# flat_jerusalem

November 22, 2016

# 1 Lyapunov exponents of the Teichmüller flow

## 1.1 Vincent Delecroix, Bordeaux (France)

This notebook is a companion to the pdf presentation made at Jerusalem in November 2016 at the occasion of Sage days 79.

```
In [8]: from IPython.display import HTML
```

## 1.2 demo 1: rational billiard

```
In [77]: # this huge piece of code is used to generate billiard and translation sur

         def solve(x,u,y,v):
             r"""
             Return (a,b) so that: x + au = y + bv
             """
             d = -u[0] * v[1] + u[1] * v[0]
             a = v[1] * (x[0]-y[0]) + v[0] * (y[1] - x[1])
             b = u[1] * (x[0]-y[0]) + u[0] * (y[1] - x[1])
             return (a/d,b/d)

         def frm(vectors, x, i, d):
             r"""
             First return map in the triangle determined by ``vectors``.

             INPUT:

             - ``x`` - a real number in (0,1)

             - ``i`` - either 0, 1 or 2 (determine the side)

             - ``d`` - an angle
             """
             d = Va(d)
             assert normals[i].dot_product(d) > 0, "dot product with {} is {}".form
             j = (i+1)%3
```

```
        k = (j+1)%3

        # we assume that the base of i is at (0,0)
        # solve(x,u,y,v)
        tj,bj = solve(x*vectors[i], d, vectors[i], vectors[j])
        tk,bk = solve(x*vectors[i], d, -vectors[k], vectors[k])

        assert (0 < bj < 1) + (0 < bk < 1) == 1

        if 0 < bj < 1:
            return bj,j
        else:
            return bk,k

class Triangle(object):
    r"""
    A class to handle intersection of lines with triangles
    """
    def __init__(self, v0, v1, v2, t=0, color=None):
        r"""
        INPUT:

        - ``v0``, ``v1``, ``v2`` -- vertices

        - ``t`` -- optional translate

        - ``color`` -- optional color
        """
        s = Sequence([v0,v1,v2])
        self._vector_space = V = s.universe()
        self._vectors = list(s)
        self._normals = [V((-v[1],v[0]))/v.norm() for v in self._vectors]
        self._translate = V(t)
        self._color = color

    def area(self):
        x0,y0 = self._vector_space.zero()
        x1,y1 = self._vectors[0]
        x2,y2 = -self._vectors[2]
        return ((x0+x1)*(y1-y0) + (x1+x2)*(y2-y1) + (x2+x0)*(y0-y2)) / 2



    def plot(self, color=None):
        if color is None and self._color:
            color = self._color
        V = self._vector_space
        return polygon2d([self._translate + V.zero(),
```

```
                              self._translate + self._vectors[0],
                              self._translate - self._vectors[2]], alpha=0.3,

    def frm(self, i, x, d):
        V = self._vector_space
        d = V(d)
        assert self._normals[i].dot_product(d) > 0, "dot product with {} i
        j = (i+1)%3
        k = (j+1)%3

        # we assume that the base of i is at (0,0)
        # solve(x,u,y,v)
        tj,bj = solve(x*self._vectors[i], d, self._vectors[i], self._vecto
        tk,bk = solve(x*self._vectors[i], d, -self._vectors[k], self._vect

        assert (0 < bj < 1) + (0 < bk < 1) == 1

        if 0 < bj < 1:
            return j,bj
        else:
            return k,bk

    def code_to_pos(self, i, x):
        if i == 0:
            return self._translate + x * self._vectors[0]
        elif i == 1:
            return self._translate + self._vectors[0] + x*self._vectors[1]
        elif i == 2:
            return self._translate + (x-1)*self._vectors[2]
        else:
            raise ValueError("got i={} and x={}".format(i,x))

class TriangulatedSurface:
    def __init__(self, triangles, gluings):
        self._triangles = triangles
        self._gluings = gluings

    def orbit(self, t, s, x, d, iterations=10):
        r"""
        INPUT:

        - ``t`` -- triangle

        - ``s`` -- side 0, 1 or 2

        - ``d`` -- direction
        """
        orbit = []
```

```
        for _ in range(iterations):
            s1,x1 = self._triangles[t].frm(s,x,d)
            orbit.append(((t,s,x,s1,x1)))
            x = 1-x1
            t,s = self._gluings[t][s1]
        return orbit

    def plot(self, labels=False):
        G = Graphics()
        for i,t in enumerate(self._triangles):
            G += t.plot()
            if labels:
                barycenter = t._translate + (2*t._vectors[0] + t._vectors[
                G += text(str(i), barycenter)
        return G

    def plot_orbit(self, t, s, x, d, iterations=10):
        G = self.plot()
        for t,s0,x0,s1,x1 in self.orbit(t,s,x,d,iterations):
            p0 = self._triangles[t].code_to_pos(s0,x0)
            p1 = self._triangles[t].code_to_pos(s1,x1)
            G += line2d([p0,p1], color='black')
        return G

class TriangularBilliard(Triangle):
    def reflexion(self, d, i):
        r"""
        Reflexion of the direction ``d`` on the side ``i``
        """
        c,s = self._normals[i]
        c2 = c*c - s*s
        s2 = 2*c*s
        return self._vector_space((c2*d[0] + s2*d[1], s2*d[0] - c2*d[1]))

    def orbit(self, i, x, d, iterations=10):
        orbit = [(i,x)]
        for _ in range(iterations):
            i,x = self.frm(i,x,d)
            orbit.append((i,x))
            d = -self.reflexion(d,i)
        return orbit

    def plot_orbit(self, i, x, d, iterations=10):
        G = self.plot()
        orbit = self.orbit(i,x,d,iterations)
        pts = [self.code_to_pos(j,x) for j,x in orbit]
        return G + line2d(pts, color='black')
```

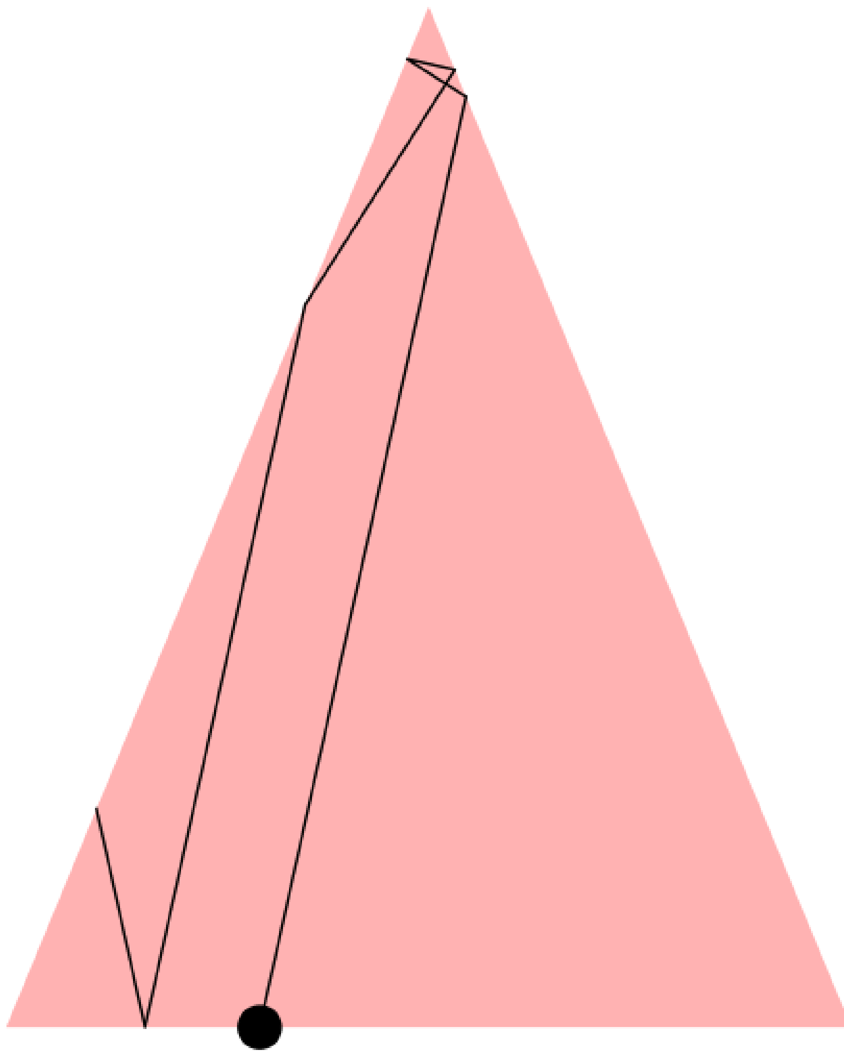
```
def rot(angle, x):
    angle = angle.n()
    c = angle.cos()
    s = angle.sin()
    return x.parent()((c*x[0] - s*x[1], s*x[0] + c*x[1]))
```

```
In [78]: colors = [(1.0, 0.0, 0.0),
          (0.5, 1.0, 0.0),
          (0.0, 1.0, 0.25),
          (1.0, 0.0, 0.75),
          (0.0, 0.25, 1.0),
          (0.0, 1.0, 1.0),
          (1.0, 0.75, 0.0),
          (0.5, 0.0, 1.0)]
```

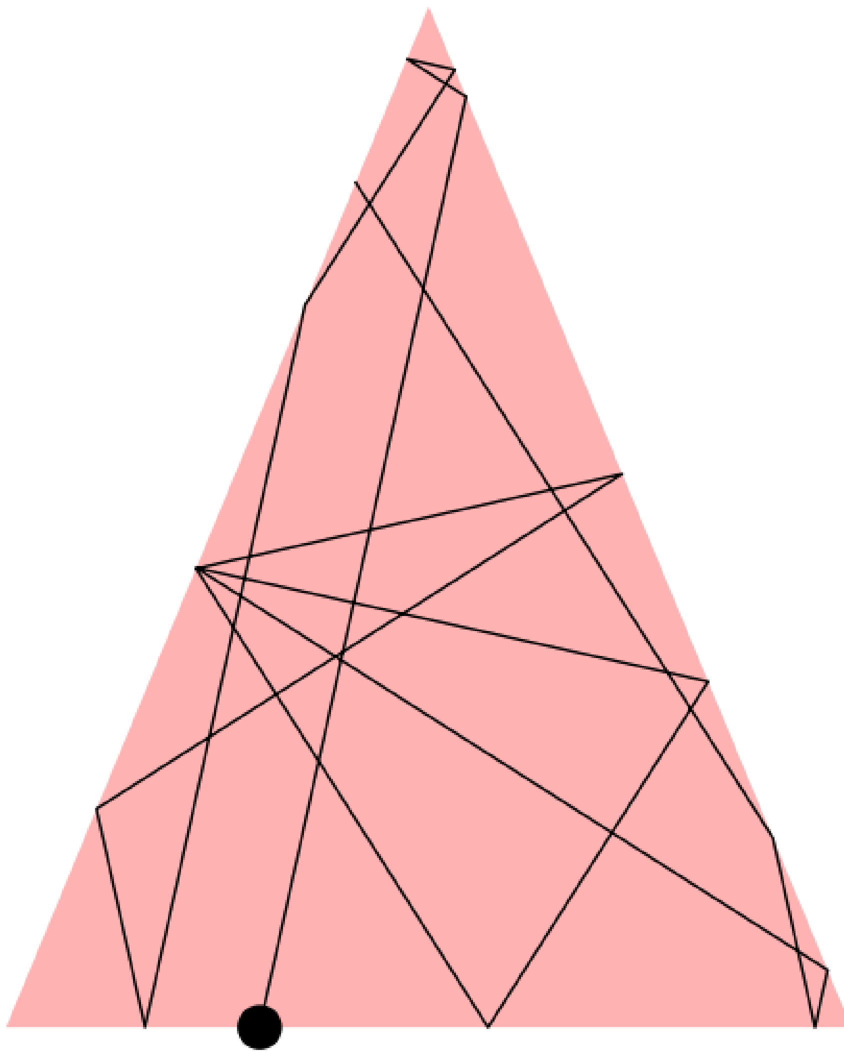
```
In [79]: x = polygen(ZZ)
         K.<sqrt2> = NumberField(x^2 - 2, embedding=AA(2).sqrt())
         Va = VectorSpace(RDF,2)
         vectors0 = [ Va((2,0)), Va((-1,1+sqrt2)), Va((-1,-1-sqrt2))]
         T0 = Triangle(*vectors0, color=colors[0])
         T1 = Triangle(*[rot(pi/4, v) for v in vectors0],   t=(2,0), color=colors[1
         T2 = Triangle(*[rot(pi/2, v) for v in vectors0],   t=(2+sqrt2,sqrt2), colo
         T3 = Triangle(*[rot(3*pi/4, v) for v in vectors0], t=(2+sqrt2,2+sqrt2), co
         T4 = Triangle(*[rot(pi, v) for v in vectors0],     t=(2,2+2*sqrt2), color=
         T5 = Triangle(*[rot(5*pi/4, v) for v in vectors0], t=(0,2+2*sqrt2), color=
         T6 = Triangle(*[rot(3*pi/2, v) for v in vectors0], t=(-sqrt2,2+sqrt2), col
         T7 = Triangle(*[rot(7*pi/4, v) for v in vectors0], t=(-sqrt2,sqrt2), color
         gluings = [[(4,0),(1,2),(7,1)],
                    [(5,0),(2,2),(0,1)],
                    [(6,0),(3,2),(1,1)],
                    [(7,0),(4,2),(2,1)],
                    [(0,0),(5,2),(3,1)],
                    [(1,0),(6,2),(4,1)],
                    [(2,0),(7,2),(5,1)],
                    [(3,0),(0,2),(6,1)]]

         O = TriangulatedSurface([T0,T1,T2,T3,T4,T5,T6,T7], gluings)
         T = TriangularBilliard(*vectors0, color=colors[0])
```

```
In [80]: # first picture of the beamer
         x = 0.3
         d = Va((0.27324123005978,1.2314486735))
         p = T.code_to_pos(0,x)
         G = T.plot_orbit(0,x,d,iterations=6)
         #(circle(p, 0.05, fill=True, color='black') + G).save('/tmp/billiard1.pdf
         (circle(p, 0.05, fill=True, color='black') + G).show(aspect_ratio=1, axes=
```



In [81]:
```
# the unfolded picture
x = 0.3
d = Va((0.27324123005978,1.2314486735))
p = T.code_to_pos(0,x)
G = T.plot_orbit(0,x,d,iterations=15)
#(circle(p, 0.05, fill=True, color='black') + G).save('/tmp/billiard2.pdf
(circle(p, 0.05, fill=True, color='black') + G).show(aspect_ratio=1, axes=
```



```
In [82]: G2 = O.plot_orbit(0,0,x,d,iterations=15)
         (circle(p, 0.05, fill=True, color='black') + G2).save('/tmp/billiard3.pdf'

In [83]: # compute a sequence of graphics to build a video
         x = 0.3
         d = Va((0.27324123005978,1.2314486735))
         limits = [-sqrt2-.1, 2+sqrt2+.1, -.1, 2+2*sqrt2]
         G = []
         for n in range(30):
             G1 = O.plot_orbit(0,0,x,d,iterations=n)
             G1.axes(False)
             G1.set_axes_range(*limits)
             G2 = T.plot_orbit(0,x,d,iterations=n)
```

```
            G2.axes(False)
            G2.set_axes_range(*limits)
            G.append(graphics_array([G2,G1]))
```

```
In [84]: # assembles the graphics in a video
         A = animate(G)
         A.ffmpeg('demo1.mp4', delay=40)
```

```
In [85]: %%HTML
         <video width="640" height="480" controls>
           <source src="demo1.mp4" type="video/mp4">
         </video>
```

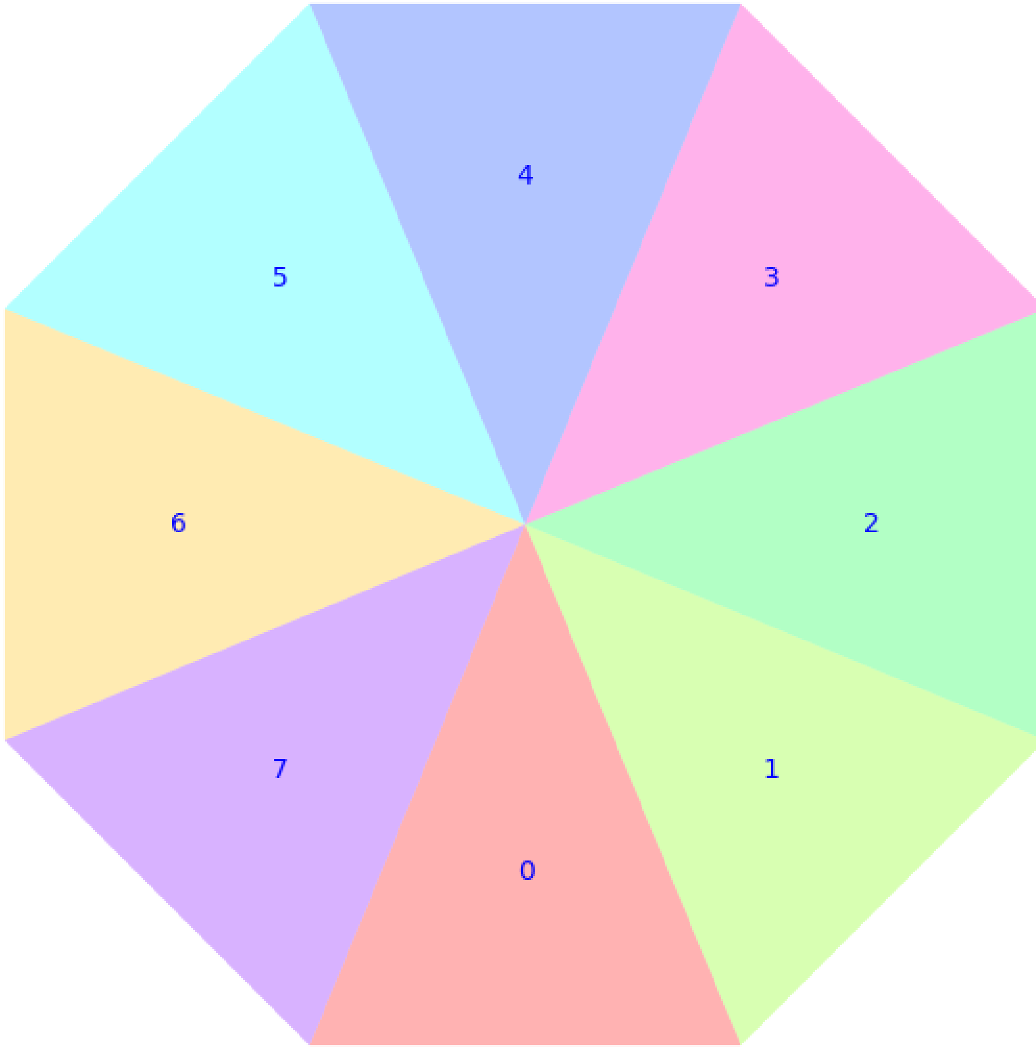
```
<IPython.core.display.HTML object>
```

```
In [ ]:
```

### 1.3 Deviation of Birkhoff sums in the octagon

We consider the horizontal side in the octagon (the edge common to triangle 0 and 4 below). And compute how many intersections we have up to a given time (Birkhoff sums).

```
In [86]: O.plot(labels=True).show(axes=False)
```




```
In [87]: # total area of the octagon
         area = 8 * O._triangles[0].area()
         print area
```

```
19.313708499
```

```
In [88]: def birkhoff_sums_octagon(x, d, num_intersection=1000):
             r"""
             Compute the number of intersections of the trajectory
             from ``x`` in the vertical direction ``d`` and the horizontal
             segment in the octagon.
```

```
        OUTPUT:

        A list of pairs `(length, intersection)`.
        """
        bs = [(0,0)]
        d /= d.norm()   # normalize the direction
        v = 0
        length = 0.0
        for i, sin, xin, sout, xout in O.orbit(0, 0, 0.11214, d, num_intersect
            t = O._triangles[i]
            p0 = t.code_to_pos(sin, xin)
            p1 = t.code_to_pos(sout, xout)
            length += (p1 - p0).norm()
            if i == 0 and sin == 0:
                v += 1
                bs.append((length,v))
        return bs
```

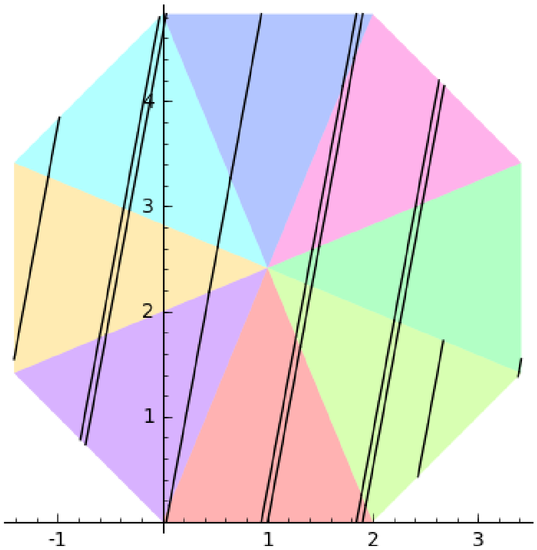
In [ ]:

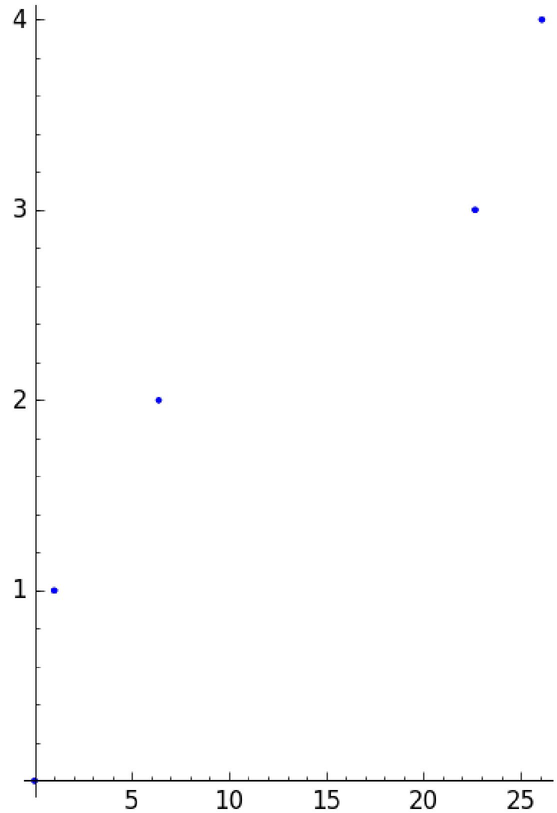
Equidistribution (Kerckhoff-Masur-Smillie theorem): we print with the code below the number of intersections of a trajectory divided by the time versus the transversal measure of the side

In [ ]:

In [89]:
```
d = Va((0.12314311, 0.657123141))
x = 0.5
n = 35
bs = birkhoff_sums_octagon(x,d,n)
graphics_array([O.plot_orbit(0,0,x,d,iterations=n), point2d(bs)])
```

Out[89]:



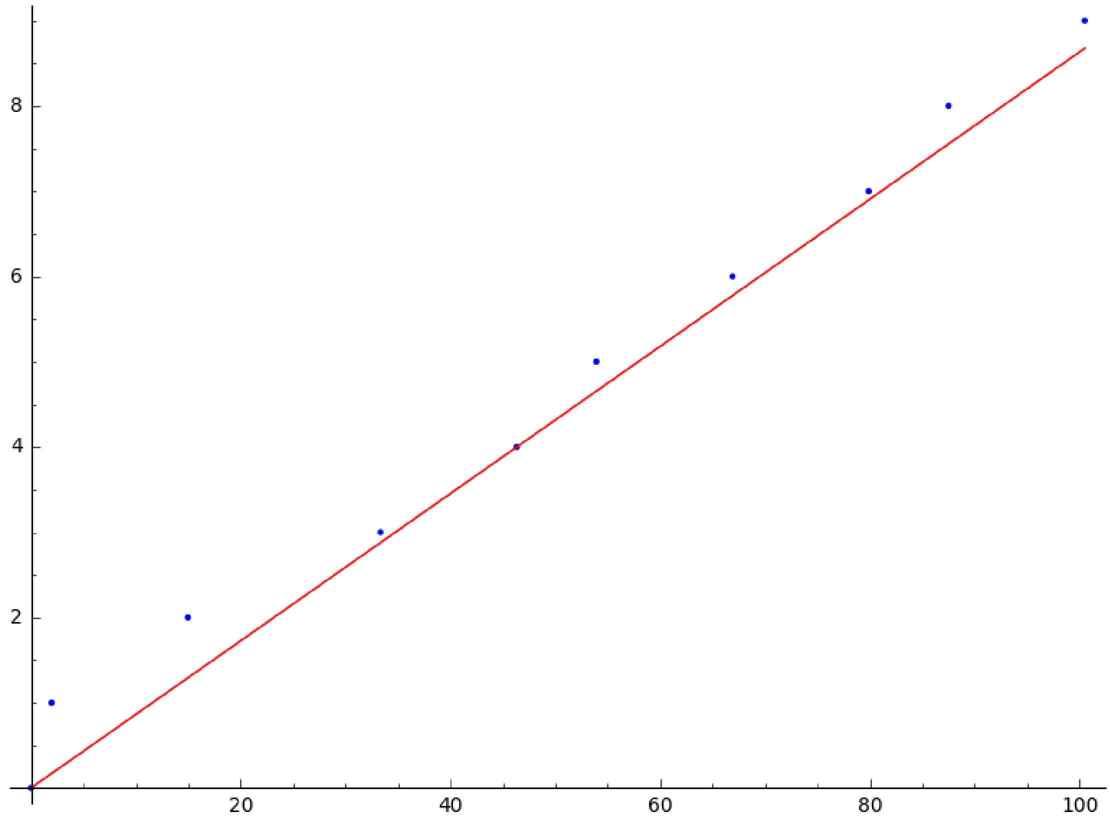




In [ ]:

In [ ]:

In [98]:
```
d = Va((0.471231120312, 0.711212768921))
d = d / d.norm()
bs = birkhoff_sums_octagon(0, d, 100)
l = bs[-1][0]
mean = d.dot_product(vector((0,2))) / area
point2d(bs) + line2d([(0,0), (l,mean*l)], color='red')
```

Out[98]:




In [ ]:

```
In [114]: print "{:25} {:17} {:15}".format("direction", "empirical mean", "transver
          for _ in range(10):
              d = (RDF**2).random_element(min=0,max=1) # random direction
              d /= d.norm()                             # normalize it
              x = RDF.random_element(0,1)               # initial point
              bs = birkhoff_sums_octagon(x, d)          # the number of intersectio
              mean = d.dot_product(vector((0,2))) / area
              empirical = bs[-1][1] / bs[-1][0]
              print "{:25} {:17} {:15}".format(d.n(digits=4), empirical.n(digits=5)
```

```
direction                 empirical mean    transversal measure
(0.8033, 0.5955)          0.056278          0.061669
(0.6575, 0.7534)          0.080703          0.078018
(0.8602, 0.5100)          0.051629          0.052810
(0.7882, 0.6155)          0.056878          0.063735
(0.07193, 0.9974)         0.11069           0.10329
(0.3802, 0.9249)          0.11372           0.095778
(0.8467, 0.5321)          0.052366          0.055096
(0.8180, 0.5752)          0.061633          0.059569
```

```
(0.7301, 0.6833)          0.068280          0.070759
(0.06648, 0.9978)         0.10504           0.10332
```

In [ ]:

In [ ]:

In the cell below we compute and plot the renormalized Birkhoff sums

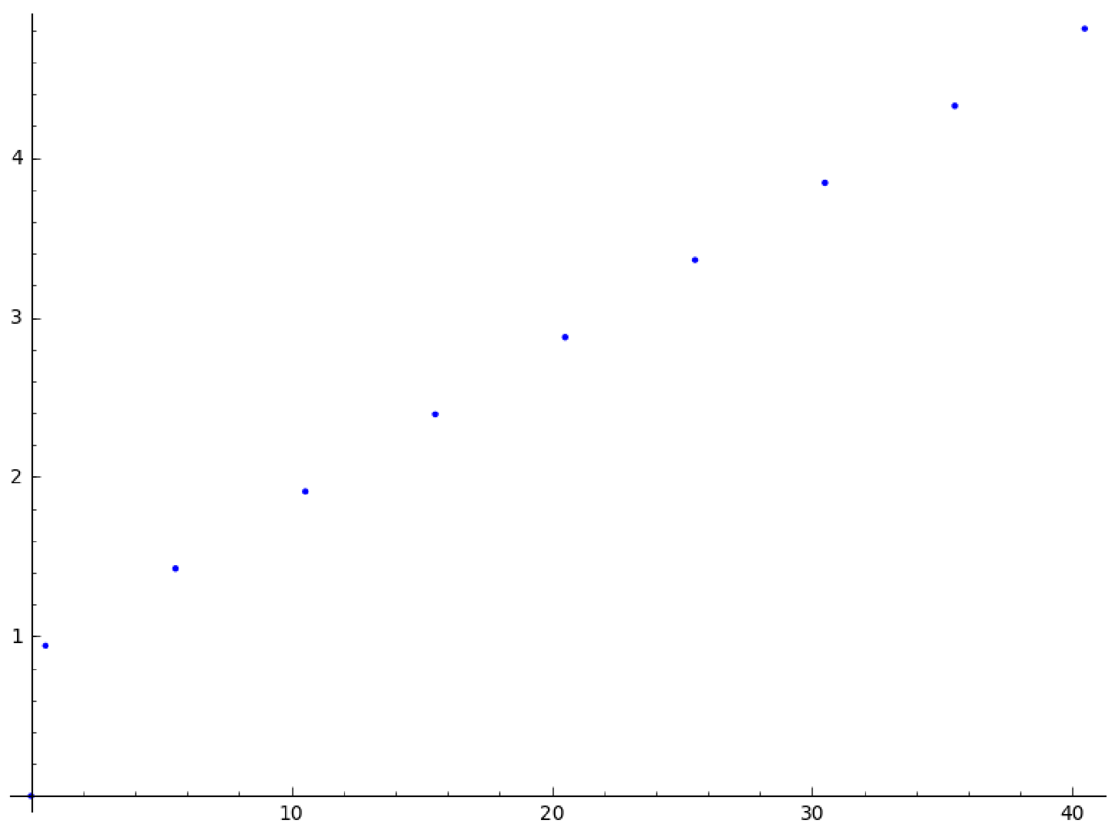
$$i(S, \gamma, \theta, x, t) - t\mu_\theta(\gamma)$$

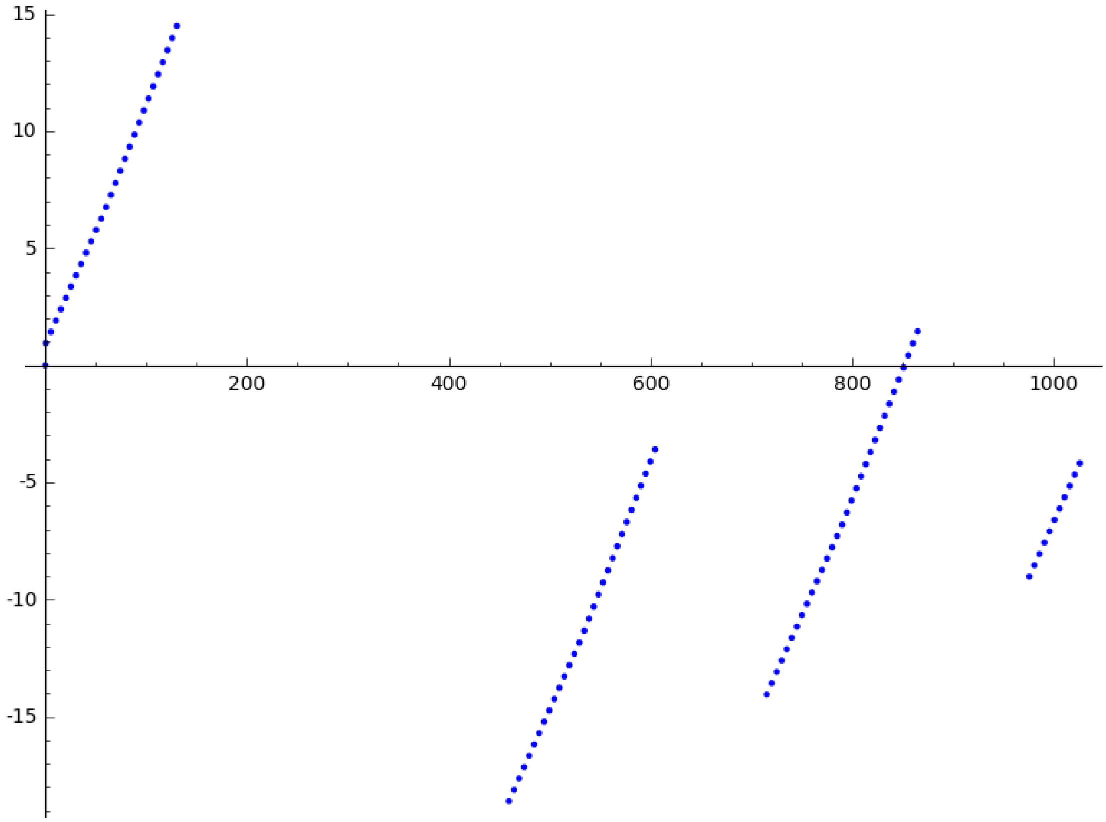
for $t$ in wider and wider ranges.

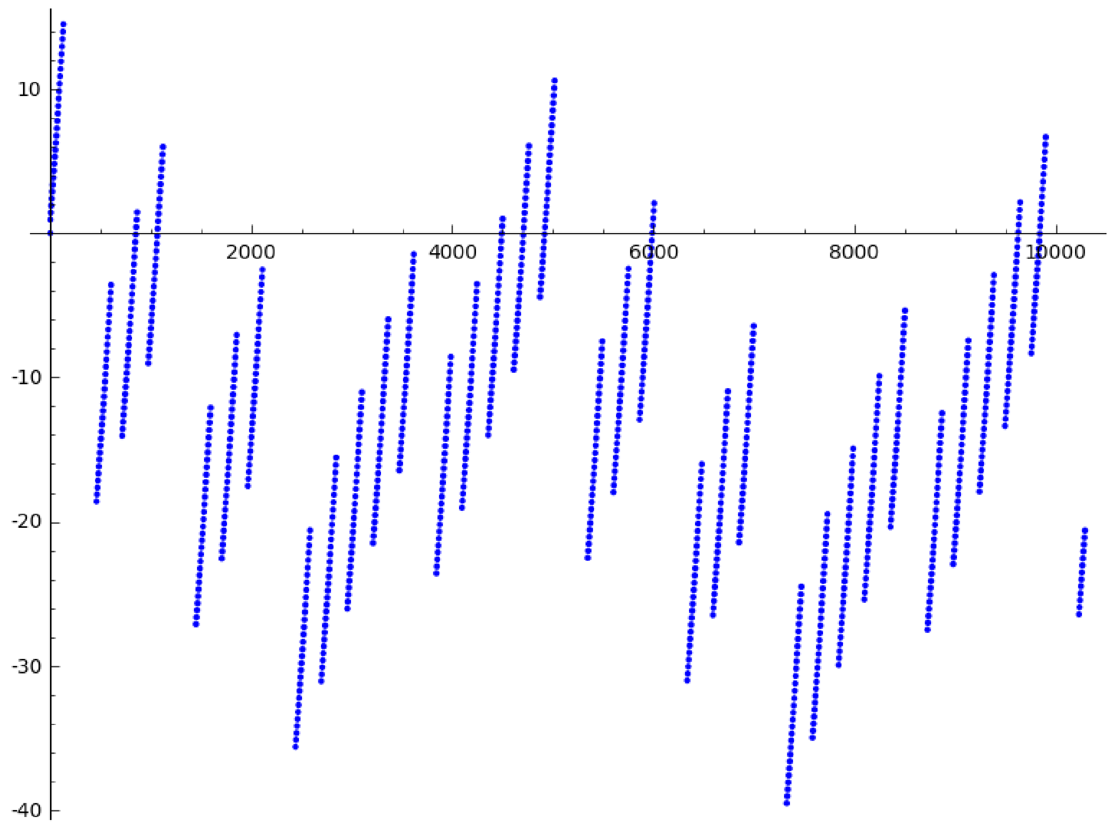
In [ ]:

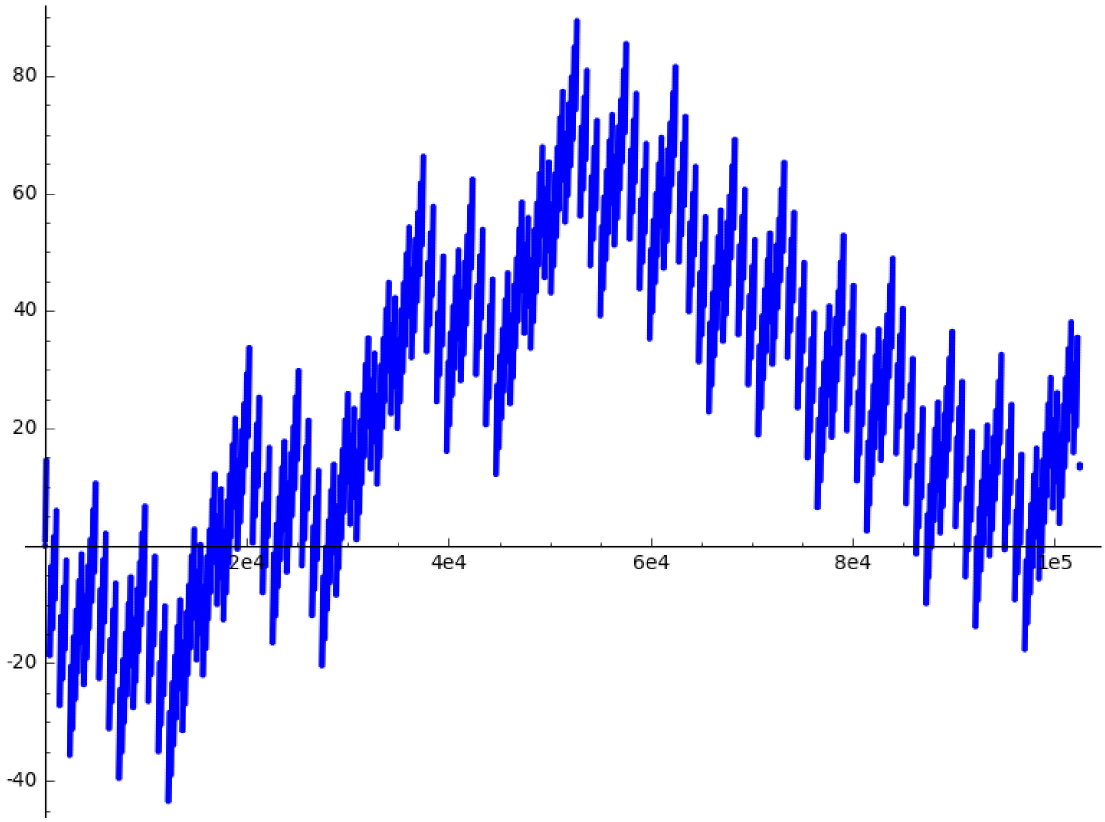
```
In [129]: d = (RDF**2).random_element(min=0,max=1)     # random direction
          d /= d.norm()                                # normalize it
          mean = d.dot_product(vector((0,2))) / area   # transversal length
          bs = birkhoff_sums_octagon(0.5, d, 10**6)    # Birkhoff sum
          rbs = [(x, y - x*mean) for x,y in bs]        # renormalized Birkhoff sums
          print "direction: {}   mean={}".format(d, mean)
          for n in [(RR(len(rbs))**(k/5.)).floor() for k in range(1,6)]:
              point2d(rbs[:n]).show()
```

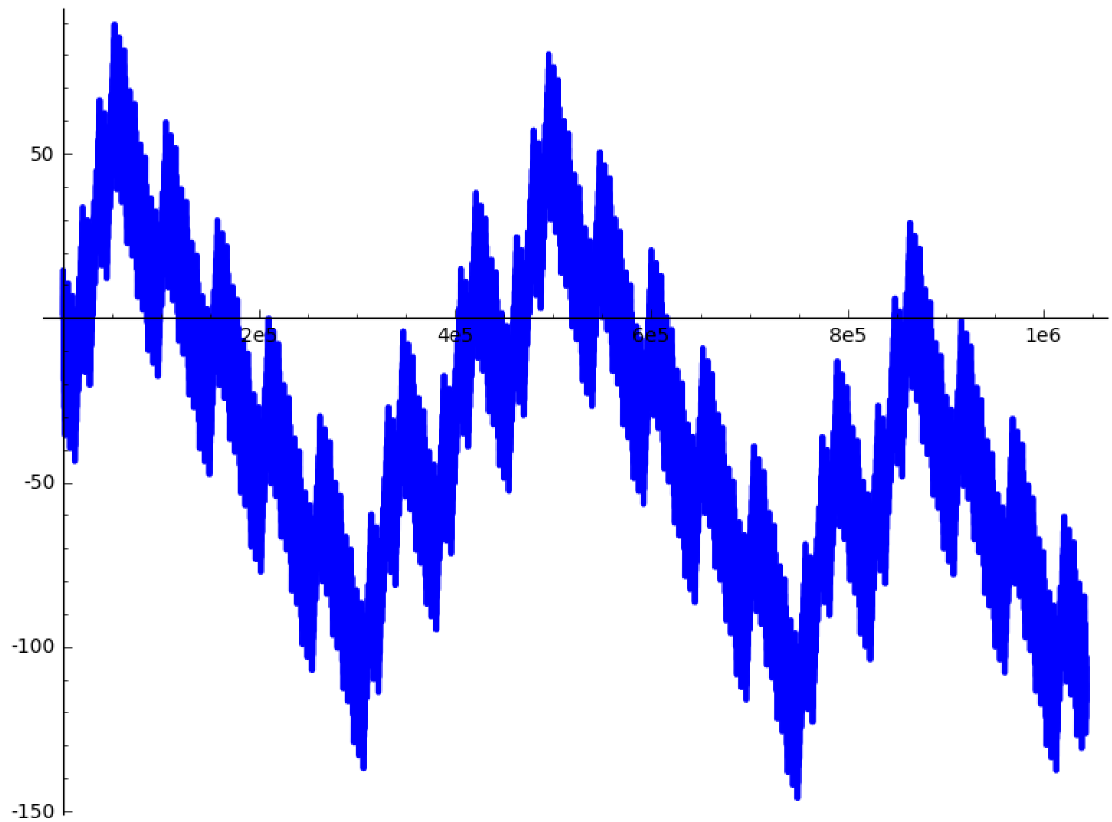
```
direction: (0.013206355342489\83, 0.9999127922866913)   mean=0.103544359939
```




15
10
5
-5
-10
-15
200
400
600
800
1000


10
-10
-20
-30
-40
2000
4000
6000
8000
10000


80
60
40
20
-20
-40
2e4
4e4
6e4
8e4
1e5




```
In [58]: # to save the picture of the renormalized Birkhoff sum, just uncomment the
         # point2d(rbs, pointsize=1).save('/tmp/birkhoff2.pdf', figsize=(4,2))
```

```
In [ ]:
```

```
In [ ]:
```

## 2 Computing Lyapunov exponents using the surface dynamics package

This part needs the surface dynamics package to be installed. To that purpose, open a terminal and run the following command

```
sage -pip install surface_dynamics
```

The code in this package was written mostly by me and the part about Lyapunov exponents by Charles Fougeron.

```
In [114]: from surface_dynamics.all import *
```

```
In [ ]:
```

In [121]:
```
# here is more or less the Lyapunov exponents associated to the octagon
# (exact values are 1, 1/3)
p = iet.Permutation('A B C D', 'B D A C')
p.stratum()
```

Out[121]:
```
H_2(2)
```

In [117]:
```
p.lyapunov_exponents_H_plus()
```

Out[117]:
```
[1.0055853164501598, 0.3296131518846267]
```

In [ ]:

In [122]:
```
# and here the Lyapunov exponents associated to the windtree
# (exact values are 1, 2/3, 2/3, 1/3, 1/3)
S4 = SymmetricGroup(4)
i = S4('')
h = S4('(1,2)(3,4)')
v = S4('(1,3)(2,4)')
q = p.cover([i,h,v,i])
```

In [123]:
```
q.lyapunov_exponents_H_plus()
```

Out[123]:
```
[1.0019240494550279,
 0.676566352078252,
 0.6656948165428316,
 0.35055284930066477,
 0.33435555212474305]
```

In [ ]: