Solving the S-unit equation in Sage

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joint with

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Goal

Have Sage solve the equation $x + y = 1$ in an *infinite family of rational numbers* the S-units.

Idea

The S-integers are

integers where we're allowed to divide by some primes.

Definition

Let $S = \{p_1, \ldots, p_n\}$, a finite set of primes. Define the S-integers

$$\mathcal{O}_S := \{a/b \ : \ a, b \in \mathbb{Z}, \ \gcd(a, b) = 1, \ b = p_1^{e_1} \cdots p_n^{e_n}\}$$

The S-units are the units $\mathcal{O}_S^\times$.

Example

$$S = \{2, 3\}, \ \mathcal{O}_S^\times = \{(-1)^a 2^{e_1} 3^{e_2}\}$$

Sage - trac ticket #22148

(Alvarado, Koutsianas, Malmskog, Rasmussen, Vincent, W.)

```
sage: K.<a> = NumberField(x)
....: S = (K.ideal(2), K.ideal(3))
....: %time solns = solve_S_unit_equation(K, S)
CPU times: user 24min 15s, sys: 10.6 s, total: 24min 26
Wall time: 24min 17s
sage: len(solns)
11
```

Why??

- (Original Motivation) Classify Picard curves over $\mathbb{Q}$ with good reduction away from 3
- Sums of products of primes
- Finitely generated subgroups of $\mathbb{C}^\times$
- Recurrence sequences of complex or algebraic numbers
- Irreducible polynomials and arithmetic graphs
- Decomposable form equations (Thue-Mahler equations)
- Algebraic number theory
- Transcendental number theory

S-unit Structure

$$S = \{p_1, \ldots, p_n\}$$
$$\mathcal{O}_S^\times = \{(-1)^a p_1^{e_1} \cdots p_n^{e_n} \ : \ e_1, \ldots, e_n \in \mathbb{Z}\}.$$

“Theorem” (Hasse)

$$\mathcal{O}_S^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^n$$

Theorem (Baker-Wüstholz, Smart, Pethö-de Weger)

Finitely many pairs $(\tau_0, \tau_1) \in \mathcal{O}_S^\times$ *satisfy* $\tau_0 + \tau_1 = 1$.

Proof.

A bound on the exponents exists. □

Preliminary bound

$$\sigma + \tau = 1$$

$$\sigma = (-1)^a p_1^{e_1} \cdots p_n^{e_n} \quad \tau = (-1)^b p_1^{f_1} \cdots p_n^{f_n}$$

$$H = \max\{|e_1|, \ldots, |e_n|, |f_1|, \ldots, |f_n|\}$$

Baker-Wüstholz

$$\log|\sigma| = e_1 \log(p_1) + \cdots + e_n \log(p_n) > e^{-c_4 \log(H)}$$

Smart

$$\log|\sigma| < c_5 e^{-c_6 H}$$

$$c_4 \log(H) > -\log(c_5) + c_6 H$$

Preliminary bound

Pethö-de Weger

There is a constant K_0 such that

$$\max(|\text{exponents}|) < K_0$$

Bad News

The K_0 constructed this way are HUGE.

Example

$$S = \{2, 3\},\ \mathcal{O}_S^\times = \{(-1)^a 2^{e_1} 3^{e_2}\}$$

```
sage: K.<a> = NumberField(x)
....: S = (K.ideal(2), K.ideal(3))
....: Sunits = UnitGroup(K, S=S)
....: %time K0_func(Sunits, [1,-1])
CPU times: user 232 ms, sys: 8 ms, total: 240 ms
Wall time: 237 ms
7.15036996673845702861312543 06e17
```

LLL Reduction

LLL allows us to construct a significantly “better” basis.

LLL uses the Gram Schmidt process but restricts to a lattice.

The perk of LLL is that it acts like magic to reduce our bound!

****IN POLYNOMIAL TIME****

Example

$$S = \{2, 3\},\ \mathcal{O}_S^\times = \{(-1)^a 2^{e_1} 3^{e_2}\}$$

```
sage: K.<a> = NumberField(x)
....: S = (K.ideal(2), K.ideal(3))
....: Sunits = UnitGroup(K, S=S)
....: K0_func(Sunits, [1,-1])
7.1503699667384570286131254306e17
....: cx_LLL_bound(Sunits, [1,-1])
CPU times: user 568 ms, sys: 24 ms, total: 592 ms
Wall time: 575 ms
30
```

Small detail (p-adics)

Baker bound and standard LLL only guaranteed to work if the maximum exponent *occurs at an infinite prime.*
i.e. The absolute value is bigger than the exponents.
Malmskog–Rasmussen: We can assume this is true if S contains but one finite prime.
Yu: There is a p-adic Baker bound that works for this finite place.
Koutsianas: Coded Yu's bound as part of his PhD work.

p-adic Bound and LLL

Example

$$S = \{2, 3\}, \ \mathcal{O}_S^\times = \{(-1)^a 2^{e_1} 3^{e_2}\}$$

```
sage: K.<a> = NumberField(x)
....: S = (K.ideal(2), K.ideal(3))
....: Sunits = UnitGroup(K, S=S)
....: v = K.places()[0]
....: %time K1_func(Sunits, v, [1,-1])
CPU times: user 100 ms, sys: 0 ns, total: 100 ms
Wall time: 95.3 ms
2.204650291205225666538006217583e15
sage: p_adic_LLL_bound(Sunits, [1,-1])
CPU times: user 1.65 s, sys: 20 ms, total: 1.67 s
Wall time: 1.68 s
52
```

Part 2 of the story - Sieve

Now that we have an upper bound, what are the actual solutions?

$$\max(|\text{exponents}|) \leq H = 52$$

The number of pairs (σ, τ) in this range is:

$$\frac{(2H+1)^{2n}}{2} = (2(52)+1)^4/2 \approx 6.7 \times 10^7$$

Time to be creative!

Preliminary steps

Let $q \in \mathbb{Z}$ be a prime such that $q \notin S$.
Then $\mathbb{Z}/q\mathbb{Z} \cong \mathbb{F}_q$, and we can define

$$\begin{array}{rcl} \Phi_q \colon \mathcal{O}_S^\times & \to & \mathbb{F}_q^\times \\ \sigma & \mapsto & \sigma \pmod{q}. \end{array}$$

Notice that if $\sigma, \tau \in \mathcal{O}_S^\times$ such that $\sigma + \tau = 1$ then

$$\Phi_q(\sigma) + \Phi_q(\tau) = 1.$$

Let $Y_q \subseteq \mathbb{F}_q^\times$ be the intersection of the image of Φ_q with the solutions to $x + y = 1$ in $\mathbb{F}_q^\times$.

Sieve

$$\mathcal{O}_S^\times = \{(-1)^a p_1^{e_1} \cdots p_n^{e_n}\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^n, \ q \in \mathbb{Z} \setminus S$$

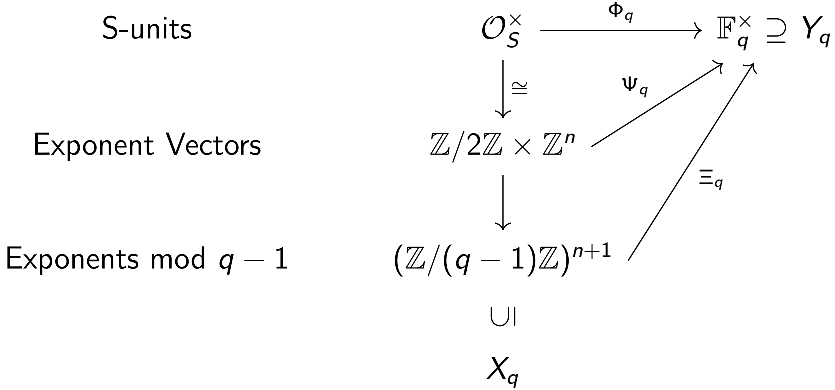
$$Y_q = \mathrm{im}(\Phi_q) \cap \text{solutions}$$

$$\begin{array}{lccc} \text{S-units} & \mathcal{O}_S^\times & \xrightarrow{\Phi_q} & \mathbb{F}_q^\times \\ & \downarrow \cong & \nearrow \Psi_q & \\ \text{Exponent Vectors} & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^n & & \end{array}$$

$$\alpha \in \mathbb{F}_q^\times \Rightarrow \alpha^{q-1} = 1$$

New Vertical Map: Take exponent vectors modulo $q-1$

Sieve



$X_q =$ all possible vectors mod $q - 1$

Narrowing using X_q and Y_q

$$X_q \subseteq (\mathbb{Z}/(q-1)\mathbb{Z})^{n+1} \xrightarrow{\Xi_q} \mathbb{F}_q^\times \supseteq Y_q$$

Definitions

- Two vectors $x, x' \in X_q$ are *complementary* if
$$\Xi_q(x) + \Xi_q(x') = 1.$$
- Let r be another prime not in S. The vectors $x \in X_q$ and $x' \in X_r$ are *compatible* if there is a $y \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^n$ s.t.
$$y \equiv x \pmod{q-1} \text{ and } y \equiv x' \pmod{r-1}.$$

Next Step: Do complementary and compatibility check for all $x \in X_q$ and drop them as we go.

We have solutions!

$$S = \{2,3\},\ \mathcal{O}_S^\times = \{(-1)^a 2^{e_1} 3^{e_2}\}$$

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Solutions

$$S = \{2, 3\},\ \mathcal{O}_S^\times = \{(-1)^a 2^{e_1} 3^{e_2}\}$$

```
sage: solns
[[(0, -1, 1), (1, -1, 0), 3/2, -1/2],
 [(0, 1, 0), (1, 0, 0), 2, -1],
 [(0, 0, -1), (0, 1, -1), 1/3, 2/3],
 [(1, 1, 0), (0, 0, 1), -2, 3],
 [(0, 2, 0), (1, 0, 1), 4, -3],
 [(0, 0, -2), (0, 3, -2), 1/9, 8/9],
 [(1, 0, -1), (0, 2, -1), -1/3, 4/3],
 [(0, -2, 1), (0, -2, 0), 3/4, 1/4],
 [(0, 0, 2), (1, 3, 0), 9, -8],
 [(1, -3, 0), (0, -3, 2), -1/8, 9/8],
 [(0, -1, 0), (0, -1, 0), 1/2, 1/2]]
```

A Larger Number Field

```
sage: K.<xi> = NumberField(x^2+x+1)
....: S = K.primes_above(3)
....: %time solve_S_unit_equation(K,S)
CPU times: user 872 ms, sys: 56 ms, total: 928 ms
Wall time: 1.81 s
[[(2, 1), (4, 0), xi + 2, -xi - 1],
 [(5, -1), (4, -1), 1/3*xi + 2/3, -1/3*xi + 1/3],
 [(5, 0), (1, 0), -xi, xi + 1],
 [(1, 1), (2, 0), -xi + 1, xi]]
```

A Larger Number Field (cont)

Thus taking $\mathbb{Q}(\xi)$ to be the number field defined by $x^2 + x + 1$, and $S = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ where $\mathfrak{p}_1\mathfrak{p}_2 = (3)$, the solutions to $x + y = 1$ in $\mathcal{O}_S^\times$ are:

$$(\xi+2, -\xi-1), (\frac{1}{3}\xi+\frac{2}{3}, -\frac{1}{3}\xi+\frac{1}{3}), (-\xi, \xi+1), \text{ and } (-\xi+1, \xi).$$