Differential Geometry $\rightsquigarrow$ Algebra $\rightsquigarrow$ Combinatorics (& back?)

SageDays 94 – Zaragoza 2018

Vít Tuček

Faculty of Science, University of Zagreb

Supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

EUROPSKI STRUKTURNI I INVESTICIJSKI FONDOVI

Operativni program KONKURENTNOST I KOHEZIJA

Europska unija
Zajedno do fondova EU

Outline

1. Differential Geometry $\rightsquigarrow$ Algebra
2. Algebra $\rightsquigarrow$ Combinatorics
3. Combinatorics $\rightsquigarrow$ Differential Geometry

Differential Geometry $\rightsquigarrow$ Algebra

Invariant differential operators

- For a homogeneous space G/P, P-representation $\mathbb{V}$ the associated homogeneous bundle is $\mathcal{V} = G \times_P \mathbb{V} \to G/P$

Invariant differential operators

- For a homogeneous space G/P, P-representation $\mathbb{V}$ the associated homogeneous bundle is $\mathcal{V} = G \times_P \mathbb{V} \to G/P$
- *invariant differential operators* between sections of two such bundles $\mathcal{V}$ and $\mathcal{W}$ must respect natural induced actions of G

$$\mathcal{D}\colon \Gamma^\infty(G/P, \mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$
$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}.$$

Invariant differential operators

- For a homogeneous space G/P, P-representation $\mathbb{V}$ the associated homogeneous bundle is $\mathcal{V} = G \times_P \mathbb{V} \to G/P$
- *invariant differential operators* between sections of two such bundles $\mathcal{V}$ and $\mathcal{W}$ must respect natural induced actions of G

$$\mathcal{D}\colon \Gamma^\infty(G/P, \mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$
$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}.$$

- (linear) differential operator $\mathcal{D}$ of order k is given by a linear map from the k-th jet prolongation

$$D\colon \Gamma^\infty(G/P, \mathcal{J}^k\mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$

Invariant differential operators

- For a homogeneous space G/P, P-representation $\mathbb{V}$ the associated homogeneous bundle is $\mathcal{V} = G \times_P \mathbb{V} \to G/P$
- *invariant differential operators* between sections of two such bundles $\mathcal{V}$ and $\mathcal{W}$ must respect natural induced actions of G

$$\mathcal{D} \colon \Gamma^\infty(G/P, \mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$
$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}.$$

- (linear) differential operator $\mathcal{D}$ of order k is given by a linear map from the k-th jet prolongation

$$D \colon \Gamma^\infty(G/P, \mathcal{J}^k\mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$

- From invariance we equivalently have homomorphism $J^k\mathbb{V} \to \mathbb{W}$, where $J^k\mathbb{V}$ denotes the algebraic jet prolongation of $\mathbb{V}$.

Invariant differential operators

- For a homogeneous space G/P, P-representation $\mathbb{V}$ the associated homogeneous bundle is $\mathcal{V} = G \times_P \mathbb{V} \to G/P$
- *invariant differential operators* between sections of two such bundles $\mathcal{V}$ and $\mathcal{W}$ must respect natural induced actions of G

$$\mathcal{D}\colon \Gamma^\infty(G/P, \mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$

$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}.$$

- (linear) differential operator $\mathcal{D}$ of order k is given by a linear map from the k-th jet prolongation

$$D\colon \Gamma^\infty(G/P, \mathcal{J}^k\mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$

- From invariance we equivalently have homomorphism $J^k\mathbb{V} \to \mathbb{W}$, where $J^k\mathbb{V}$ denotes the algebraic jet prolongation of $\mathbb{V}$.
- Passing to dual maps and taking the limit $k \to \infty$ we get

$$\mathrm{Hom}_{\mathfrak{p}}\left(\mathbb{W}^*, \mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{p})}\mathbb{V}^*\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{p})}\mathbb{W}^*, \mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{p})}\mathbb{V}^*\right)$$

Invariant differential operators

- Classification in the homogeneous case G/P
  $\rightsquigarrow$ homomorphisms (resolutions) of parabolic Verma modules
  $$M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{F}_\lambda$$
- $H^i(\mathfrak{p}_+, L) = H_i(\mathfrak{p}_-, L) = \mathrm{Tor}_i^{\mathfrak{p}_-}(\mathbb{C}, L)$ & Vermas are $\mathfrak{p}_-$-free
- Natural extension to Cartan geometries modeled on (G, P)
  $\rightsquigarrow$ multidifferential operators and curved A_∞ / L_∞ structures

Algebra $\rightsquigarrow$ Combinatorics

Highest weight theory

Complex simple Lie algebras have root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

$$\Phi \subseteq \mathfrak{h}^*$$

Highest weight theory

Complex simple Lie algebras have root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

$$\Phi \subseteq \mathfrak{h}^*$$

parabolic generalization

$$\mathfrak{g} = \mathfrak{l}_\Sigma \oplus \bigoplus_{\alpha \in \Phi_\Sigma} \mathfrak{g}_\alpha$$

Highest weight theory

Complex simple Lie algebras have root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

$$\Phi \subseteq \mathfrak{h}^*$$

parabolic generalization

$$\mathfrak{g} = \mathfrak{l}_\Sigma \oplus \bigoplus_{\alpha \in \Phi_\Sigma} \mathfrak{g}_\alpha$$

big and interesting class of their representations have highest weight $\rightsquigarrow$

$\mathbb{F}_\lambda$ or $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$

Kostant's formula [Kos61]

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{p}$ be a parabolic subalgebra with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_+$. Let $W^{\mathfrak{l}}$ be the poset of minimal coset representatives. For every $\mathfrak{g}$-integral and $\mathfrak{g}$-dominant weight λ there is isomorphism of $\mathfrak{l}$ modules

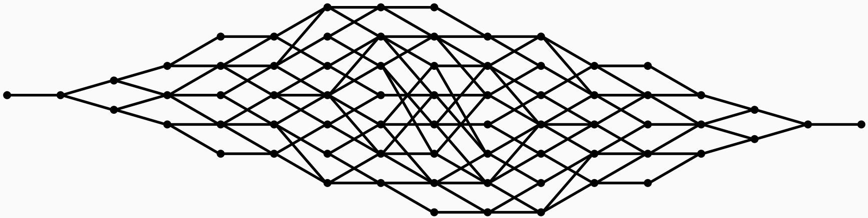
$$H^i(\mathfrak{p}_+, L(\lambda)) \simeq \bigoplus_{\substack{w \in W^{\mathfrak{l}} \\ l(w) = i}} \mathbb{F}_{w(\lambda+\rho)-\rho},$$

where $L(\lambda)$ is the finite dimensional $\mathfrak{g}$-module with highest weight λ and $\mathbb{F}_\mu$ are finite dimensional $\mathfrak{l}$ modules with highest weights μ.

Nilpotent cohomology / BGG resolution for $\mathrm{SU}(2,2)$

$$
\begin{array}{ccccccccc}
 & & & \nearrow & (2,\ -3,\ 0) & \searrow & & & \\
(0,\ 0,\ 0) & \longrightarrow & (1,\ -2,\ 1) & & & & (1,\ -4,\ 1) & \longrightarrow & (0,\ -4,\ 0) \\
 & & & \searrow & (0,\ -3,\ 2) & \nearrow & & &
\end{array}
$$

The BGG graph of type $(A_7, A_3 \times A_3)$



sage

Enright's formula

Definition

Let Ψ_λ be the set of roots orthogonal to $\lambda + \rho$.

Denote by $\Phi^+_{n,\lambda}$ the roots which satisfy the following conditions

1. $\alpha \in \Phi^+_n$ and $(\lambda + \rho, \alpha^\vee)$ is a positive integer;
2. α is orthogonal to Ψ_λ;
3. α is short if there exist a long root in Ψ_λ.

Let W_λ be the subgroup of W which is generated by reflections s_α for $\alpha \in \Phi^+_{n,\lambda}$.

Enright's formula – continued

Let Φ_λ be the subset of Φ of elements β with $s_\beta \in W_\lambda$ and let $\Phi_{\lambda,c} = \Phi_c \cap \Phi_\lambda$, $\Phi^+_{\lambda,c} = \Phi_{\lambda,c} \cap \Phi^+$.

Φ_λ is a root subsystem and $(\Phi_{\lambda,c}, \Phi_{\lambda,n})$ is (*reduced*) Hermitian symmetric pair

Theorem (3.7 of [DES91])
For unitarizable highest weight modules $L(\lambda)$ and for $i \in \mathbb{N}$ we have

$$H^i(\mathfrak{p}_+, L(\lambda)) \simeq \bigoplus_{w \in W^{c,i}_\lambda} F(\overline{w(\lambda+\rho)} - \rho)$$

where $\overline{\lambda}$ is the unique Φ^+_c-dominant element in the W_c orbit of λ and $W^{c,i}_\lambda = \{w \in W_\lambda : w\rho \text{ is } \Phi^+_{\lambda,c}\text{-dominant and } l_\lambda(w) = i\}$.

$\lambda = -(t+n-1)\omega_1 + (t+1)\omega_n$

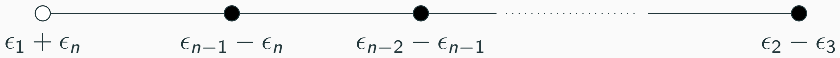
For $t = 0$ we obtain set of singular roots $\Psi^+_\lambda = \{\epsilon_1 - \epsilon_n\}$ and the set of generating roots $\Phi^+_{n,\lambda} = \{\epsilon_1 + \epsilon_n\}$. This gives the subsystem of type A_1

$$\Phi_\lambda = \{\epsilon_1 + \epsilon_n, -\epsilon_1 - \epsilon_n\}$$

and the only nontrivial cohomology is in degree 1 with weight $-n\omega_1 + \omega_{n-1}$.

$\lambda = -(t+n-1)\omega_1 + (t+1)\omega_n$

For $t \geq 1$ we get no singular roots $\Psi_\lambda^+ = \emptyset$ and the generated subsystem is of type A_{n-1}.

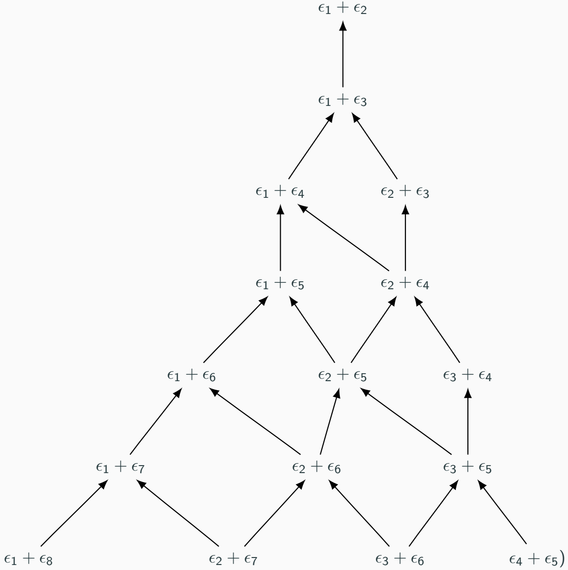


$\lambda = -(t+n-1)\omega_1 + (t+1)\omega_n$

$$
\begin{array}{c}
(-t-n+1, 0, 0, \ldots, 0, 0, t+1) \\
\downarrow \\
(-t-n, 0, 0, \ldots, 0, 1, t) \\
\downarrow \\
(-t-n-1, 0, 0, \ldots, 1, 0, t) \\
\vdots \\
(-t-2n+4, 0, 1, \ldots, 0, 0, t) \\
\downarrow \\
(-t-2n+3, 1, 0, \ldots, 0, 0, t) \\
\downarrow \\
(-t-2n+3, 0, 0, \ldots, 0, 0, t)
\end{array}
$$

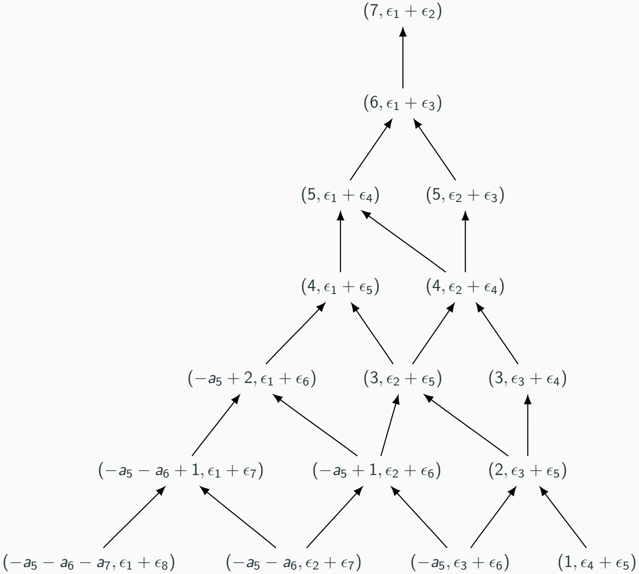
Example – scalar products with positive roots

SO*(16): $\lambda = (a_5 + 1)\,\omega_5 + a_6\omega_6 + a_7\omega_7 - (2\,a_5 + 2\,a_6 + a_7 + 8)\,\omega_8$

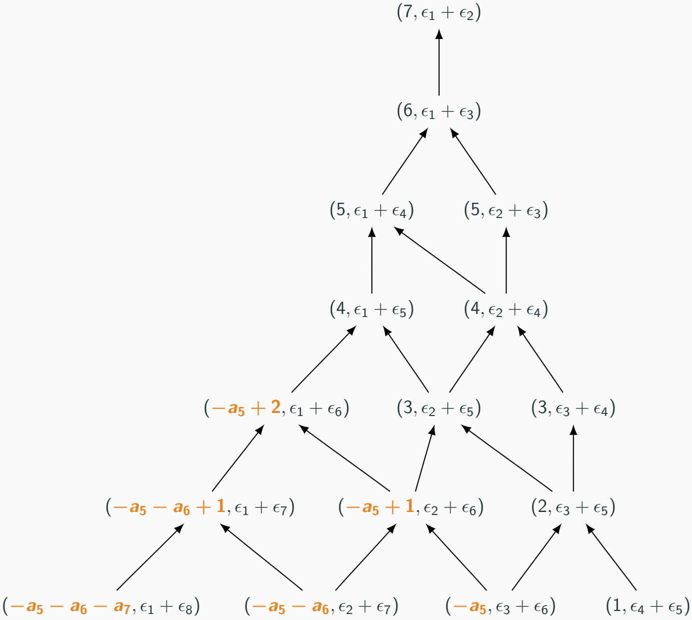


Example – scalar products with positive roots

$\mathrm{SO}^*(16)$: $\lambda = (a_5 + 1)\,\omega_5 + a_6\omega_6 + a_7\omega_7 - (2\,a_5 + 2\,a_6 + a_7 + 8)\,\omega_8$



Example – scalar products with positive roots



Combinatorics $\rightsquigarrow$ Differential Geometry

Combinatorics $\rightsquigarrow$ Differential Geometry

- homomorphisms of Verma modules are determined by so called *singular vectors* $\rightsquigarrow$ system of polynomial coefficient PDEs on vector valued polynomials
- Weyl algebra $\rightsquigarrow$ sparse linear system
- SageManifolds?

Thank you for attention!

References

- Mark G. Davidson, Thomas J. Enright, and Ronald J. Stanke. "Differential operators and highest weight representations". In: *Memoirs of the American Mathematical Society* 94.455 (1991), pp. iv+102.
- Bertram Kostant. "Lie Algebra Cohomology and the Generalized Borel-Weil Theorem". In: *The Annals of Mathematics*. Second Series 74.2 (1961). ArticleType: research-article / Full publication date: Sep., 1961 / Copyright © 1961 Annals of Mathematics, pp. 329–387.