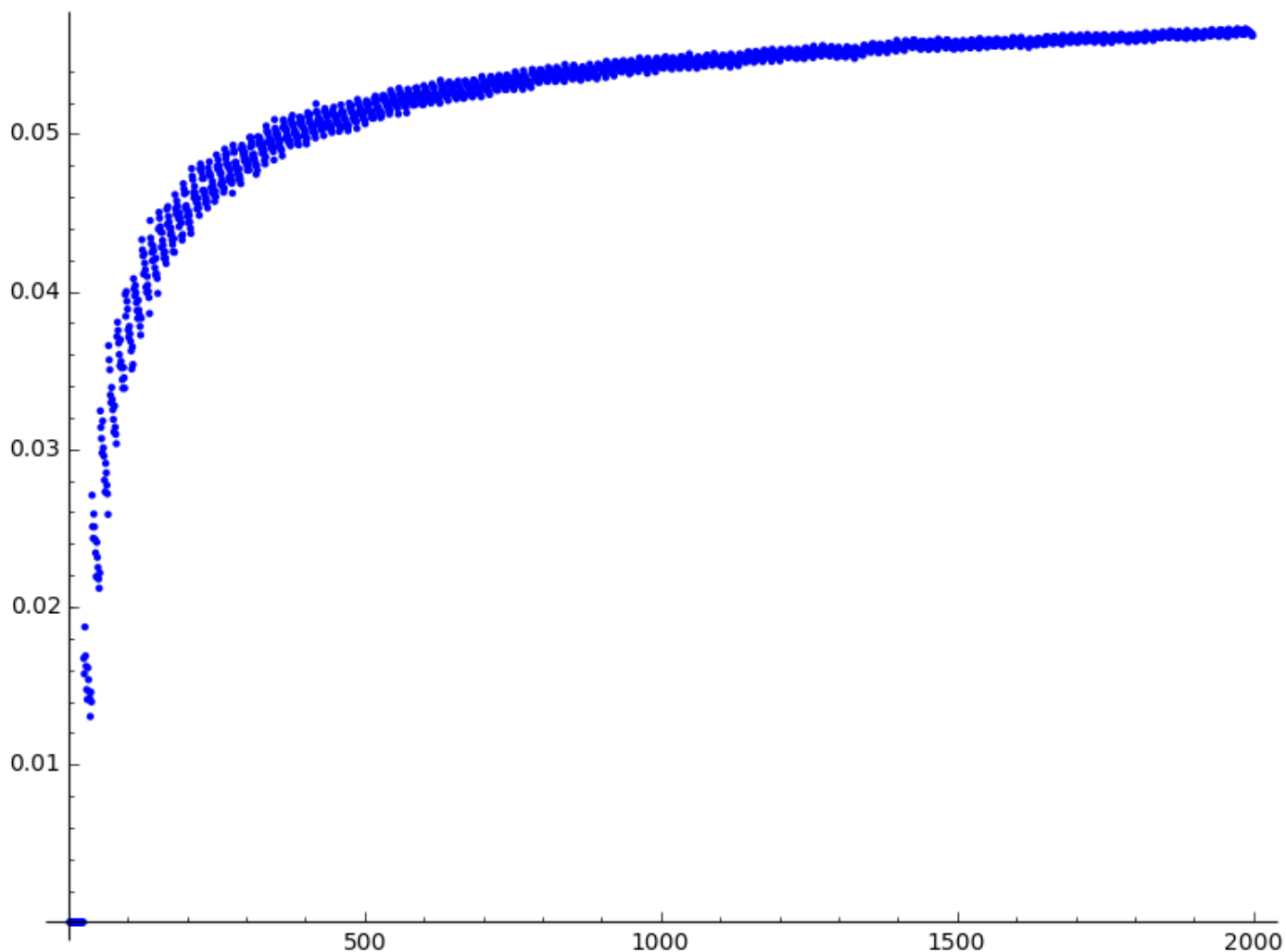
$$S(a, b, c, d, N) := \{(a + bk)(c + dk) : 1 \leq a + bk, c + dk \leq N\},$$

then what can we say about

$$\frac{|S(a, b, c, d, N)|}{|S(N)|}$$

when  $a, b, c, d$  are fixed positive integer and  $N \rightarrow \infty$

Below is a graph of  $\frac{|S(3,5,12,14,N)|}{|S(N)|}$  for  $N \in [1, 2000]$ .



```
def arith_tab(a, b, c, d, n):
    return [(a + b*i)*(c + d*j) for i in range(1, (n-a)/b + 1) for j in
range(1, (n-c)/d+1)]

def multiplication_list(n):
    return [i*j for i in range(1,n+1) for j in range(1,n+1)]

def ratio_arith(a, b, c, d, n):
    n_1=len(set(arith_tab(a, b, c, d, n)))
    n_2=len(set(multiplication_list(n)))
    return float(n_1)/float(n_2)
```

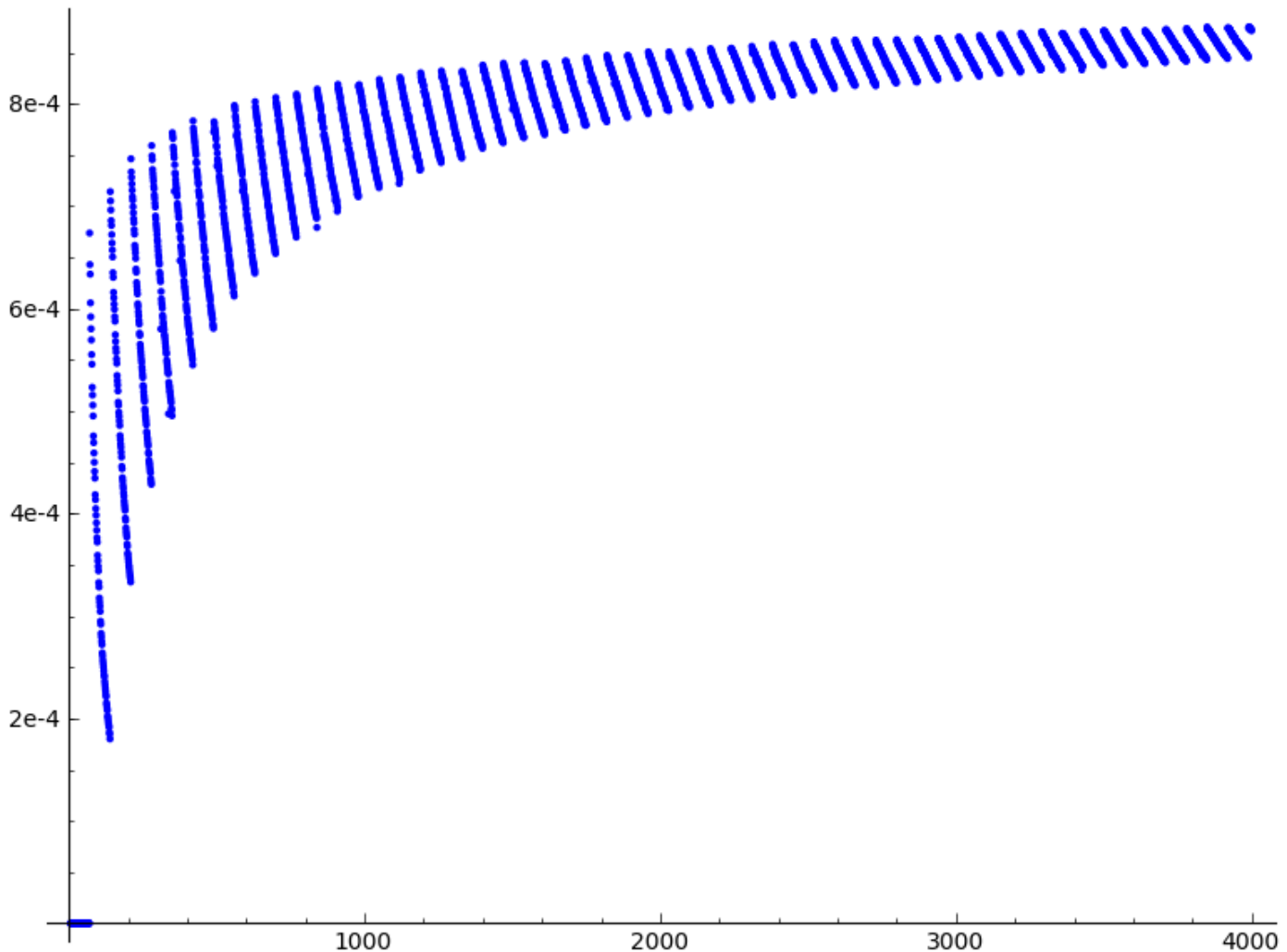
```
@interact
def _(n=('Range', 300), a_1=('a', 2), q_1=('b', 5), a_2=('c', 1), q_2=('d',
3)):
    list=[ratio_arith(a_1, q_1, a_2, q_2, i) for i in range(2, n)]
    show(point([(i+2,list[i]) for i in range(n-2)], figsize=6))
```

With some modification in Ford's proof, we show that  $\frac{|S(a,b,c,d,N)|}{|S(N)|} \asymp_{b,d} 1$  as  $N \rightarrow \infty$

But what if we take two **arbitrary** subsets  $A(N), B(N) \subseteq \{1, \dots, N\}$  such that  $|A(N)|, |B(N)| \geq \delta N$  and consider the ratio  $\frac{|A(N)*B(N)|}{|S(N)|}$ .

$$A(N) * B(N) := \{ab : a \in A(N), b \in B(N)\}$$

Below is a plot of  $\frac{|A(N)*B(N)|}{|S(N)|}$  for  $N \leq 4000$ , where  $A(N), B(N)$  are random subsets  $S(N)$  having density  $\delta = \frac{1}{70}$ .



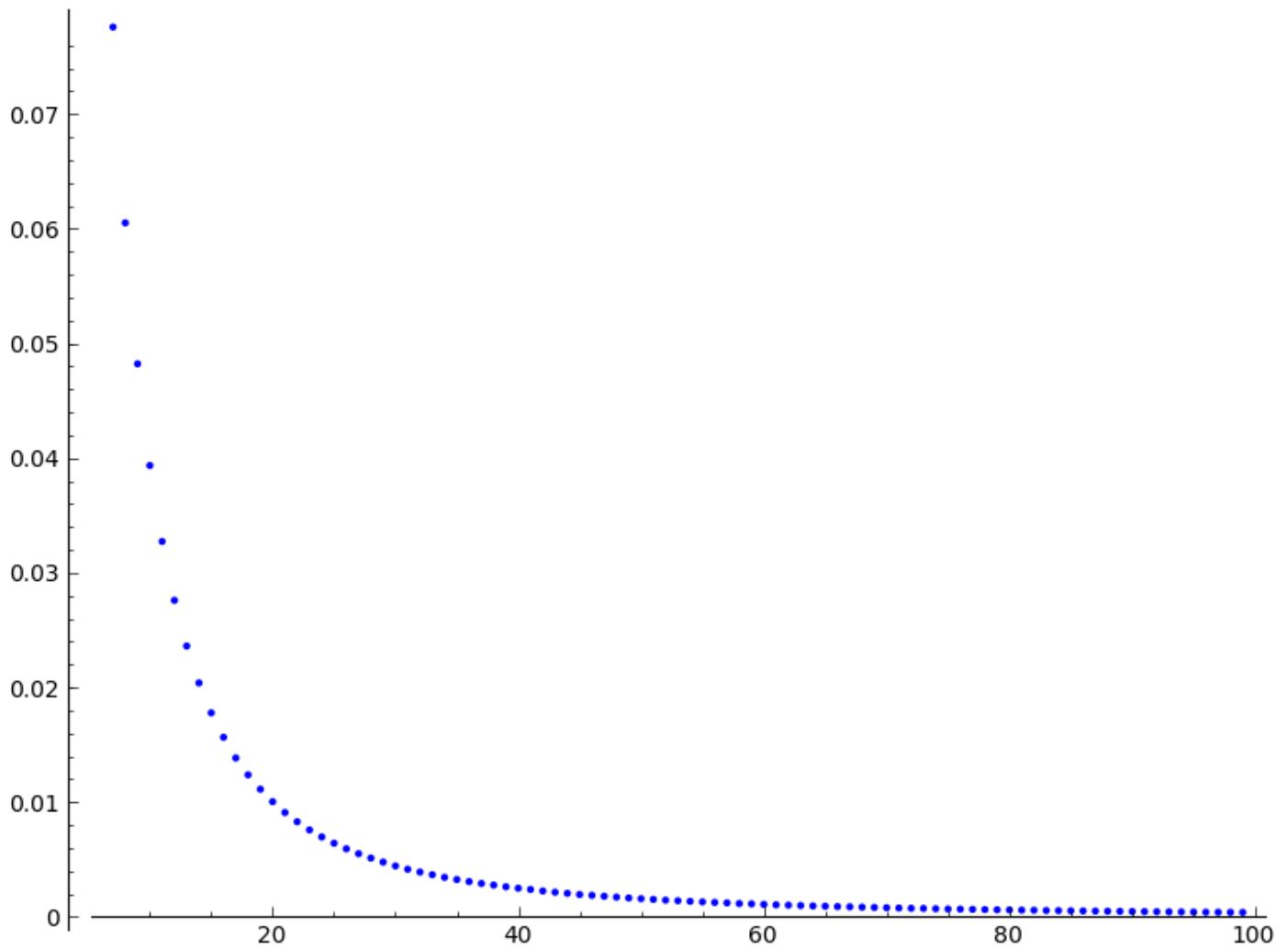
From the above graph it seems  $\frac{|A(N)*B(N)|}{|S(N)|} \asymp_{\delta} 1$  when  $\delta$  fixed and  $N \rightarrow \infty$ . For now I don't have a proof for this claim.

In the graph below, for a given  $\delta$ , the point is at  $(\delta, f(\delta))$  if:

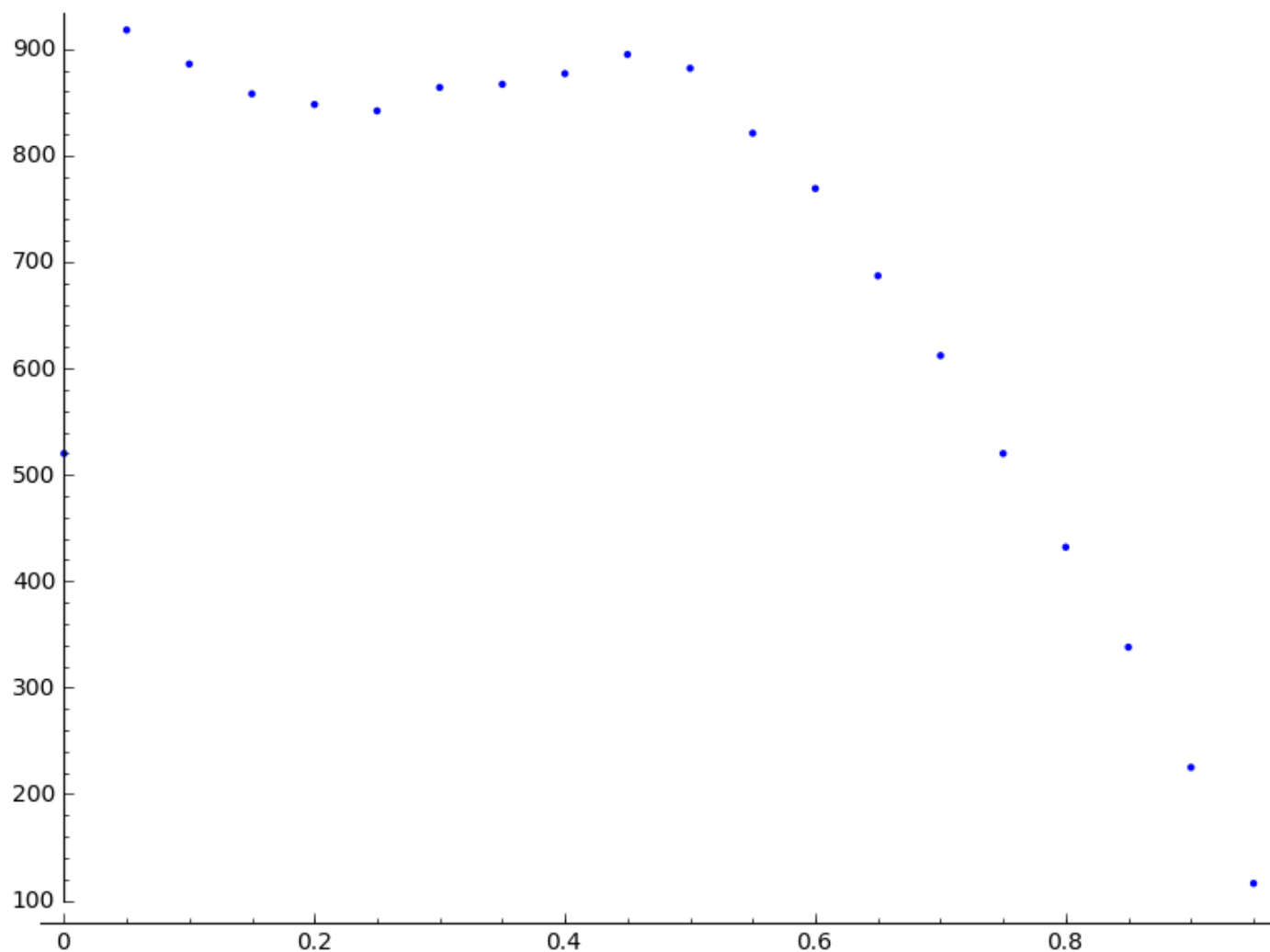


$$f(\delta) = \frac{1}{30} \sum_{i=1}^{30} \frac{|A(N_i) * B(N_i)|}{|S(N_i)|}$$

where  $\{N_1, \dots, N_{30}\} \subseteq [3000, 4000]$  randomly chosen and  $A(N_i), B(N_i)$  are random subsets of  $\{1, \dots, N_i\}$  having density  $\delta$ .



Histogram of elements of multiplicity 2 in  $300 \times 300$  multiplication table.



Here is an interesting entire function given by  $f(n, s) = \sum_{d|n} d^s$ .

```
def div_sum(n, s):
    L=divisors(n)
    return sum([d^s for d in L])
```

```
complex_plot(div_sum(13*2, x), (-20, 20), (-20, 20))
```

