Computing examples of generalized Legendre curves that admit QM in Sage

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Sage Days 62, July 28 - August 1

For integers $2 \leq e_1, e_2, e_3 \leq \infty$, the *triangle group* (e_1, e_2, e_3) is defined by the presentation

$$\langle x, y \mid x^{e_1} = y^{e_2} = (xy)^{e_3} = id \rangle.$$

A triangle group is called *arithmetic* if it has a unique embedding to $\mathrm{SL}(2, \mathbb{R})$ with image either commensurable with $\mathrm{PSL}(2, \mathbb{Z})$ or with an order of a quaternion algebra B over a totally real field K.

One of the best-known arithmetic triangle groups is $(2, 3, \infty)$; it is isomorphic to the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

An arithmetic triangle group Γ acts on the upper half plane, $\mathcal{H}$, via linear fractional transformation; $\Gamma \backslash \mathcal{H}$ is a modular curve when at least one of e_i is ∞; otherwise, it is a *Shimura curve.* While modular curves parametrize certain isomorphism classes of elliptic curves, Shimura curves parametrize isomorphism classes of certain 2-dimensional abelian varieties with quaternionic multiplication (QM). Arithmetic triangle groups have been classified by Takeuchi [2] into 19 commensurability classes.

Each arithmetic triangle group can be realized as a monodromy group of some integral of the form

$$\int_0^1 \frac{dx}{\sqrt[N]{x^i (1-x)^j (1-lx)^k}},$$

with $N, i, j, k \in \mathbb{Z}$. Wolfart [3] realized these integrals as periods of the generalized Legendre curves

$$C_\lambda^{[N;i,j,k]} : y^N = x^i (1-x)^j (1-\lambda x)^j,$$

where λ is a constant and $1 \leq i, j, k < N$.

In a recent WIN3 project [1], we study the generalized Legendre family of curves:

$$C_\lambda^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^j,$$

where λ is a constant and $1 \leq i, j, k < N$, and give an explicit method for determining whether they admit QM.

This Sage Days project is to compute explicit examples of this method in action!

References

[1] A. Deines, J. Fuselier, L. Long, H. Swisher, and F. Tu. Generalized legendre curves and quaternionic multiplication. preprint.

[2] K. Takeuchi. Arithmetic triangle groups. *J. Math. Soc. Japan*, 29(1):91–106, 1977.

[3] J. Wolfart. Werte hypergeometrischer Funktionen. *Invent. Math.*, 92(1):187–216, 1988.