Bridges between Automatic Sequences and Algebra and Number Theory
Centre de Recherches Mathématiques (Montréal) – April 24–28, 2017

Exercises to be done before the talks

Exercise 1 (Stepan Starosta)**.** Let $\alpha \in \mathbb{R}$ such that its continued fraction expansion is purely periodic of period ℓ. That is $\alpha = [(a_0, a_1, a_2, a_3, ..., a_{\ell-1})^\omega]$, where the superscript ω denotes infinite repetition. Set

$$M(\alpha) = \text{ accumulation points of } \left\{(-1)^{n+1}\left([a_{n+1}, a_{n+2}, \ldots] + [0, a_n, a_{n-1}, a_{n-2}, \ldots, a_1]\right) : n \in \mathbb{N}\right\}.$$

1. Find $M(\alpha)$ for some α of your choice.
2. Write an algorithm which has α (or the periodic part of the continued fraction of α) as an input and which outputs a list of all elements of $M(\alpha)$.
3. Find $\alpha \in \mathbb{R}$ with ℓ even such that $\#M(\alpha) \neq \ell$.
4. Find $\alpha \in \mathbb{R}$ such that $M(\alpha) = M(2\alpha)$.

Sample values :

— $M\left(\frac{1+\sqrt{5}}{2}\right) = \left\{\frac{1}{5}\sqrt{5}, -\frac{1}{5}\sqrt{5}\right\}$;

— $M\left(\sqrt{2}+1\right) = \left\{-\frac{1}{4}\sqrt{2}, \frac{1}{4}\sqrt{2}\right\}$;

— $M\left(2\sqrt{2}+2\right) = \left\{-\frac{1}{8}\sqrt{2}, \frac{1}{2}\sqrt{2}\right\}$;

— $M\left(\frac{\sqrt{37}+4}{7}\right) = \left\{\frac{2}{37}\sqrt{37}, -\frac{2}{37}\sqrt{37}, -\frac{3}{74}\sqrt{37}, -\frac{7}{74}\sqrt{37}, \frac{3}{74}\sqrt{37}, \frac{7}{74}\sqrt{37}\right\}$;

Remark : Computer experiments will be carried out in the computer algebra system `SageMath` which is a recommended system for this exercise (although it may also be done by hand). Its cloud (SageMathCloud) version does not require installation and its basic features are free. The details on how to work with continued fractions in `SageMath` can be found in the official documentation.

Exercise 2 (Narad Rampersad (to be done **before** the lecture))**.** In this exercise we will use Hamoon Mousavi's Walnut software to compute the lengths of squares occurring in certain sequences.

1. Download the Walnut Prover software and the Walnut Manual from

   `https://cs.uwaterloo.ca/~shallit/papers.html`

   (search the page for "Walnut").
2. Install the software on your system. Installation instructions are in Section 6 of the manual. You need Java and Graphviz to be installed on your system to use Walnut.
3. Read Section 2.3 of the manual on the Thue–Morse word (and as much else of the manual as you need to understand the basic operation of Walnut). Perform the example given at the bottom of p. 23 to compute the orders of the squares in the Thue–Morse word.
4. Now let $T_3 = (t_n)_{n\geq 0}$ be the base-3 generalization of the Thue–Morse word defined as follows. First let $s_3(n)$ denote the sum of the digits of the base-3 expansion of n. Then define $t_n = s_3(n) \bmod 3$. So

   $$T_3 = 012120201120201012 20\cdots.$$

   Use Walnut to compute the orders of the squares in T_3. [Suggestion : Copy the file T.txt in the "Word Automata Library" directory and modify it so that it defines an automaton that computes T_3.]

Exercise 3 (Narad Rampersad (to be done **before** the lecture))**.** The Thue–Morse word T can also be defined as the fixed point of the morphism μ that maps $0 \to 01$ and $1 \to 10$; i.e. $T = \mu^{\omega}(0) = \mu(T)$. The word T is *overlap-free* ; i.e., it contains no occurrences of words of the form $axaxa$, where a is a letter and x is a (possible empty) word. Use this property to prove "by hand" that any square in the Thue–Morse word is of the form $\mu^k(xx)$ where $x \in \{0, 1, 010, 101\}$.

Exercise 4 (Élise Vandomme (to do before the lecture))**.** The k-kernel of a sequence $(s_n)_{n\geq 0}$ is a set $\mathcal{K}_k(s)$ of subsequences of s defined by

$$\mathcal{K}_k(s) := \{s(k^e n + r)_{n\geq 0} \mid e \geq 0, 0 \leq r < k^e\}.$$

Determine the 2-kernel of

— the Thue–Morse sequence $(t_n)_{n\geq 0}$ where $t_0 = 0$, $t_{2n} = t_n$ and $t_{2n+1} = 1 - t_n$,

— the period-doubling sequence that is the fixed point of the morphism f such that $f(0) = 01$ and $f(1) = 00$.