Iwasawa theory II

talk in Oxford

§ HISTORIC BACKGROUND : Tate's BSD in the function field case

$\bar{\mathcal{E}} \to \bar{C}/\bar{\mathbb{F}}_q$

$\mathcal{E} \to C/\mathbb{F}_q$

Everything is translated to geometry we have an elliptic surface over a curve over a finite field. Then we extend to $\bar{\mathbb{F}}_q$. Tate takes a certain étale cohomology group $X_{/C}$. Goup-over-down. All terms in BSD magically fall into place.

Poincaré duality $\bar{\mathcal{E}}$ $\rightsquigarrow$ regulator

action of Frob on X $\rightsquigarrow$ L-function

Tamagawa numbers, torsion, Sha all appear where they should.

Iwasawa theory of elliptic curves is the failed attempt to do this over $\mathbb{Q}$

§ $\mathbb{Z}_p$- EXTENSION (analogue of $\overline{\mathbb{F}}_q(C)/\mathbb{F}_q(C)$)

Fix a prime p. For each $n \geq 1$

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}) \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^n\mathbb{Z}$$

→ There is a unique $K_n/\mathbb{Q}$ of Galois group $G_n \cong \mathbb{Z}/p^n\mathbb{Z}$ unram. outside p.

Set $K_\infty = \bigcup K_n$ and $\Gamma = \varprojlim_n G_n$

Aside: sage should have better functionality for abelian fields. Given a Dicharacter χ it should give the abelian field $\overline{\mathbb{Q}}^{\ker \chi}$ etc

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§ IWASAWA THEORY FOR ELLIPTIC CURVES

why do I restrict to this? Probably because William was involved, sage can do this but nothing else

Let $E/\mathbb{Q}$ be an elliptic curve, p is assumed to be of good ordinary reduction.

$\forall n \geq 1$ There is a naturally defined $\mathbb{Z}_p$-module X_n such that [if is finite]

$$0 \to Ш(E/K_n)[p^\infty] \to X_n \to \mathrm{Hom}(E(K_n), \mathbb{Z}_p) \to 0$$

$$\mathrm{Hom}(E(K_n), \mathbb{Z}_p) = \mathbb{Z}_p^{\mathrm{rank}\, E(K_n)}$$

is exact.

Now G_n acts on it. It's a $\mathbb{Z}_p[G_n]$-module

$$\mathbb{Z}_p[G_n] \cong \mathbb{Z}_p[X]/(X^{p^n}-1) = \mathbb{Z}_p[T]/((T+1)^{p^n}-1)$$

so $T=0$ corresponds to G_n-fixed part.

Set $X = \varprojlim X_n$ which is a $\varprojlim \mathbb{Z}_p[G_n] =: \Lambda$-module

$$\varprojlim \mathbb{Z}_p[G_n] \cong \mathbb{Z}_p[[T]]$$

We associate to X a <u>characteristic element</u> $f_E \in \Lambda$

think of a generating function, like the ζ-function of $\mathcal{E}$, obtained by the action of Fr on X

Go up - over - down: $X_0 \rightsquigarrow X^{\Gamma} \longrightarrow X_{\Gamma} \longrightarrow X_0$

largest quotient Γ acts trivially

Magically all the terms of BSD appear, ... well not exactly.

Global duality Gal coh $\rightsquigarrow$ p-adic regulator Reg_p

$\langle , \rangle_p : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{Q}_p$

<u>Theorem</u> Perrin-Riou, Schneider

If $Reg_p \neq 0$ then

- $ord_{T=0} f_E \geq \operatorname{rank} E(\mathbb{Q})$
- If we have $=$, then
  - $Ш(E/\mathbb{Q})[p^\infty]$ is finite
  - $f_E^*(0) \overset{\times}{=} \dfrac{N_p^2 \cdot \prod c_v \cdot \# Ш(E/\mathbb{Q}) \cdot Reg_p}{(\# E(\mathbb{Q})_{tors})^2}$

leading term

up to a p-adic unit

§2 Modular Symbols

There is a map $[\cdot]^+ : \mathbb{Q} \to \mathbb{Q}$ such that

$$[0]^+ = \frac{L(E,1)}{\Omega^+} \leftarrow \text{Néron period}$$

and

twisted L-function

$$\sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \chi(a) \left[\frac{a}{p^{n+1}}\right]^+ = \text{Gausssum}(\chi) \cdot \frac{L(E,\chi,1)}{\Omega_+}$$

for any $\chi : G_n \to \bar{\mathbb{Q}}_p^\times$ (primitive)

Putting $[\,]^+$ together, we get a p-adic L-function $\mathcal{L}_E \in \Lambda$ such that

$$\mathcal{L}_E(0) = \underbrace{\left(1 - \frac{1}{\alpha}\right)^2}_{\doteq N_p} \cdot \frac{L(E,1)}{\Omega^+}$$

and $\alpha \in \mathbb{Z}_p^\times$ with $\alpha^2 - a_p \alpha + p = 0$

and

$$\chi(\mathcal{L}_E) = \frac{1}{\alpha^{n+1}} \, Ga(\chi) \cdot \frac{L(E,\chi,1)}{\Omega_+}$$

Theorem (Kato + Skinner–Urban)

$$f_E = \mathcal{L}_E \cdot u \quad \text{some } u \in \Lambda^\times$$

$\Rightarrow$ same order of vanishing, same $\doteq$ leading term

p-adic BSD: $\operatorname{ord}_{T=0} \mathcal{L}_E = \operatorname{rank} E(\mathbb{Q})$

$$\mathcal{L}_E^*(0) = \left(1 - \frac{1}{\alpha}\right)^2 \cdot \frac{\prod c_v \; \# Ш \cdot \mathrm{Reg}_p}{(\# E(\mathbb{Q})_{tors})^2}$$

not just $\doteq$

§2 SHARK

- Compute s st $s \geq \mathrm{ord}_{T=0}\, \mathcal{L}_E$
$$\mathrm{ord}_{T=0}\, \mathcal{L}_E = \mathrm{ord}_{T=0}\, f_E \geq \mathrm{rank}\, E(\mathbb{Q})$$

- If $=$ then $Ш(E/\mathbb{Q})[p^\infty]$ is finite!

and $\#Ш \stackrel{?}{=}$ explicit formula involving $\mathcal{L}_E^*(0)$