# FUNDAMENTAL DOMAINS OF SOME ARITHMETIC GROUPS OVER FUNCTION FIELDS

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## 0. Introduction

Let $\Gamma$ be the group $GL(2, A)$, where $A$ is the polynomial ring $\mathbb{F}_q[T]$ over a finite field $\mathbb{F}_q$, and $\Delta \subset \Gamma$ a congruence subgroup. As is described in J.-P. Serre's "Trees" [10], the study of $\Delta$ and its group theoretic properties is largely the same as the study of the natural action of $\Delta$ on the Bruhat–Tits building $\mathcal{T}$ of $PGL(2, K_\infty)$, where $K_\infty$ is the completion of $K = \mathrm{Quot}(A)$ at the infinite place. For example, the abelian rank $\dim_{\mathbb{Q}}(\Delta^{ab} \otimes \mathbb{Q})$ equals the number $g(\Delta \setminus \mathcal{T})$ of independent cycles on the quotient graph $\Delta \setminus \mathcal{T}$ of $\mathcal{T}$ by $\Delta$. Further, the homology $H_1(\Delta \setminus \mathcal{T}, \mathbb{C})$ is (essentially) a space of automorphic forms over the global field $K$ (see e.g. [7]), and carries information about the arithmetic of $K$. We also mention that some of the graphs $\Delta \setminus \mathcal{T}$ have interesting combinatorial properties that deserve further investigation [8].

It is therefore desirable to dispose of a) general assertions about the structure of the graph $\Delta \setminus \mathcal{T}$, depending perhaps on the type of $\Delta$; b) formulae on numerical invariants like genera $g(\Delta \setminus \mathcal{T})$, numbers of cusps, of endpoints, ...; c) procedures that allow, $\Delta$ being specified, to automatically calculate the associated graph $\Delta \setminus \mathcal{T}$.

Most of this is achieved in the present article, for $\Delta$ one of the congruence subgroups $\Gamma(n)$, $\Gamma_1(n)$, $\Gamma_1^*(n)$, $\Gamma_0(n)$ of $\Gamma$ (definitions below). The case of the Hecke congruence subgroups $\Gamma_0(n)$, arithmetically the most interesting, turns out to be the most complicated. This is why the largest part of the paper is dedicated to this case.

The contents are as follows: In the first section, after introducing the relevant concepts, we describe the graph $\Delta \setminus \mathcal{T}$ by means of its fibering over $\Gamma \setminus \mathcal{T}$. We also collect some properties proved in [7] and needed later on. In Sec. 2, we turn to the case

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of the Hecke congruence subgroup $\Gamma_0(n)$. Propositions 2.10, 2.11, 2.14 state structural properties of the associated graph, Theorem 2.17 gives a formula for $g(\Gamma_0(n)\setminus\mathcal{T})$. These results have been known before [4, 5], but were given there in a somewhat disguised and less coherent form.

In Sec. 3, we prove (Theorem 3.3) the bijectivity of the canonical map $j_\Delta$ from $H_1(\Delta\setminus\mathcal{T},\mathbb{Z})$ to $\underline{H}_!(\mathcal{T},\mathbb{Z})^\Delta$, the canonical integral structure in the space of automorphic forms attached to $\Delta=\Gamma_0(n)$. (It is easy to show that $j_\Delta$ is injective with finite cokernel; the surjectivity comes out by a detailed analysis of $\Delta\setminus\mathcal{T}$.) We also present an algorithm (3.8), highly specific to $\Delta=\Gamma_0(n)$, that exhibits a maximal subtree of $\Delta\setminus\mathcal{T}$. It is particularly useful since it also gives a basis for $\underline{H}_!(\mathcal{T},\mathbb{C})^\Delta$, and thus enables numerical investigations of e.g. Hecke operators on this space. Section 4 is devoted to a remarkable observation, so far not fully understood, on multiplicities of automorphic representations. In the fifth and last section, we give neat descriptions and formulae for the graphs $\Delta\setminus\mathcal{T}$, for $\Delta=\Gamma(n)$, $\Gamma_1(n)$, $\Gamma_1^*(n)$. Here we have omitted proofs, since these follow largely the same lines as in the case $\Gamma_0(n)$ (but are easier).

The paper is based on the second author's Diplomarbeit [9]. It was finished during a visit of the first author at Université Paul Sabatier, Toulouse, whose generous hospitality is gratefully acknowledged.

The following notation is used throughout the paper:

$\mathbb{F}_q=$ finite field with $q=p^n$ elements

$A=\mathbb{F}_q[T]$ polynomial ring in the indeterminate $T$, with degree function "deg" ($\deg 0=-\infty$)

$K=\mathbb{F}_q(T)=\mathrm{Quot}(A)$

$K_\infty=\mathbb{F}_q((\pi))$ completion at infinity, where

$\pi=T^{-1}$

$O_\infty=\mathbb{F}_q[[\pi]]$ $\infty$-adic integers

$G=$ group scheme GL(2) with center $Z\xrightarrow{\cong}\mathbb{G}_m$

$\mathcal{K}=G(O_\infty)=\mathrm{GL}(2,O_\infty)$

$\mathcal{I}=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{K}\mid c\equiv 0(\mathrm{mod}\,\pi)\}$ Iwahori subgroup

$\Gamma=G(A)=\mathrm{GL}(2,A)$

$\mathcal{T}=$ Bruhat–Tits tree of $\mathrm{PGL}(2,K_\infty)$

If $n$ is a non-constant element of $A$, $\Gamma(n)\hookrightarrow\Gamma_1(n)\hookrightarrow\Gamma_1^*(n)\hookrightarrow\Gamma_0(n)\hookrightarrow\Gamma$ denote the subgroups of matrices that are congruent to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right)$ modulo $n$, respectively.

For a graph $\mathcal{S}$, we let $X(\mathcal{S})$ and $Y(\mathcal{S})$ be the sets of vertices, of oriented edges, respectively. For $e\in Y(\mathcal{S})$, $o(e)$, $t(e)\in X(\mathcal{S})$ and $\bar{e}\in Y(\mathcal{S})$ denote its origin, terminus, and inversely oriented edge. For all properties of graphs not explicitly cited, we refer to [10].



## 1. The Tree $\mathcal{T}$, Fundamental Domains, Harmonic Cochains

The tree $\mathcal{T}$ is described in [10, Ch. II, Sec. 1]. It is $(q+1)$-regular and provided with a transitive action of $G(K_\infty)$. Its vertices and edges are given by

$$\begin{aligned}X(\mathcal{T})&=G(K_\infty)/\mathcal{K}\cdot Z(K_\infty)\\ Y(\mathcal{T})&=G(K_\infty)/\mathcal{I}\cdot Z(K_\infty),\end{aligned}\tag{1.1}$$

where by our choice of orientation, the canonical map $Y(\mathcal{T})\longrightarrow X(\mathcal{T})$ associates with each edge its origin. For $i\in\mathbb{Z}$, let $v_i$ (resp. $e_i$) be the vertex (resp. edge) represented by the matrix $\left(\begin{smallmatrix}\pi^{-i}&0\\0&1\end{smallmatrix}\right)$. Then $o(e_i)=v_i$, $t(e_i)=v_{i+1}$. Since $\Gamma$ is discrete in $G(K_\infty)$, stabilizers in $\Gamma$ of edges or vertices are finite. More specifically, put

$$\begin{aligned}G_0&=G(\mathbb{F}_q)\hookrightarrow\Gamma\\ G_i&=\left\{\begin{pmatrix}a&b\\0&d\end{pmatrix}\in\Gamma\,\middle|\,\deg b\le i\right\}\qquad(i\ge 1).\end{aligned}\tag{1.2}$$

Then for $i\ge 0$, $G_i=$ stabilizer of $v_i$ in $\Gamma$ and $G_i\cap G_{i+1}=$ stabilizer of $e_i$. Note that $G_i\cap G_{i+1}=G_i$ if $i\ge 1$ and $G_0\cap G_1=\{$upper triangular matrices over $\mathbb{F}_q\}$.

**1.3.** As is easy to prove ([11] or [10]), the subgraph formed of the $v_i$ and $e_i$ with $i\ge 0$ is a *fundamental domain* for $\Gamma$, i.e., maps isomorphically onto the quotient graph $\Gamma\setminus\mathcal{T}$.

**1.4.** Next, let $\Delta$ be a *congruence subgroup* of $\Gamma$, which means it contains some $\Gamma(n)$, $n\in A$. We try to recover the quotient $\Delta\setminus\mathcal{T}$ by examining the "ramified covering"

$$\pi_\Delta:\Delta\setminus\mathcal{T}\longrightarrow\Gamma\setminus\mathcal{T},$$

from which we calculate $X(\Delta\setminus\mathcal{T})=\Delta\setminus G(K_\infty)/\mathcal{K}\cdot Z(K_\infty)$ and $Y(\Delta\setminus\mathcal{T})=\Delta\setminus G(K_\infty)/\mathcal{I}\cdot Z(K_\infty)$. It suffices to know the edges $e$ of $\Delta\setminus\mathcal{T}$ oriented such that $\pi_\Delta(e)=e_i$, some $i$. We thus define for $e\in Y(\Delta\setminus\mathcal{T})$, $v\in X(\Delta\setminus\mathcal{T})$

$$\begin{aligned}\mathrm{type}(e)=i&\Leftrightarrow\pi_\Delta(e)=e_i\\ \mathrm{type}(v)=i&\Leftrightarrow\pi_\Delta(v)=v_i,\\ X_i:=X_i(\Delta\setminus\mathcal{T})=\{v\mid\mathrm{type}(v)=i\},\ &Y_i:=Y_i(\Delta\setminus\mathcal{T})=\{e\mid\mathrm{type}(e)=i\}.\end{aligned}\tag{1.5}$$

From (1.2) we get the bijection

$$\begin{aligned}\Delta\setminus\Gamma/G_i&\xrightarrow{\cong}X_i(\Delta\setminus\mathcal{T})\\ \gamma&\longmapsto\gamma(v_i)\end{aligned}$$

and similarly for $Y_i$. For reasons that become apparent later, we choose the opposite identifications

$$\begin{aligned}G_i\setminus\Gamma/\Delta&\xrightarrow{\cong}X_i(\Delta\setminus\mathcal{T})\\ (G_i\cap G_{i+1})\setminus\Gamma/\Delta&\xrightarrow{\cong}Y_i(\Delta\setminus\mathcal{T})\end{aligned}\tag{1.6}$$

induced by $\gamma \longmapsto \gamma^{-1}(v_i)$, $\gamma^{-1}(e_i)$, respectively.

**1.7.** Put $\Sigma = \Sigma_\Delta$ for the set $\Gamma/\Delta$, on which $G_i$ acts from the left. Thus the orbits of the various $G_i$ or $G_i \cap G_{i+1}$ correspond to the vertices or edges of $\Delta \setminus \mathcal{T}$ of type $i$. For $\sigma \in \Sigma$ we put

$$\begin{aligned} [\sigma]_i &= \text{class of } \sigma \bmod G_i &&= \text{vertex of type } i \text{ of } \Delta \setminus \mathcal{T}, \\ [\sigma]_{i,i+1} &= \text{class of } \sigma \bmod G_i \cap G_{i+1} &&= \text{edge of type } i \text{ of } \Delta \setminus \mathcal{T}. \end{aligned}$$

We have canonical maps (the origin and terminus maps):

$$\begin{aligned} o_i : Y_i &\longrightarrow X_i & \text{and} \quad t_i : Y_i &\longrightarrow X_{i+1} \\ [\sigma]_{i,i+1} &\longmapsto [\sigma]_i & [\sigma]_{i,i+1} &\longmapsto [\sigma]_{i+1}. \end{aligned}$$

For $i \geq 1$, $o_i$ is bijective, and

$$\begin{aligned} \rho_i : X_i &\longrightarrow X_{i+1} \\ [\sigma]_i &\longmapsto [\sigma]_{i+1} \end{aligned}$$

is well-defined.

**1.8.** Let $n$ have minimal degree $d$ such that $\Gamma(n)$ is contained in $\Delta$. Since $G_i$ acts on $\Sigma$ via $p_n : G_i \longrightarrow \Gamma/\Gamma(n)$ and $p_n(G_{d-1}) = p_n(G_d) = \ldots$, $t_i$ and $\rho_i$ are bijective for $i \geq d-1$. Hence the subgraph of $\Delta \setminus \mathcal{T}$ consisting of the edges of type $\geq d-1$ is a disjoint union of $X_{d-1}(\Delta \setminus \mathcal{T}) = G_{d-1} \setminus \Sigma$ half-lines •——•——•—— ···, called the *cusps* of $\Delta \setminus \mathcal{T}$. We often identify the set cusp $(\Delta)$ of cusps with $X_{d-1}$. Our primary interest is in the *genus* $g(\Delta \setminus \mathcal{T})$ (= rank of the first homology group $H_1(\Delta \setminus \mathcal{T}, \mathbb{Z})$) and in constructions of maximal subtrees of $\Delta \setminus \mathcal{T}$. Now $g(\Delta \setminus \mathcal{T})$ certainly doesn't change if we cut off all the edges of type $\geq d$ (i.e., essentially the cusps). The Euler formula then gives

$$\begin{aligned} g(\Delta \setminus \mathcal{T}) &= 1 + \sharp\{\text{non-oriented edges}\} - \sharp\{\text{vertices}\} \\ &\qquad \text{of the truncated graph} \\ &= 1 + \sum_{0 \leq i \leq d-1} \sharp Y_i(\Delta \setminus \mathcal{T}) - \sum_{0 \leq i \leq d} \sharp X_i(\Delta \setminus \mathcal{T}) \qquad (1.9) \\ &= 1 + \sharp Y_0(\Delta \setminus \mathcal{T}) - \sharp X_0(\Delta \setminus \mathcal{T}) - \sharp\text{cusp}(\Delta) \end{aligned}$$

in view of the bijectivity of $o_i$, $i \geq 1$. To that numerical identity corresponds the geometrical fact that $\Delta \setminus \mathcal{T}$ is homotopic with the graph $(\Delta \setminus \mathcal{T})^*$ obtained by collapsing each edge of type $\geq 1$ to one vertex. Note that $(\Delta \setminus \mathcal{T})^*$ is a bipartite graph with vertices $X_0(\Delta \setminus \mathcal{T})$ and cusp $(\Delta)$ and edges $Y_0(\Delta \setminus \mathcal{T})$. Hence:

**1.10.** A maximal subtree of $\Delta \setminus \mathcal{T}$ may be obtained by deleting certain edges of type zero.

**1.11.** Let $\gamma \in \Gamma$ and $i \geq 0$. The stabilizer $\Gamma_v$ of $v := \gamma^{-1}(v_i)$ in $\Gamma$ is $G_i^\gamma = \gamma^{-1} G_i \gamma$, the stabilizer in $\Delta$ is $G_i^\gamma \cap \Delta$. On the other hand, the stabilizer of $\gamma\Delta \in \Sigma$ in $G_i$ is $G_i \cap \gamma\Delta\gamma^{-1}$, conjugate to $G_i^\gamma \cap \Delta$. Hence the fixed groups $\Delta_v$ of vertices $v$ (and, similarly, $\Delta_e$ of edges $e$) may be detected through the action of the $G_i$ on the finite set $\Sigma$. Note that $\Delta$ acts on $\mathcal{T}$ via its quotient $\bar{\Delta}$ by the finite group $\Delta \cap Z(K_\infty)$. For edges $e$ of $\mathcal{T}$, we therefore define the *weight* as



$$w(e) = \sharp\bar{\Delta}_e = [\Delta_e : \Delta_e \cap Z(K_\infty)].$$

**1.12.** We let $\underline{H}(\mathcal{T}, \mathbb{Z})$ be the group of alternating harmonic maps $\varphi : Y(\mathcal{T}) \longrightarrow \mathbb{Z}$, i.e., maps that satisfy

$$\begin{aligned} &\text{(i)} \quad \varphi(e) + \varphi(\bar{e}) = 0 \ \forall e \in Y(\mathcal{T}) \qquad \text{and} \\ &\text{(ii)} \quad \sum_{o(e)=v} \varphi(e) = 0 \ \forall v \in X(\mathcal{T}). \end{aligned}$$

Further, $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Delta$, the group of *harmonic cochains* for $\Delta$, is the subgroup of those $\varphi$ that are $\Delta$-invariant and have compact support modulo $\Delta$ (i.e., vanish on the cusps of $\Delta \setminus \mathcal{T}$). Similar notation will be used for maps $\varphi$ with values in $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$....

Proofs of the next three results may be found in [7].

**1.13. Proposition.** (*loc. cit.* 3.2). *Let $\bar{\Delta} = \Delta^{ab}/\mathrm{tor}(\Delta^{ab})$ be the maximal torsion-free abelian quotient of $\Delta$. There is a canonical isomorphism of $\bar{\Delta}$ with $H_1(\Delta \setminus \mathcal{T}, \mathbb{Z})$, the first homology group of $\Delta \setminus \mathcal{T}$, and a canonical injection with finite cokernel*

$$\begin{aligned} j_\Delta : H_1(\Delta \setminus \mathcal{T}, \mathbb{Z}) &\hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Delta, \\ \varphi &\longmapsto \varphi^* \end{aligned}$$

*the harmonic cochain $\varphi^*$ being defined through $\varphi^*(e) = w(e)\varphi(e)$.*

(Recall that $H_1(\mathcal{S}, \mathbb{Z})$ of a graph $\mathcal{S}$ is the set of $\varphi : Y(\mathcal{S}) \longrightarrow \mathbb{Z}$ that have finite support and satisfy (i) and (ii) in (1.12).)

**1.14. Proposition.** (*loc. cit.*, 3.4.5). *$j_\Delta$ is bijective whenever $\bar{\Delta} = \Delta/\Delta \cap Z(K_\infty)$ has no non-$p$-torsion elements.*

This applies in particular to the congruence subgroups $\Gamma(n)$ and $\Gamma_1(n)$.

**1.15. Proposition.** (*loc. cit.*, 3.3.3). *$j_\Delta$ is bijective whenever there exists a maximal subtree $\mathcal{S}$ of $\Delta \setminus \mathcal{T}$ such that all the edges $e \in Y(\Delta \setminus \mathcal{T}) - Y(\mathcal{S})$ have weight $w(e) = 1$.*

Later on we will see that (1.15) may be applied to the groups $\Gamma_1^*(n)$ and $\Gamma_0(n)$.

**1.16.** In what follows, we shall focus on the cases where $\Delta$ is one of the groups $\Gamma(n)$, $\Gamma_1(n)$, $\Gamma_1^*(n)$, $\Gamma_0(n)$. Let $n \in A$ be given as

$$n = \prod_{1 \leq i \leq s} f_i^{r_i} = \prod_{1 \leq i \leq s} n_i,$$

where the $f_i$ are different monic primes with $\deg f_i = l_i$, $\deg n = \Sigma r_i l_i =: d$. We further put $q_i := q^{l_i} = \sharp A/f_i$. We will have need for some arithmetic functions related to $n$, viz.,

$$\begin{aligned} \varphi(n) &= \prod_{1\le i\le s} q_i^{r_i-1}(q_i-1) \\ \epsilon(n) &= \prod_{1\le i\le s} q_i^{r_i-1}(q_i+1) \\ \sigma_0(n) &= \prod_{1\le i\le s}(r_i+1) \\ \kappa(n) &= \prod_{1\le i\le s}(q_i^{[(r_i-1)/2]}+q_i^{[r_i/2]}). \end{aligned} \tag{1.17}$$

Here [.] denotes the "greatest integer" function. Note that all of these are multiplicative, $\varphi(n)$ is the order of the multiplicative group $(A/n)^*$, $\epsilon(n)$ the order of $\mathbb{P}^1(A/n)$, the projective line over the finite ring $A/n$, and $\sigma_0(n)$ is the number of (monic) divisors of $n$.

## 2. The Fundamental Domain for Hecke Congruence Subgroups

In this section, $\Delta$ is a Hecke congruence subgroup $\Gamma_0(n)$ with $n$ as in (1.16). We let

$$\mathbb{P} := \mathbb{P}^1(A/n) = \{(u:v) | u,v \in A/n, (A/n)u + (A/n)v = A/n\}, \tag{2.1}$$

where as usual, $(u:v)$ is the equivalence class of $(u,v)$ modulo $(A/n)^*$. It is easy to see that

$$\begin{aligned} \Gamma/\Delta &\xrightarrow{\cong} \mathbb{P} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto (a:c) \end{aligned} \tag{2.2}$$

as $\Gamma$-sets, where the action of $\Gamma$ on $\mathbb{P}$ is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(u:v) = (au+bv : cu+dv)$. Here and in the sequel, we write representatives for residue classes, well-definedness being obvious. Thus $\Sigma_\Delta = \mathbb{P}$. Furthermore, $\mathbb{P}$ splits according to the prime decomposition of $n$:

$$\mathbb{P} \xrightarrow{\cong} \mathbb{P}^1(A/n_1) \times \cdots \times \mathbb{P}^1(A/n_s) \tag{2.3}$$

with diagonal $\Gamma$-action. Each component may be represented as

$$\mathbb{P}^1(A/n_i) = \bigcup_{0\le h\le r_i} \mathbb{P}^1(A/n_i)(h), \tag{2.4}$$

where

$$\mathbb{P}^1(A/n_i)(h) = \{(u : f_i^h) | u \bmod f_i^{r_i-h}, (u,f_i)=1 \text{ if } 1\le h\le r_i-1\}$$

(we require that $u=1$ for $h=r_i$).

The number $h$ is called the *height* $h(x)$ of $x=(u:f_i^h)$. Note that each $x$ with $h(x)\ge 1$ may also be represented as $x=(1:v)$ with $f_i|v$. Each $\underline{x}=(x_1,\ldots,x_s)\in \mathbb{P}=\Pi\mathbb{P}^1(A/n_i)$ has a well-defined height $(h_1,\ldots,h_s)$ with $0\le h_i = h(x_i)\le r_i$, and each component $\mathbb{P}^1(A/n_i)$ contains $\mathbb{P}^1(\mathbb{F}_q) = \{(u:1)|u\in\mathbb{F}_q\}\cup\{(1:0)\} = \mathbb{F}_q\cup\{\infty\}$.



For describing $\Delta\setminus\mathcal{T}$, we have to calculate the orbits of several groups on $\mathbb{P}$, notably those of $G_0$, $G_0\cap G_1$, and $G_{d-1}$.

In order to simplify notation, we now put

$$\begin{aligned} &G := G_0 = \mathrm{GL}(2,\mathbb{F}_q), \quad B := G_0\cap G_1, \quad Z := Z(\mathbb{F}_q) \xrightarrow{\cong} \mathbb{F}_q^*, \\ &U := \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) | b\in\mathbb{F}_q\right\}, \quad T := \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) | a,d\in\mathbb{F}_q^*\right\}, \quad T^{ns}\cong\mathbb{F}_{q^2}^* \end{aligned} \tag{2.5}$$

a fixed non-split maximal torus in $G$ (any two of these are conjugate). The following intersection properties are well-known and easily proved.

**2.6. Lemma.** *Let $g,h,k\in G$ and $x^g = g^{-1}xg$.*

i) $\sharp(B^g\cap U^h)>1 \Rightarrow U^h\subset B^g$;

ii) $B^g\ne B^h \Rightarrow \exists k\in G$ *such that* $T^k = B^g\cap B^h$;

iii) $B^g, B^h, B^k$ *pairwise different* $\Rightarrow B^g\cap B^h\cap B^k = Z$;

iv) $B^g\cap T^{ns} = Z$;

v) $(T^{ns})^g \ne (T^{ns})^h \Rightarrow (T^{ns})^g\cap(T^{ns})^h = Z$.

We now first treat the primary case.

**2.7. Lemma.** *Let $x=(u:v)\in\mathbb{P}=\mathbb{P}^1(A/n)$, where $n=f^r$, $f$ prime of degree $l$. The stabilizer of $x$ in $G=GL(2,\mathbb{F}_q)$ is conjugate to one of the following groups:* a) $B$; b) $T^{ns}$; c) $Z\cdot U = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) | a\in\mathbb{F}_q^*, b\in\mathbb{F}_q\right\}$; d) $Z$. *We have*

*case* a $\Leftrightarrow$ $x\in\mathbb{P}^1(\mathbb{F}_q)\hookrightarrow\mathbb{P}$.
*There is one orbit of this type, of length* $q+1$;

*case* b $\Leftrightarrow$ $l$ *is even and* $x\in\mathbb{P}^1(\mathbb{F}_{q^2})-\mathbb{P}^1(\mathbb{F}_q)\hookrightarrow\mathbb{P}$.
*There is one such orbit, of length* $q(q-1)$;

*case* c $\Leftrightarrow$ $x\in\{(u:1)|u\in A/n-\mathbb{F}_q, \exists e\in\mathbb{F}_q$ *s.t.* $(u-e)^2=0\}\cup\{(1:v)|0\ne v\in A/n$ *s.t.* $v^2=0\}$.
*There are* $(q_1^{[r/2]}-1)/(q-1)$ *such orbits, of length* $q^2-1$ $(q_1 := q^l)$;

*case* d $\Leftrightarrow$ *none of a, b, c. Orbits have length* $q(q^2-1)$.

**Proof.** Let $x\in\mathbb{P}$ be fixed under $\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in G-Z$. Suppose first that $h(x)=0$, $x=(u:1)$. Then

$$cu^2+(d-a)u-b=0. \tag{*}$$

If (*) is quadratic irreducible, $u\in\mathbb{F}_{q^2}-\mathbb{F}_q\hookrightarrow A/f^r$, $l=\deg f$ is even, and the two solutions are uniquely determined in $A/f^r$ by Hensel's lemma. Conversely, each $u\in\mathbb{F}_{q^2}-\mathbb{F}_q\hookrightarrow A/f^r$ satisfies some irreducible equation (*). The stabilizer of such an $x=(u:1)$ is as in b).

If $c=0$ then $u\in\mathbb{F}_q$, case a); if (*) is reducible with $c\ne 0$, there exist $e,e'$ in $\mathbb{F}_q$ s.t. $(u-e)(u-e')=0$. Thus, without restriction, $(u-e)$ is a zero-divisor, i.e., $(u-e)\in fA/f^r$, which implies either $u=e\in\mathbb{F}_q$, case a), or $u\notin\mathbb{F}_q$, $e=e'$, case c).

If $h(x)\ge 1$, $x=(1:v)$, then $bv^2+(a-d)v-c=0$, which implies $c=0$. Then either $v=0$, case a), or $v\ne 0$, $a-d=v^2=0$, i.e., case c).

The numbers and lengths of orbits of a given type come out by counting. □

Next, we study how $G$-orbits split into $B$-orbits.

**2.8. Lemma.** *The orbits in* (2.7) *split as follows into $B$-orbits:*

(a) *two $B$-orbits* a1) $\{(1:0)\}$, a2) $\{(u:1)|u \in \mathbb{F}_q\}$;
(b) *one $B$-orbit;*
(c) *two $B$-orbits* c1) *of length $q-1$, given by the elements of height $\geq 1$,* c2) *of length $q(q-1)$, given by the elements of height* 0, *for each $G$-orbit of type* c;
(d) *$q+1$ $B$-orbits of length $q(q-1)$ for each $G$-orbit of type* d.

*The stabilizers $B_x$ of elements $x \in \mathbb{P}$ are conjugate to $B$ in case* a1, *$T$ in case* a2, *$Z$ in cases* b, c2 *and* d, *and $Z \cdot U$ in case* c1.

**Proof.** The cases a, b, d are obvious. For c note that such a $G$-orbit is isomorphic with $G/Z \cdot U$. Now use the Bruhat decomposition $G = B \dot{\cup} BwU$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the fact that each $G$-orbit of type c contains elements $x$ with $h(x) = 0$ as well as $y$ with $h(y) \geq 1$. □

Still in the primary case (notations as in 2.7), we calculate the $G_{d-1}$-orbits on $\mathbb{P} = \bigcup_{0 \leq h \leq r} \mathbb{P}(h)$, $\mathbb{P}(h) = \{x \in \mathbb{P} | h(x) = h\}$. Clearly, $\mathbb{P}(h)$ is stable under $G_{d-1}$.

**2.9. Lemma.** *$\mathbb{P}(h)$ splits as follows under $G_{d-1}$:*

(a) *$h = 0$: One orbit of length $|n| = q_1^r$, with order of stabilizer of an element equal to $(q-1)^2$;*
(b) *$1 \leq h \leq [r/2]$ : $q_1^{h-1} \frac{q_1 - 1}{q-1}$ orbits of length $(q-1)q_1^{r-2h}$ with order of stabilizer $(q-1)q_1^{2h}$;*
(c) *$[r/2] < h < r$ : $q_1^{r-h-1} \frac{q_1-1}{q-1}$ orbits of length $q-1$, with order of stabilizer $(q-1)q_1^r$;*
(d) *$h = r$: One orbit of length* 1, *order of stabilizer $(q-1)^2 q_1^2$.*

**Proof.** Let $G_{d-1,h} \hookrightarrow G_{d-1}$ be the subgroup $\left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \middle| \begin{smallmatrix} a \in \mathbb{F}_q^* \\ b \equiv 0 (\bmod f^h) \end{smallmatrix} \right\}$. In view of (2.4), the $G_{d-1}$-orbits on $\mathbb{P}(h)$ correspond to the $G_{d-1,h}$-orbits on $A/f^r$ $(h = 0)$, $(A/f^{r-h})^*$ $(1 \leq h \leq r-1)$, $\{0\}$ $(h = r)$, respectively, which acts through $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) u = au + b$. The lengths and stabilizers come out by counting. □

Now we come back to the general case: $n = \pi f_i^{r_i}$ as in (1.16), $\mathbb{P} = \Pi \mathbb{P}^1(A/n_i) = \prod_{1 \leq i \leq s} \bigcup_{0 \leq h_i \leq r_i} \mathbb{P}^1(A/n_i)(h_i)$ as in (2.3), (2.4).

Let us determine the "exceptional" cases where the stabilizer $G_{\underline{x}}$ of $\underline{x} = (x_1, \ldots, x_s)$ in $G$ is strictly larger than $Z$. If this is the case, then $Z \subsetneq G_{\underline{x}} = \bigcap_{1 \leq i \leq s} G_{x_i}$, where all the $G_{x_i}$ are of type a, b or c in (2.7).



**2.10. Proposition.** *Let $\underline{x} \in \mathbb{P}$ be as above. The stabilizer $G_{\underline{x}}$ is conjugate to one of the following groups:* a) $B$; a*) $T$; b) $T^{ns}$; c) $Z \cdot U$; d) $Z$. *We have*

*case* a $\Leftrightarrow$ $\underline{x} \in \mathbb{P}^1(\mathbb{F}_q) \overset{\text{diag}}{\hookrightarrow} \mathbb{P}$.
*There is one orbit of this type, of length $q+1$;*

*case* a* $\Leftrightarrow$ *there exists a non-trivial partition $S \dot{\cup} S'$ of $\{1, 2, \ldots, s\}$ and $u \neq v \in \mathbb{P}^1(\mathbb{F}_q)$ s.t. $x_i = u$ if $i \in S$, $x_i = v$ if $i \in S'$. There are $2^{s-1} - 1$ orbits of this type, corresponding to the partitions, each of length $q(q+1)$;*

*case* b $\Leftrightarrow$ *for $i = 1, 2, \ldots, s$, $l_i$ is even, and there exists an irreducible quadratic equation $X^2 + aX + b$ over $\mathbb{F}_q$ s.t. each $x_i$ is a root. In this case, there are $2^{s-1}$ orbits of this type, each of length $q(q-1)$.*

*case* c $\Leftrightarrow$ $\underline{x} \notin \mathbb{P}^1(\mathbb{F}_q) \overset{\text{diag}}{\hookrightarrow} \mathbb{P}$, *but there exists $e \in \mathbb{P}^1(\mathbb{F}_q)$ such that for $i = 1, 2, \ldots, s$ : if $e = (c:1)$ then $x_i = (u_i : 1)$, $(u_i - e)^2 = 0$, if $e = (1:0)$ then $x_i = (1:v)$, $v^2 = 0$. There are $(q-1)^{-1} \prod_{1 \leq i \leq s} (q_i^{[r_i/2]} - 1)$ orbits of this type, each of length $q^2 - 1$;*

*case* d $\Leftrightarrow$ *none of a, a*, b, c. There are precisely*
$$\frac{\epsilon(n) - (q+1)\sharp(a) - q(q+1)\sharp(a^*) - q(q-1)\sharp(b) - (q^2-1)\sharp(c)}{q(q^2-1)}$$
*orbits of this type, each of length $q(q^2-1)$.*
($\sharp(x)$ = *number of orbits of type $x$*)

**Proof.** Our list of possible stabilizers is complete by (2.7) and (2.6). Cases a and a* are then immediate from (2.7) and (2.6) (ii) + (iii). The $\underline{x}$ with $G_{\underline{x}}$ conjugate to $T^{ns}$ are those whose components $x_i$ satisfy an irreducible quadratic equation (the same for all $i$, (2.6) (v)). Since, again by (2.7), there are precisely 2 solutions in each $A/n_i$, there are $\frac{q(q-1)}{2} \cdot 2^s$ such $\underline{x}$, which split in $2^{s-1}$ orbits of length $q(q-1)$. This gives b). Case c holds for $\underline{x}$ if and only if for all $i$, the components $x_i$ are of type a or c in (2.7), where the distinguished element $e \in \mathbb{P}^1(\mathbb{F}_q)$ is the same for all $i$, and c) appears for at least one $i$ (i.e., $x_i \neq e$). There are $q_i^{[r_i/2]}$ elements $x_i \in \mathbb{P}^1(A/n_i)$ "sufficiently close" to a given $e \in \mathbb{P}^1(\mathbb{F}_q)$, and therefore $(q+1)(\prod_{1 \leq i \leq s} q^{[r_i/2]} - 1)$ elements $\underline{x}$ giving rise to case c. □

**2.11. Proposition.** *The $G$-orbits in* (2.10) *split as follows into $B$-orbits:*

(a) *two $B$-orbits* a1) $\{(1:0)\}$, a2) $\{(u:1)|u \in \mathbb{F}_q\}$;
(a*) *the $G$-orbit associated with the partition $S \cup S' = \{1, \ldots, s\}$ splits into three $B$-orbits consisting of $\underline{x}$ with*

a1*) $x_i = \infty, i \in S, \; x_i = u, i \in S'$,
a2*) $x_i = \infty, i \in S', x_i = u, i \in S$,
a3*) $x_i = u, \; i \in S, \; x_i = v, i \in S'$,

($u \neq v \in \mathbb{F}_q$) *of lengths $q, q, q(q-1)$, respectively;*

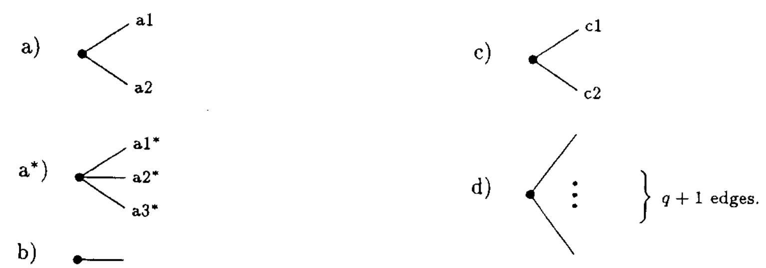
(b) *one $B$-orbit of length $q(q-1)$;*
(c) *two $B$-orbits* c1) *of length $q-1$, given by the elements $\underline{x}$ with $h_i(\underline{x}) \geq 1$, all $i$,* c2) *of length $q(q-1)$, given by the elements $\underline{x}$ with $h_i(\underline{x}) = 0$, all $i$, in the given $G$-orbit of type* c;
(d) *$q+1$ $B$-orbits of length $q(q-1)$ for each $G$-orbit of type* d.

*The stabilizers are conjugate to $B$ in case* a1, *to $T$ in cases* a2, a1*, a2*, *to $Z$ in cases* a3*, b, c2, d, *and to $Z \cdot U$ in case* c1.

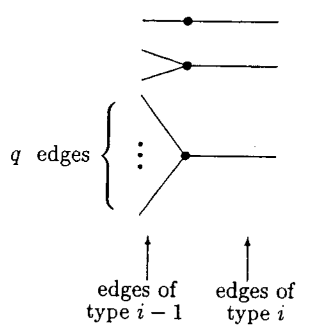
**Proof.** Clear from (2.10) and (2.8). □

Recalling that the orbits given in (2.10) and (2.11) describe the vertices $X_0(\Delta\backslash\mathcal{T})$ and edges $Y_0(\Delta\setminus\mathcal{T})$ of type zero of $\Delta\setminus\mathcal{T}$, we see:

**2.12.** The vertices of type 0 of $\Delta\setminus\mathcal{T}$ ($\Delta = \Gamma_0(n)$) together with their adjacent edges have the following shape:




**2.13.** By similar (but less complicated) considerations, one gets for the vertices of types $i \geq 1$ the following possible shapes, which all occur:




This will however not be used in what follows.

There rests the description of the cusps, i.e., of $G_{d-1}\setminus\mathbb{P}$. Let $\underline{h} = (h_1,\dots,h_s)$, $0 \leq h_i \leq r_i$ for $i = 1,2,\dots,s$, and

$$\mathbb{P}(\underline{h}) = \{\underline{x}\in\mathbb{P} \mid h_i(\underline{x}) = h_i \forall i\}.$$



**2.14. Proposition.** *$\mathbb{P}(\underline{h})$ splits as follows under $G_{d-1}$:*
(a) *Each $h_i = 0$ or $r_i$: one orbit;*
(b) *There exists $i$ such that $0 < h_i < r_i$: number of orbits is*

$$(q-1)\prod_{\substack{1\leq i\leq s\\ 0<h_i<r_i}} (q_i-1)q_i^{m_i-1} \qquad \text{with } m_i = \min\{h_i, r_i-h_i\}.$$

*In case* a, *the stabilizer of an element is $T\times p$-group, in case* b, *it is $Z\times p$-group (semi-direct products).*

**Proof.** Case a is clear. Thus assume $\underline{h}$ is as in b). $\mathbb{P}(\underline{h})$ has

$$\prod_{\substack{1\leq i\leq s\\ h_i=0 \text{ or } r_i}} q_i^{r_i-h_i} \prod_{\substack{1\leq i\leq s\\ 0<h_i<r_i}} (q_i-1)q_i^{r_i-h_i-1}$$

elements, and $G_{d-1}$ operates on it through a faithful action of a factor group of order

$$(q-1)\prod_{\substack{i\\ h_i=0 \text{ or } r_i}} q_i^{r_i-h_i} \prod_{\substack{i\\ 1\leq h_i\leq r_i/2}} q_i^{r_i-2h_i}$$

(see proof of (2.9)). The quotient gives the stated formula. □

**2.15.** Cusps corresponding to $G_{d-1}$-orbits of type a above are called *regular*; there are $2^s$ of them. Cusps of type b are called *irregular*. We let $G_{d-1}\backslash\mathbb{P} = (G_{d-1}\backslash\mathbb{P})_{\text{reg}} \cup (G_{d-1}\setminus\mathbb{P})_{\text{irr}}$ be the associated decomposition and select representatives as follows: For $\underline{h} = (h_1,\dots,h_s)$ consider the set of all $\underline{x} = (x_1,\dots,x_s)\in\mathbb{P}$ for which

$$x_i = \begin{cases} (0:1), & \text{if } h_i = 0,\\ (u_i : f_i^{h_i}),\ \begin{array}{l} u_i \bmod f_i^{m_i}, (u_i,f_i)=1,\\ m_i \text{ as in (2.14)}\end{array} & \text{if } 0<h_i<r_i,\\ (1:0), & \text{if } h_i = r_i.\end{cases}$$

The multiplicative group $\mathbb{F}_q^*$ acts diagonally on this set; if $\underline{h}$ is irregular (i.e., $\exists\, i$ s.t. $0<h_i<r_i$), we choose representatives mod $\mathbb{F}_q^*$. The resulting $\underline{x}$ form a system of representatives $\mathfrak{X}(\underline{h})$ for $G_{d-1}\setminus\mathbb{P}(\underline{h})$, and thus a system $\mathfrak{X} = \mathfrak{X}_{\text{reg}}\cup\mathfrak{X}_{\text{irr}}$ for $G_{d-1}\setminus\mathbb{P}$ if $\underline{h}$ varies. We further label the regular cusps represented by $((0:1),\dots,(0:1))$ and $((1:0)\dots,(1:0))$ by cusp (0) and cusp ($\infty$), respectively.

**2.16. Lemma.** *The number of irregular cusps of $\Delta\backslash\mathcal{T}$ is given by $(q-1)^{-1}(\kappa(n)-2^s)$. (See (1.17) for $\kappa(n)$.)*

**Proof.** Let

$$\sigma(\underline{h},i) = \begin{cases} 1, & \text{if } h_i = 0 \text{ or } r_i,\\ (q_i-1)q_i^{m_i-1}, & \text{if } 0<h_i<r_i.\end{cases}$$

A straightforward calculation yields

$$\sum_{\substack{h_i\\ 0\leq h_i\leq r_i}} \sigma(\underline{h},i) = \kappa(n_i).$$

Now the number in question is $(q-1)^{-1}$ times the following:

$$\sum_{\underline{h}\text{ irregular}} \prod_i \sigma(\underline{h}, i) = \sum_{\text{all } \underline{h}} \prod_i \sigma(\underline{h}, i) - 2^s$$

$$= \prod_i \sum_{0 \le h_i \le r_i} \sigma(\underline{h}, i) - 2^s = \prod_i \kappa(n_i) - 2^s = \kappa(n) - 2^s$$

since $\kappa$ is multiplicative. □

We have now accumulated all the ingredients to calculate the genus of $\Delta \setminus \mathcal{T}$.

**2.17. Theorem.** *Let $n = \prod_{1 \le i \le s} f_i^{r_i}$ be the prime decomposition of $n \in A$ and $\Delta = \Gamma_0(n)$. The genus of $\Delta \setminus \mathcal{T}$ is given by*

$$g(\Delta \setminus \mathcal{T}) = 1 + \frac{\epsilon(n) - (q+1)\kappa(n) - 2^{s-1}[r(n)q(q-1) + (q+1)(q-2)]}{q^2-1}.$$

*Here $r(n) = 1$ if the prime divisors $f_i$ all have even degree, and $r(n) = 0$ otherwise. The quantities $\epsilon(n)$ and $\kappa(n)$ are defined in (1.17).*

**Proof.** We apply (1.9), where the relevant cardinalities of $X_0(\Delta \setminus \mathcal{T})$, $Y_0(\Delta \setminus \mathcal{T})$ and cusp $(\Delta) = X_{d-1}(\Delta \setminus \mathcal{T})$ are given by Propositions 2.10, 2.11, 2.14 and Lemma 2.16, respectively. □

**2.18. Corollary.** $g(\Delta \setminus \mathcal{T}) = q^{d-2} + o(q^{d-3})$.

**2.19. Corollary.** *Let $n$ be square-free. Then*

$$g(\Delta \setminus \mathcal{T}) = 1 + \frac{\epsilon(n) - 2^{s-1}q[r(n)(q-1) + q + 1]}{q^2-1}.$$

*In particular, if $n$ is prime of degree $d$ then $g(\Delta \setminus \mathcal{T}) = (q^d - q^2)/(q^2-1)$ for even and $(q^d - q)/(q^2-1)$ for odd $d$.*

**2.20. Corollary.** *We have $g(\Delta \setminus \mathcal{T}) = 0 \Leftrightarrow d = \deg n \le 2$ and $g(\Delta \setminus \mathcal{T}) = 1 \Leftrightarrow q = 2$, $d = 3$ and $n$ divisible by $f^2$, where $\deg f = 1$ (i.e., $n$ one of $T^2(T+1)$, $(T+1)^2T$, $T^3$, $(T+1)^3$ over $\mathbb{F}_2$).*

**Proof.** The "if" direction is immediately verified. For the other direction, we apply $\epsilon(n) = q^d + o(q^{d-1})$, $\kappa(n) = o(q^{d/2})$ and perform some case considerations. □

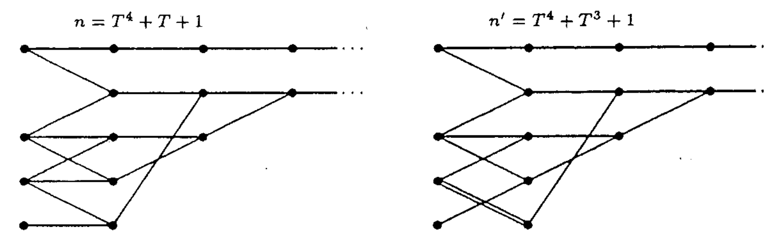
In the list to follow, we give the genera for all $n$ of degree $d \le 4$. The symbols $f$, $g$, $h$, $k$ denote different prime polynomials in $A$; their subscript indicates the degree.



**2.21. Table of genera.** $g(\Delta \setminus \mathcal{T})$, $\Delta = \Gamma_0(n)$, $\deg n \le 4$.

| Splitting type of $n$ | degree | $g(\Delta \setminus \mathcal{T})$ |
|---|---|---|
| $\deg n \le 2$ | $\le 2$ | $0$ |
| $f_1 \cdot g_1 \cdot h_1$, $f_2 \cdot g_1$, $f_3$ | 3 | $q$ |
| $f_1^2 \cdot g_1$, $f_1^3$ | 3 | $q-1$ |
| $f_1 \cdot g_1 \cdot h_1 \cdot k_1$ | 4 | $q^2+4q$ |
| $f_2 \cdot g_1 \cdot h_1$ | | $q^2+2q$ |
| $f_2 \cdot g_2$, $f_4$ | | $q^2$ |
| $f_3 \cdot g_1$ | | $q^2+q$ |
| $f_1^2 \cdot g_1 \cdot h_1$ | | $q^2+3q-3$ |
| $f_1^2 \cdot g_2$ | | $q^2+q-1$ |
| $f_1^2 \cdot g_1^2$ | | $q^2+q-2$ |
| $f_1^3 \cdot g_1$ | | $q^2+2q-3$ |
| $f_2^2$ | | $q^2-q$ |
| $f_1^4$ | | $q^2-1$ |

**2.22. Remark.** As follows from the theorem, $g(\Delta \setminus \mathcal{T})$ depends only on the splitting type of $n$, as do the numbers of vertices, edges and cusps given by (2.10), (2.11), (2.14). But even with the same splitting type of $n$ and $n'$, the graphs $\Gamma_0(n) \setminus \mathcal{T}$ and $\Gamma_0(n') \setminus \mathcal{T}$ are in general not isomorphic. As an example, we draw without further explanation the graphs for $n = T^4 + T + 1$ and $n' = T^4 + T^3 + 1 \in \mathbb{F}_2[T]$, both prime, with $g(\Delta \setminus \mathcal{T}) = 4$. Vertices are ordered with type $0, 1, 2, \ldots$ from the left to the right.




## 3. A Maximal Subtree Algorithm

We keep the setting of the last section and develop an algorithm for a maximal subtree of $\Gamma_0(n) \setminus \mathcal{T}$.

**3.1. Lemma.** *The edges of type* a1, a2, c1, c2 *in $Y_0(\Delta \setminus \mathcal{T})$ (see (2.12)) are situated on cusps of $\Delta \setminus \mathcal{T}$. More precisely, deleting an edge of type* a1 *or* c1 *disconnects the graph, where the component not containing the attached vertex $x \in X_0$ is a half-line.*

**Proof.** It suffices to show that the corresponding $B$-orbit on $\mathbb{P}$ is actually stable under $G_{d-1}$. This is clear from the description given in (2.10) and (2.11). □

As follows from the above, any $\varphi \in H_1(\Delta \setminus \mathcal{T}, \mathbb{Z})$ vanishes on edges $e \in Y_0(\Delta \setminus \mathcal{T})$ of type a1, a2, c1, c2, and it vanishes *a priori* on those of type b.

**3.2. Proposition.** *Let $\varphi \in H_1(\Delta \setminus \mathcal{T}, \mathbb{Z})$ be zero on all $e \in Y_0(\Delta \setminus \mathcal{T})$ of type* d. *Then $\varphi = 0$.*

**Proof.** As in (1.9), let $(\Delta \setminus \mathcal{T})^*$ be obtained from $\Delta \setminus \mathcal{T}$ by collapsing the edges of type $\geq 1$. We consider the canonical isomorphism of $H_1(\Delta \setminus \mathcal{T}, \mathbb{Z})$ with $H_1((\Delta \setminus \mathcal{T})^*, \mathbb{Z})$ as an identification. Then the support of $\varphi$ is contained in the subgraph $\mathcal{C}$ of $(\Delta \setminus \mathcal{T})^*$ generated by the edges of type a* (i.e., a1*, a2*, a3*) in $Y_0(\Delta \setminus \mathcal{T})$. Now $\mathcal{C}$ is connected since the $G_{d-1}$-orbits determined by $B$-orbits of type a3* are those of $((0:1), \ldots, (0:1)) \in \mathbb{P}$, i.e., all the vertices of $\mathcal{C}$ are connected to the regular cusp cusp(0) (see (2.15)).

By Euler's formula, we thus have $g(\mathcal{C}) = 1 + \sharp Y(\mathcal{C}) - \sharp X(\mathcal{C})$ with $\sharp Y(\mathcal{C}) = 3(2^{s-1} - 1)$. Further,

$$\begin{aligned}\sharp X(\mathcal{C}) &= \sharp\{v \in X_0(\Delta \setminus \mathcal{T}) | \text{type of } v = a^*\} + \sharp \left\{\begin{array}{l}\text{classes} \in G_{d-1} \setminus \mathbb{P} \text{ represented} \\ \text{by those } \underline{x} \in \mathbb{P} \text{ described in} \\ \text{Proposition 2.10, case a}^*\end{array}\right\} \\ &= 2^{s-1} - 1 + \sharp\{\text{regular cusps different from cusp}(\infty)\} \\ &= 2^{s-1} - 1 + 2^s - 1.\end{aligned}$$

Together, this yields $g(\mathcal{C}) = 0$. Hence $\varphi$ lives on the tree $\mathcal{C}$, and must therefore vanish identically. □

**3.3. Theorem.** *The canonical map $j_\Delta : H_1(\Delta \setminus \mathcal{T}, \mathbb{Z}) \longrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Delta$ of* (1.13) *is an isomorphism.*

**Proof.** By the proposition, a maximal subtree of $\Delta \setminus \mathcal{T}$ may be obtained by deleting certain edges $\in Y_0(\Delta \setminus \mathcal{T})$ of type d. By Proposition 2.11, the associated stabilizers are $Z = Z(\mathbb{F}_q) \hookrightarrow G_0$, and so the weight of these edges (1.11) is one. We conclude by Proposition 1.15. □

**3.4.** Now consider the system of representatives $\mathfrak{X}$ for $G_{d-1} \setminus \mathbb{P}$ chosen in (2.15). It defines maps, depending on $\mathfrak{X}$,

$$\begin{aligned}\Psi' : G_{d-1} \setminus \mathbb{P} &\longrightarrow Y_0(\Delta \setminus \mathcal{T}) \\ [\underline{x}]_{d-1} &\longmapsto [\underline{x}]_{01} = \text{class of } \underline{x} \bmod B = G_0 \cap G_1 \quad \text{and} \\ \Psi = o_0 \circ \Psi' : G_{d-1} \setminus \mathbb{P} &\longrightarrow X_0(\Delta \setminus \mathcal{T}).\end{aligned}$$

**3.5. Lemma.** *The map $\Psi$ restricted to $(G_{d-1} \setminus \mathbb{P})_{\text{irr}}$ is injective.*

**Proof.** (i) $\Psi'$ is injective since $G_0 \cap G_1 \hookrightarrow G_{d-1}$.

(ii) The edges $e \in Y_0(\Delta \setminus \mathcal{T})$ with $e \neq [\underline{x}]_{01}$, $o(e) = o([\underline{x}]_{01})$ are represented by $w\gamma(\underline{x})$ with $w = \binom{0\,1}{1\,0}$, $\gamma \in U$ (notations as in (2.5); consider the Bruhat decomposition $G = B \cup BwU$). But for $\underline{x} \in \mathfrak{X}$, each component $w\gamma(x_i)$ has height 0 or $r_i$, so $[w\gamma(\underline{x})]_{d-1} \in (G_{d-1} \setminus \mathbb{P})_{\text{reg}}$.



(iii) So let $\Psi([\underline{x}]_{d-1}) = \Psi([\underline{y}]_{d-1})$ with $\underline{x}, \underline{y} \in \mathfrak{X}_{\text{irr}}$. By (ii) we have $\Psi'([\underline{x}]_{d-1}) = \Psi'([\underline{y}])$, so by (i), $\underline{x} = \underline{y}$. □

**3.6.** We now define certain sets of edges and vertices that will be distinguished in our algorithm, see below. Let

$$\tilde{Y}(\Delta \setminus \mathcal{T}) := \{e \in Y_0(\Delta \setminus \mathcal{T}) | e \text{ a } B\text{-orbit of type d in (2.11)}\} \cap \Psi'((G_{d-1} \setminus \mathbb{P})_{\text{irr}})$$

and

$$\tilde{X}(\Delta \setminus \mathcal{T}) := o_0(\tilde{Y}(\Delta \setminus \mathcal{T})).$$

**3.7. Lemma.** *$\tilde{X}(\Delta \setminus \mathcal{T})$ and $\tilde{Y}(\Delta \setminus \mathcal{T})$ both have precisely*

$$\frac{\kappa(n) - 2^s - (\Pi q_i^{[r_i/2]} - 1)}{q - 1}$$

*elements.*

**Proof.** As directly follows from the description of $G$-orbits on $\mathbb{P}$ given in (2.10), elements of $\mathfrak{X}_{\text{irr}}$ can *never* represent orbits of type a, a*, or b. On the other hand, each $G$-orbit of type c is represented by some element of $\mathfrak{X}_{\text{irr}}$. The assertion now follows from (2.16) and (2.10), case c. □

Let now $\mathcal{S}$ be the subgraph of $\Delta \setminus \mathcal{T}$ obtained by the following

**3.8. Algorithm.** For each $v \in X_0(\Delta \setminus \mathcal{T})$ of type d, do:

- $v \notin \tilde{X}(\Delta \setminus \mathcal{T})$: Delete $q$ of the $q + 1$ edges adjacent to $v$.
- $v \in \tilde{X}(\Delta \setminus \mathcal{T})$: Delete $q - 1$ of the $q$ edges $e \notin \tilde{Y}(\Delta \setminus \mathcal{T})$ and adjacent to $v$.

**3.9. Theorem.** *$\mathcal{S}$ is a maximal subtree of $\Delta \setminus \mathcal{T}$.*

**Proof.** (i) Clearly, $\mathcal{S}$ contains all the vertices of $\Delta \setminus \mathcal{T}$.

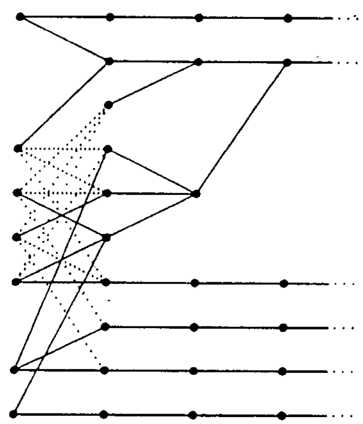
(ii) $\mathcal{S}$ is connected.

Since each vertex of $\mathcal{S}$ is connected to a cusp of $\Delta \setminus \mathcal{T}$, it suffices to show that all the cusps of $\Delta \setminus \mathcal{T}$ are connected in $\mathcal{S}$ to the cusp cusp(0). Now: cusp($\infty$), represented by a1), is connected to cusp(0) through the edge of type a2; regular cusps $\neq$ cusp(0), cusp($\infty$) are connected to cusp(0) through vertices $v \in X_0(\Delta \setminus \mathcal{T})$ of type a* (through the edge of type a3*); irregular cusps cusp(?) are connected to cusp(0) through $\Psi$(cusp(?)).

(iii) We can easily count the number of edges omitted in $\mathcal{S}$, using (2.10) case d and (3.7). The result is that precisely $g(\Delta \setminus \mathcal{T})$ edges have been deleted. Thus $\mathcal{S}$ is a tree, and therefore a maximal subtree. □

**3.10. Example.** The following example exhibits (except for the lack of "isolated" vertices of type b in $X_0(\Delta \setminus \mathcal{T})$) all the features of the general case. We let $q = 3$, $n = T^2(T^2 + T + 2) \in A = \mathbb{F}_3[T]$, $s = 2$, $f_1 = T$, $f_2 = T^2 + T + 2$. The graph $\Gamma_0(n) \setminus \mathcal{T}$ has genus 11, it is represented by

$n = T^2(T^2 + T + 2)$



where vertices are of type $0, 1, 2, \ldots$ from left to right. Representatives of $X_i(\Delta \setminus \mathcal{T})$ in $\mathbb{P}$ are given by

$$\begin{aligned}
X_0(\Delta \setminus \mathcal{T}) = \{&(1:0,1:0),(T:1,T:1),(1:0,T:1),(T:1,0:1),\\
&(T:1,T+2:1),(1:0,0:1),(1:T,1:0)\}\\
X_1(\Delta \setminus \mathcal{T}) = \{&(1:0,1:0),(0:1,0:1),(0:1,1:1),(0:1,T:1),\\
&(0:1,T+1:1),(0:1,T+2:1),(1:T,0:1),\\
&(0:1,1:0),(1:0,0:1),(1:T,1:0)\}\\
X_2(\Delta \setminus \mathcal{T}) = \{&(1:0,1:0),(0:1,0:1),(0:1,T:1),(1:T,0:1),\\
&(0:1,1:0),(1:0,0:1),(1:T,1:0)\}\\
X_3(\Delta \setminus \mathcal{T}) = \{&(1:0,1:0),(0:1,0:1),(1:T,0:1),(0:1,1:0),\\
&(1:0,0:1),(1:T,1:0)\}
\end{aligned}$$

ordered "from top to bottom" in the picture. The dotted lines are edges deleted by our algorithm.

## 4. An Observation on Multiplicities of Automorphic Representations

Again, $\Delta = \Gamma_0(n)$. As indicated in the introduction, the $\mathbb{C}$-vector space $H(n) := \underline{H}_!(\mathcal{T}, \mathbb{C})^{\Delta}$ is a space of automorphic cusp forms (of a rather special type) in the sense of Langlands. In particular, it is provided with Hecke operators $T_m$ ($m \in A$ monic) and a Petersson scalar product $(.,.)$ with the usual properties. All of this is described in detail in [7, Ch. IV]. For the moment, we only need the scalar product, defined by

$$(\varphi, \psi) := [\Gamma : \Delta]^{-1} \sum_{e \in Y(\Delta \setminus \mathcal{T})} w(e)^{-1} \varphi(e) \overline{\psi(e)}. \tag{4.1}$$



The normalization is such that $(\varphi, \psi)$ is compatible with the inclusions $H(n) \hookrightarrow H(n')$, $n'$ a multiple of $n$. Next, let $m$ be a divisor of $n$ and $a$ one of $n/m$ (always assumed to be monic), and

$$\begin{aligned} i_{a,m} : H(m) &\longrightarrow H(n). \\ i_{a,m}(\varphi)(e) &= \varphi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) e\right) \end{aligned}$$

We put

$$H^{\text{new}}(n) := \underline{H}_!^{\text{new}}(\mathcal{T}, \mathbb{C})^{\Gamma_0(n)} := [\sum_{\substack{m|n \\ m \neq n}} \sum_{a|n/m} \operatorname{im}(i_{a,m})]^{\perp} \tag{4.2}$$

(orthogonal complement in $H(n)$).

Then $H^{\text{new}}(n)$ is the space of newforms (*loc. cit.*) of the automorphic representations determined by $H(n)$.

**4.3. Proposition.**

$$H(n) = \bigoplus_{m|n} \bigoplus_{a|n/m} i_{a,m} H^{\text{new}}(m)$$

(*orthogonal direct sum*).

**Proof.** This is obvious from the automorphic description of $H(n)$: We can first orthogonally decompose $H(n)$ according to different automorphic representations $\pi$. Then we split $\pi$ into a tensor product of local representations, thus reducing the problem to a local one. The assertion then follows from [2, Cor. to Theorem 1], see also [3, Theorem 2.2.6]. □

Presumably, an elementary direct proof in the spirit of [1] can be given for the proposition, but most likely it will be unpleasant.

**4.4.** The dimensions

$$\begin{aligned} g(n) &:= \dim H(n) = g(\Gamma_0(n) \setminus \mathcal{T}) \text{ and} \\ g^{\text{new}}(n) &:= \dim H^{\text{new}}(n) \end{aligned}$$

are therefore related by

$$g(n) = \sum_{m|n} \sigma_0(n/m) g^{\text{new}}(m),$$

which allows to recursively calculate $g^{\text{new}}$ from $g$. We let

$$g^{\text{stab}}(n) := \sum_{m|n} g^{\text{new}}(m),$$

which counts the number of automorphic representations attached to $\underline{H}_!(\mathcal{T}, \mathbb{C})^{\Gamma_0(n)}$, i.e., those whose conductor is a divisor of $(n) \cdot \infty$, and whose $\infty$-component is the special representation of $\mathrm{PGL}(2, K_\infty)$ ([7], Ch. IV).

**4.5. Proposition.** *If $n$ is square-free of degree $d$, $g^{\mathrm{stab}}$ is given by*

$$g^{\mathrm{stab}}(n) = \frac{q^d - q^2}{q^2 - 1},\ d \text{ even}$$
$$= \frac{q^d - q}{q^2 - 1},\ d \text{ odd}.$$

*In particular, it does not depend on the splitting of $n$.*

**Proof.** This follows by an easy induction on the number $s$ of prime divisors of $n$, using (2.19) and Möbius inversion. □

**4.6. Remark.** Using the results of [6], we can get rid of the special role of the place $\infty$, i.e., consider divisors $\mathfrak{n}$ of $K = \mathbb{F}_q(T)$ not necessarily divisible by $\infty$, for which a generalization of (4.5) remains true. The assertion is then, vaguely, that the number of automorphic representations of conductor a divisor of $\mathfrak{n}$ does only depend on the degree of $\mathfrak{n}$, provided that $\mathfrak{n}$ is square-free. For this numerical fact we presently have no "structural" explanation.

## 5. Results on Other Groups

The congruence subgroups $\Delta = \Gamma(n), \Gamma_1(n), \Gamma_1^*(n)$ of $\Gamma$ may be treated following the same line of argument as in the case $\Delta = \Gamma_0(n)$. That is, we have to calculate $G_i$-orbits $(i = 0, 01, d-1)$ on $\Sigma := \Gamma/\Delta$. This will yield $g = g(\Delta \setminus \mathcal{T})$, and a closer look on the $G_{01}$-orbits on $\Sigma$ shows that the proof of Theorem 3.3 also applies to the other cases; i.e., a maximal subtree of $\Delta \setminus \mathcal{T}$ may be obtained by deleting edges $\in Y_0(\Delta \setminus \mathcal{T})$ of weight one. We restrict to write down the results, whose proofs are left to the reader. We use the notation $G, B \ldots$ introduced in (2.5).

**5.1.** Let first $\Delta = \Gamma_1^*(n)$. Then

$$\Sigma = \Gamma/\Gamma_1^*(n) \xrightarrow{\cong} \{(u,v) \in A/n \times A/n | (A/n)u + (A/n)v = A/n\},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a,c)$$

which we regard as an identification. Note that $\Sigma$ has $\varphi(n)\epsilon(n)$ elements. Since all the groups $G_i$ contain the center $Z \cong \mathbb{F}_q^*$, it suffices to determine the orbits on $\Sigma/Z = \{\text{classes } [u,v] \bmod \mathbb{F}_q^* \text{ of elements } (u,v) \in \Sigma\}$.

**5.2. Proposition.** (i) *The orbits of $G$ on $\Sigma/Z$ are as follows:*

(a) $\frac{\varphi(n)}{q-1}$ *orbits of length $(q+1)$, given by the elements $[u,v] \in \Sigma/Z$ such that $u$ and $v$ are $\mathbb{F}_q$-linearly dependent. The stabilizer of $[u,v]$ in $G$ is conjugate to $B$;*

(b) $\frac{\varphi(n)[\epsilon(n)-(q+1)]}{(q^2-1)(q-1)q}$ *orbits of length $(q^2-1)q$, with stabilizers $Z$.*

(ii) *Splitting into $B$-orbits:*

(a) *Each $G$-orbit of type* a *splits into two $B$-orbits* a1) *of length* 1*, represented by $[u,0]$, with stabilizer $B$, and* a2) *of length $q$, represented by $[0,v]$, with stabilizer $T$, respectivley;*



(b) *Each $G$-orbit of type* b *splits into $q+1$ $B$-orbits of length $(q-1)q$, with stabilizer $Z$.*

(iii) *The orbits of $G_{d-1}$ on $\Sigma/Z$ are:*

(a) $\frac{\varphi(n)}{q-1}$ *orbits, each consisting of one element $[u,0]$, $u \in (A/n)^*$;*

(b) *Let $\underline{h} = (h_1, \ldots, h_s)$ be an irregular height vector, i.e., $0 \le h_i \le r_i$ for $i = 1, 2, \ldots, s$, $\underline{h} \ne (0, \ldots, 0), (r_1, \ldots, r_s)$. The set $\{[u,v] | h(v_i) = h_i\}$ is $G_{d-1}$-stable and splits into $\frac{\varphi(n)}{(q-1)^2} \prod_{1 \le i \le s} \frac{q_i - 1}{q_i}$ orbits, each of length $(q-1)\prod q_i^{r_i - h_i}$ and with stabilizer of order $(q-1)\prod q_i^{h_i}$. ($h(v_i) \in \{0, 1, \ldots, r_i\}$ is the truncated valuation of $v_i$). Summing over the $\sigma_0(n) - 2$ irregular height vectors $\underline{h}$, this yields*

$$\frac{\varphi(n)}{(q-1)^2}\Big[\prod_{1 \le i \le s}\Big(r_i + 1 - \frac{r_i - 1}{q_i}\Big) - 2\Big]$$

*orbits of $G_{d-1}$ of type* b*;*

(c) $\frac{\varphi(n)}{q-1}$ *orbits on $\{[u,v] | v \text{ unit in} A/n\}$ each of length $q^d$, and with stabilizer conjugate to $T$.* □

**5.3. Corollary.** *The genus $g(\Delta \setminus \mathcal{T})$, where $\Delta = \Gamma_1^*(n)$, is given by*

$$g(\Delta \setminus \mathcal{T}) = 1 + \frac{\varphi(n)}{(q^2-1)(q-1)}\Big[\epsilon(n) - (q+1)\Big(q - 2 + \prod_{1 \le i \le s}\Big(r_i + 1 - \frac{r_i - 1}{q_i}\Big)\Big)\Big].$$

**Proof.** (1.9) + (5.2). □

We see that e.g. $g(\Delta \setminus \mathcal{T}) = q^{2d-3} + o(q^{2d-4})$ and $g(\Delta \setminus \mathcal{T}) = 0 \Leftrightarrow d \le 2$.

As in the case of $\Gamma_0(n)$, edges of type a1, a2 are situated on cusps of $\Delta \setminus \mathcal{T}$. Thus a maximal subtree is obtained by deleting edges of type b, which have weight $w(e) = 1$. Thus:

**5.4. Corollary.** *The map $j_\Delta : H_1(\Delta \setminus \mathcal{T}, \mathbb{Z}) \longrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Delta$ is bijective for $\Delta = \Gamma_1^*(n)$.*

**5.5.** We now treat the case of $\Gamma_1(n)$ and study orbits on $\Gamma/\Gamma_1(n) \cdot Z$ via its quotient mapping $p$ onto $\Gamma/\Gamma_1^*(n) \cdot Z$, which has degree $q - 1$.

**5.6. Proposition.** *The inverse image $p^{-1}(W)$ of one of the orbits $W$ of $G, B, G_{d-1}$ mentioned in (5.2) splits into $q-1$ orbits except for the following:*

i) $p^{-1}(W) =$ *one $G$-orbit if $W$ is a $G$-orbit of type* a*;*

ii) $p^{-1}(W) =$ *one $B$-orbit if $W$ is a $B$-orbit of type* a1, a2*;*

iii) $p^{-1}(W) =$ *one $G_{d-1}$-orbit if $W$ is a $G_{d-1}$-orbit of type* a *or* c. □

Inserting the cardinalities of $X_0, Y_0, X_{d-1}$ so obtained into (1.9) yields:

**5.7. Corollary.** *The genus $g(\Delta \setminus \mathcal{T})$, where $\Delta = \Gamma_1(n)$, is given by*

$$g(\Delta \setminus \mathcal{T}) = 1 + \frac{\varphi(n)}{q^2-1}\Big[\epsilon(n) - (q+1)\prod_{1 \le i \le s}\Big(r_i + 1 - \frac{r_i - 1}{q_i}\Big)\Big].$$

Let now finally $\Delta = \Gamma(n)$. Here, $G, B$ and $G_{d-1}$ are subgroups of $\Sigma = \Gamma/\Gamma(n) = \{\gamma \in \mathrm{GL}(2, A/n) | \det\gamma \in \mathbb{F}_q^*\}$. Hence the numbers of orbits are just the indices. We get

**5.8. Corollary.** *The genus $g(\Delta \setminus \mathcal{T})$, where $\Delta = \Gamma(n)$, is given by*

$$g(\Delta \setminus \mathcal{T}) = 1 + \frac{\varphi(n)\epsilon(n)}{q^2 - 1}(q^d - q - 1).$$

As in the preceding cases, we could use (1.15) to show the bijectivity of $j_\Delta$ for $\Delta = \Gamma(n)$ and $\Gamma_1(n)$. It is however easier to refer to (1.14), since these two groups have no non-$p$-torsion. In these two cases, the genus can also be calculated by the formula in Ex. 2, p. 136 in [10], provided the number of cusps is known.

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# EXAMPLES OF COMPACT HOLOMORPHIC SYMPLECTIC MANIFOLDS WHICH ARE NOT KÄHLERIAN III

Dedicated to Professor W. Browder on the occasion of his 60th birthday

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## 1. Introduction

In [14, 15] we have found irreducible compact holomorphic symplectic manifolds of dimension $2n$ for $n \geq 2$ which do not admit any Kähler structure. Those manifolds can be either simply connected or not simply connected. In this work we will consider deformations of some of them, thus producing more examples. We can see the effect of topological argument in the proof of the nonkählerness. We also give a poisson-symplectic proof of a generalization of the Bogomolov unobstructed theorem. By applying the same argument one can prove the unobstructed theorem for other holomorphic symplectic manifolds, e.g., see the Remark 3 in Sec. 4. Further results will be found in [16].

Let $M$ be a complex manifold of dimension $2n$. A holomorphic symplectic structure or form is a closed holomorphic 2-form $\omega$ on $M$ with maximal rank (see [18, p.47]).

One might ask that under what condition a compact holomorphic symplectic manifold is Kählerian, i.e., admits a positive closed (1,1) form. For example, by [25] and [22], we know that every K-3 surface is Kählerian. In [24], Todorov asked if every irreducible compact holomorphic symplectic manifold of dimension more than 4 is Kählerian. Some counter-examples have been found in [14, 15]. To get more examples, we first generalize the method in [15], then prove that many of their deformations are holomorphic symplectic manifolds which are not Kählerian.

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