p-Adic Precision

Theory, examples and application to some p-adic differential equations

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p-adic methods

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Study of p-adic objects

This is my motivation, but we might need more tools...

1. Differential precision (w. X.Caruso and D.Roe)
   - Direct analysis
   - Application in linear algebra
   - The main lemma

2. *p*-adic differential equations with separation of variables
   - Isogeny computation
   - The original scheme

3. Application of differential precision (w. P.Lairez)
   - Applying the lemma
   - A more subtle approach

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Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=-l}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $l \in \mathbb{Z}$ and p a prime number.
While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$, with $l \in \mathbb{Z}$.

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Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

p-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

Proposition (p-adic errors don't add)

Indeed,

$$(a + O(p^k)) + (b + O(p^k)) = a + b + O(p^k).$$

That is to say, if a and b are known up to precision $O(p^k)$, then so is $a + b$.

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Remark

It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

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Precision formulae

Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1} p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

In particular,
$$\frac{1}{p^c y + O(p^d)} = y^{-1} p^{-c} + O(p^{d-2c})$$

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A little warm-up on computing determinants : expansion

An example of determinant computation

$$\begin{bmatrix} p^5 + O(p^{10}) & 1 + O(p^{10}) & 1 + p^3 + O(p^{10}) \\ O(p^{10}) & 1 + O(p^{10}) & 1 + O(p^{10}) \\ 2p^6 + O(p^{10}) & 2p + O(p^{10}) & 2p + p^5 + O(p^{10}) \end{bmatrix}$$

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If we expand directly using the expression of the determinant in terms of the coefficients, we get:

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If we expand directly using the expression of the determinant in terms of the coefficients, we get:

$$-2p^9 + O(p^{10}),$$

because of $1 \times 1 \times O(p^{10})$.

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If we compute **approximate** SNF, we now get:

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If we compute **approximate** SNF, we now get:

$$-2p^9+O(p^{13}),$$

because of $1 \times p^3 \times O(p^{10}) = O(p^{13})$.

Application to Hilbert's matrix

Definition

For any $n \in \mathbb{N}^*$, we define Hilbert's n-dimensional matrix, $H^{(n)} \in M_n(\mathbb{Q})$, with $H^{(n)}_{i,j} = \frac{1}{i+j-1}$ for $1 \leq i, j \leq n$. One can prove $H^{(n)-1} \in M_n(\mathbb{Z})$.

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Example

$$H^{(6)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \end{pmatrix}.$$

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Hilbert's matrix inversion

Over **Real Double Float**:

$$|H^{(6)-1}_{\text{exact}} - H^{(6)-1}_{\text{approx}}| = \begin{pmatrix} 10^{-09} & 10^{-08} & 10^{-07} & 10^{-07} & 10^{-07} & 10^{-07} \\ 10^{-08} & 10^{-06} & 10^{-06} & 10^{-05} & 10^{-05} & 10^{-06} \\ 10^{-07} & 10^{-06} & 10^{-05} & 10^{-04} & 10^{-04} & 10^{-05} \\ 10^{-07} & 10^{-05} & 10^{-04} & 10^{-04} & 10^{-04} & 10^{-04} \\ 10^{-07} & 10^{-05} & 10^{-04} & 10^{-04} & 10^{-04} & 10^{-04} \\ 10^{-07} & 10^{-06} & 10^{-05} & 10^{-04} & 10^{-04} & 10^{-05} \end{pmatrix}.$$

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Over **Real Double Float**:

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$$H_{\text{exact}}^{(20)-1}[1,1] = 400, \text{ zero significant digit.}$$

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Over $\mathbb{Q}_2$, with initial precision $O(2^{30})$:

$$H^{(6)} = (P + O(2^{31})) \begin{bmatrix} 2^{-3} & & & & & 0 \\ & 2^{-3} & & & & \\ & & 2^{-3} & & & \\ & & & 2^{-3} & & \\ & & & & 2^{-1} & \\ 0 & & & & & 2^{-1} \end{bmatrix} (Q + O(2^{31})).$$

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$$H_{\text{approx}}^{(20)-1}[1,1] = 400 + O(2^{34}), \text{ 30 significant digits}$$

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- No idea on what is **optimal**.

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Lemma (CRV14)

Let $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ be a (strictly) ***differentiable*** *mapping.*

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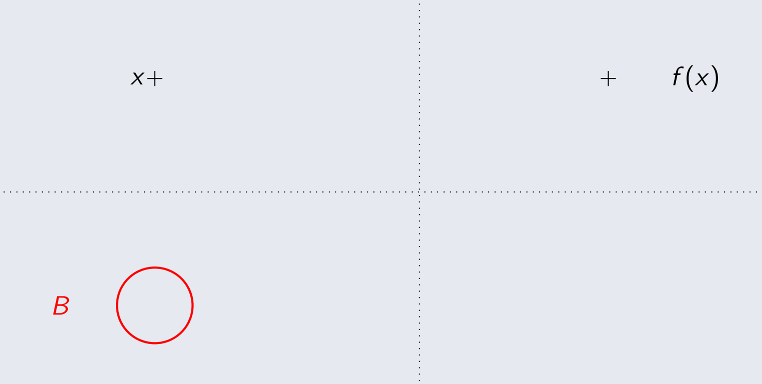
Interpretation



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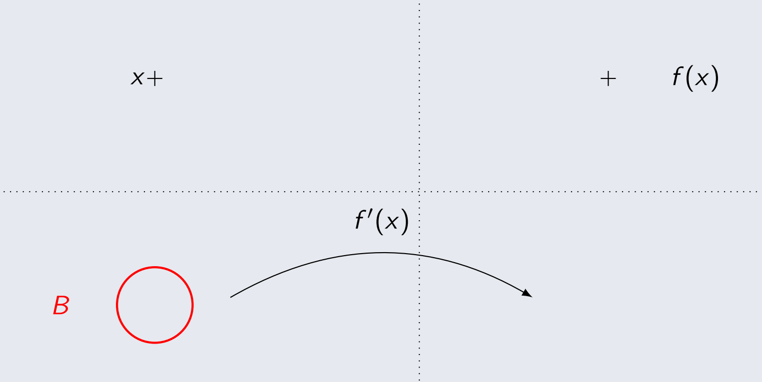

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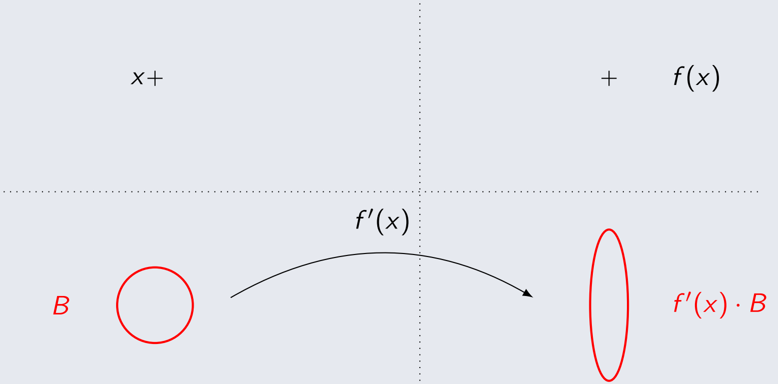

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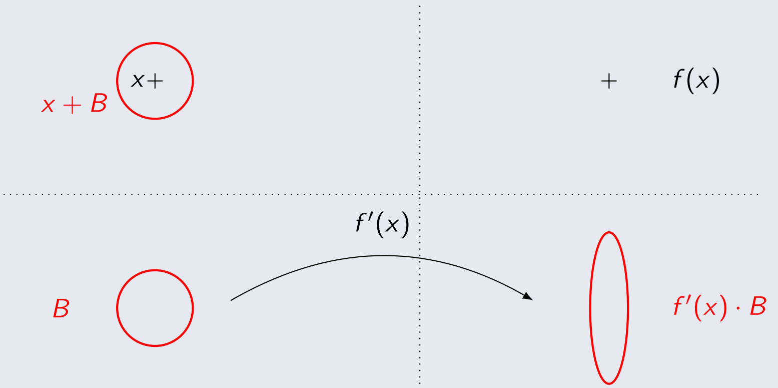

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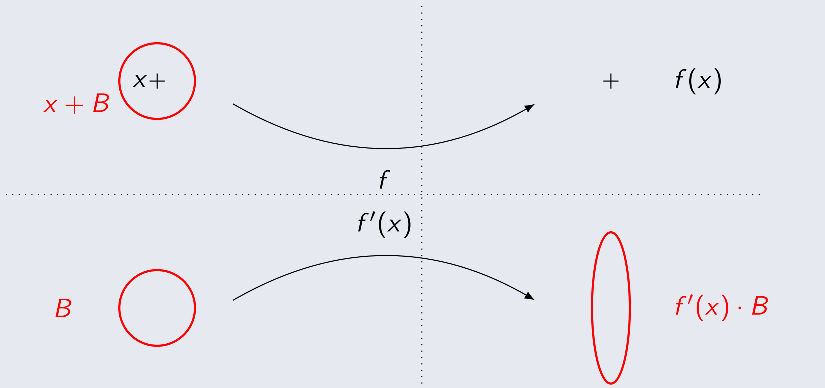

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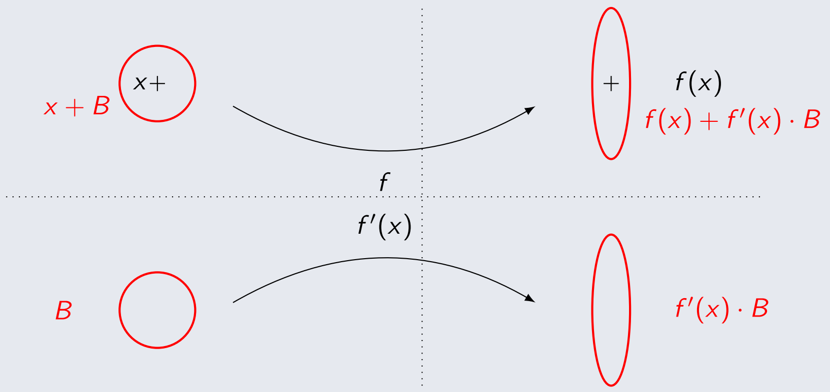

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Our framework can be extended to **(complete) ultrametric K-vector spaces** (*e.g.* $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$).

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This can be determined with **Newton-polygon** techniques.

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- **Approximate SNF is optimal.**

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- On matrices: characteristic polynomial, LU factorization, inverse... (see CRV15)
- On $\mathbb{Q}_p[X]$: evaluation, interpolation, GCD, factorization... (see CRV16)

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- On $\mathbb{Q}_p[X]$: evaluation, interpolation, GCD, factorization... (see CRV16)
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Some differentiable operations

Some more examples

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Let E and $\tilde{E}$ be two elliptic curves over $\mathbb{Z}/p\mathbb{Z}$:

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Writing $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$, we get :

$$(Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6.$$

A p-adic computation of a solution

Computing the isogeny

Given E and $\widetilde{E}$, the goal is to compute the isogeny I *via* the differential equation:

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3. Reduce mod p to get the solution in $\mathbb{Z}/p\mathbb{Z}$.

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To compute $y \mod x^{2^N+1}$, we need an initial precision of $O(N^2)$ digits.

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Theorem

Let $\Phi : (g, h) \mapsto y$ *such that* $y(0) = 0$ *and* $y' = gh(y)$*. Then,*

$$\Phi'(g, h) \cdot (\delta g, \delta h) = h(y) \int \left(\delta g + \frac{g \delta h(y)}{h(y)} \right).$$

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In our case, $p \neq 2$, $y, g, h \in \mathbb{Z}_p[\![x]\!]$, $g(0) = h(0) = 1$*. If* $\delta g = \delta h = O(p^k)$*, then*

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In our case, $p \neq 2$, $y, g, h \in \mathbb{Z}_p[\![x]\!]$, $g(0) = h(0) = 1$. *If* $\delta g = \delta h = O(p^k)$, *then*

$$\Phi'(y) \cdot (\delta g, \delta h) \mod x^{2^N+1} \in \frac{O(p^k)}{p^N} \mathbb{Z}_p[\![x]\!].$$

First conclusion on the application of the lemma

Proposition

$\Phi(g,h) \mod (p, t^{2^n})$ *is determined by* $g, h \mod (p^{1+\log_p 2^n}, t^{2^n}))$. *In other words, we have a logarithmic loss in precision.*

What happens in practice ?

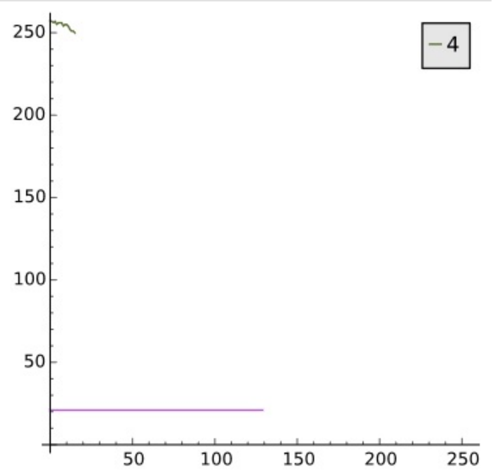

Figure: Precision over the output

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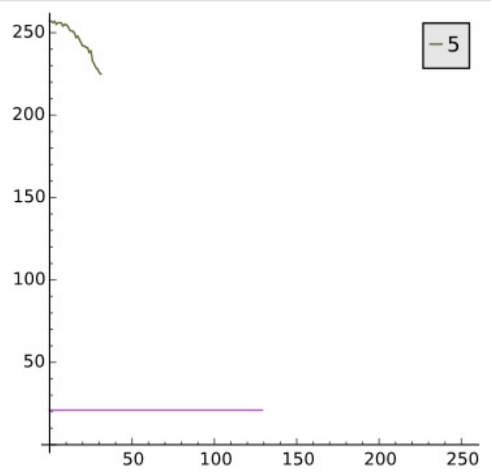

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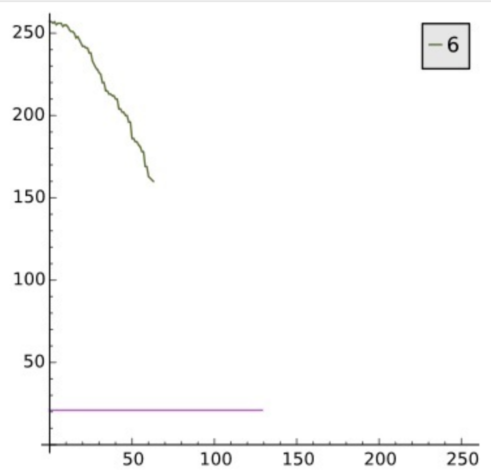

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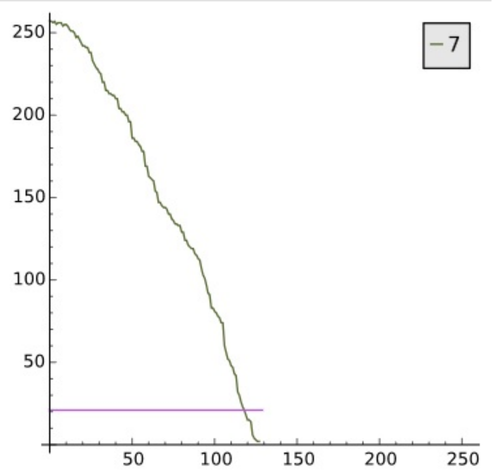

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Different way of representing the *p*-adics

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- In the previous computation, we start with some given approximations of g, h, u_0 and try **to follow** the algorithm for the exact counterparts of g, h, u_0. This is somehow **much stronger** than our desire: computing a good approximate solution.
- Another way is then to modify the current g, h, u_0 **at each step**, in a consistent way, so as to keep on getting better approximate solutions.
- A third way here will be to work entirely in $\mathbb{Z}/p^{\kappa}\mathbb{Z}$.

New framework

In this new computation, we consider h as given, and not varying for the lemma.

Lemma

Let $Y : g \mapsto y$ *such that* $y(0) = 0$ *and* $y' = gh(y)$. *Then,*

$$Y'(g) \cdot (\delta g) = h(y) \int \delta g.$$

A consequence of the lemma

Corollary

Let $n > 0$ *and* $\kappa > 1$ *be integers, and let* $g \in \mathbb{Z}_p[\![t]\!]$ *such that* $Y(g) \pmod{t^{n+1}}$ *has integer coefficients. For any* $y \in \mathbb{Q}_p[\![t]\!]$ *the following are equivalent:*

1. $y = Y(\bar{g}) \pmod{t^{n+1}}$ *for some power series* $\bar{g} \in \mathbb{Z}_p[\![t]\!]$ *such that* $\int(\bar{g} - g) = 0 \pmod{p^\kappa}$*;*
2. $y = Y(g) \pmod{p^\kappa, t^{n+1}}$.

Final take on the Newton scheme

As a consequence, we can prove that it is harmless to work in $\mathbb{Z}/p^k\mathbb{Z}$ for our computation.

Proposition

We can obtain the solution $\Phi(g,h) \mod (p, t^{n+1})$ *knowing* $g, h \mod (p^{\lfloor \log_p n \rfloor + 1}, t^{n+1})$ *and applying the following iteration:*

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modulo $p^{\lfloor \log_p n \rfloor + 1}$ *and growing order of truncation.*

Timings

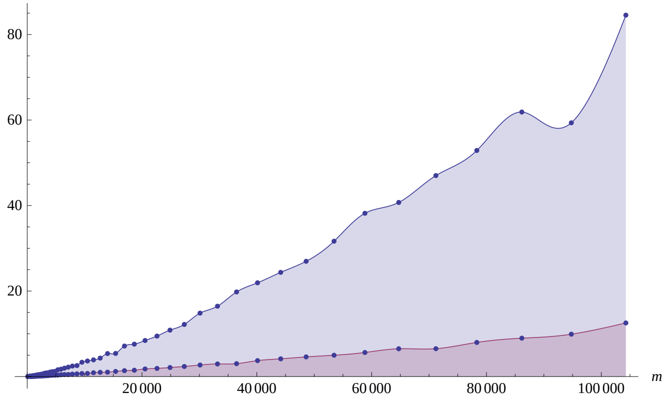

Figure: Timings in seconds, measured on a laptop, of our Algorithm run at precision λ_{old} (upper curve) and λ_{new} (lower curve) in order to compute an approximation modulo $(5, t^{4m+1})$ of some given m-isogenies.

Speedup

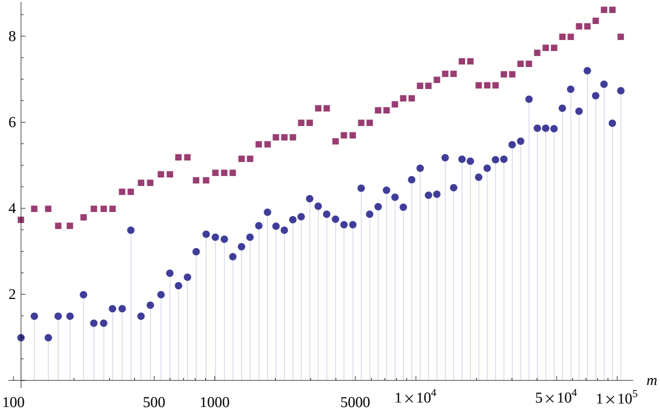

Figure: Practical speedup obtained with the new precision analysis compared with the theoretical improvement (m-axis in logarithmic scale). (■) is the ratio on precisions, (•) is the actual speedup.

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References

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Linear Algebra

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Thank you for your attention

Thanks