PROBLEMS FOR THE SAGE DAYS

Ernst-Ulrich Gekeler
Universität des Saarlandes
gekeler@math.uni-sb.de

In what follows, I will use the standard notation:

$\mathbb{F}_q$ is a finite field of characteristic p, $q = p^f$
$A = \mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$ in an indeterminate T
$K = \mathbb{F}_q(T)$, with completion K_∞ at infinity
C_∞ the completed algebraic closure of K_∞
$\Omega = C_\infty \setminus K_\infty$ the Drinfeld upper half-plane, fibered over the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$
$\Gamma(1) = \mathrm{GL}(2, A)$ the modular group with its standard congruence subgroups $\Gamma(n)$, $\Gamma_1(n)$, $\Gamma_0(n)$
Usually $\mathfrak{p}$ denotes a prime polynomial in A.

PROBLEM 1: Bernoulli-Carlitz and Bernoulli-Goss "numbers".

These are elements $B(k)$ of k (resp. $\beta(k)$ of A) that contain the essential parts of the Carlitz-Goss zeta function

$$\zeta_{CG}(s) = \sum_{a \in A \text{ monic}} a^{-s}$$

at positive integers k (resp. negative integers $-k$). Both of these, a priori rather different and unrelated, are related to class number questions of the cyclotomic field extension $K(\mathfrak{p})$ of K generated by the $\mathfrak{p}$-division points of the Carlitz module, where $\mathfrak{p}$ is a prime of A. See [7] for a compact overview . (All of this is also in [11], but in much larger generality and technicality, and certainly more difficult to read.) While the $B(k)$ satisfy a Von-Staudt property, the $\beta(k)$ are subject to Kummer congruences. The precise form of the relation to class numbers ([7], Theorems 2.8 resp. 2.9, due to Goss resp. Okada) suggests both a Conjecture 1 (loc. cit. 6.9) about a possible reversal of Theorem 2.9 and divisibility implications between $B(k)$ and $\beta(k')$. More precisely, we ought to have:

Conjecture 2: For any prime $\mathfrak{p}$ of degree d and $0 < k < q^d - 1$: if $\mathfrak{p}|B(k')$ for each k' $\mathfrak{p}$-equivalent (loc. cit. 1.9) with k, then $\mathfrak{p}|\beta(q^d - 1 - k)$.

Thus, make a numerical study of the numbers $B(k)$ and $\beta(k)$ and test the two described conjectures! The second one is of course a consequence of the first one; it has the advantage that it can be checked in the framework of Bernoulli-* numbers only and doesn't require the (expensive) calculation of isotypical components of class groups.

PROBLEM 2: Goss polynomials for A.

Let $G_k(X) \in K[X]$ ($k = 0, 1, 2 ...$) be the series of Goss polynomials of the lattice $L = \overline{\pi}A$ in C_∞ as defined and studied in [6], sect. 3. Here $\overline{\pi}$ is the Carlitz period, i.e., L is the lattice that describes the Carlitz module. The $G_k(X)$ arise in many different contexts, for example in describing the t-expansions of Drinfeld modular forms (loc. cit., or [2]) or in the study of Goss-type L-series.

The most basic and important question is about the highest exponent $\gamma(k)$ such that $X^{\gamma(k)}$ divides $G_k(X)$. Bosser and Pellarin have proposed a complicated formula for $\gamma(k)$ in terms of the q-adic expansion of $k-1$, which holds for $q = p$ (and unfortunately fails for $q \neq p$). Determine $\gamma(k)$ for general q! More generally, determine the Newton polygon of G_k over the local field K_∞ (the case $q = p$ is settled, work in preparation) and, even more generally, study the arithmetic of $\mathrm{G}_k(X)$ (splitting into irreducible factors, ramification of the splitting field, etc.).

PROBLEM 3: Hecke operators on Drinfeld modular forms.

Let Γ be a congruence subgroup of $\Gamma(1)$, e.g., one of the groups $\Gamma_*(n)$ with $* = 0, 1$, or empty and a conductor $n \in A$. (Even the case $\Gamma = \Gamma(1)$ is interesting!). The Hecke operators $T_{\mathfrak{p}}$ on spaces of modular (or quasi-modular, see [2]) forms are described by means of Goss polynomials for finite $\mathbb{F}_q$-vector spaces, but are awfully complicated to evaluate ([6], sect. 7). Some results on that evaluation are given in C. Armana's thesis [1], sect. 4.3. Complete these results! Under which conditions on (Γ, k, m) is the action of the algebra of Hecke operators on the space of modular forms of weight k, type m for Γ semisimple, (absolutely) irreducible? What is the meaning of eigenforms, eigenspaces? Are there generic (independent of q) splitting patterns?

PROBLEM 4: t-expansions of specific modular forms.

Let t be the canonical uniformizer at ∞ for modular forms for $\Gamma(1)$, and let f be one of the (quasi-) modular forms g, h, Δ, E, g_k as defined in [6], or monomials in such, or derivatives (w.r.t. one of the operators Θ or ∂) of one of the preceding. (E = logarithmic derivative of Δ is only quasi-modular, as are the ordinary derivatives of modular forms.) What can be said about the t-expansions of such f? For which ones are there simple expansions with a comprehensive combinatorial meaning? In [6], multiplicative formulas for Δ and h are given, while B. Lopez in [12] found beautiful additive formulas for Δ and h which parallel the additive expansion of E.

PROBLEM 5: Splitting of Hecke algebras on Drinfeld automorphic forms.

In contrast with the modular forms in Problems 3 and 4, Drinfeld automorphic forms are characteristic-zero valued objects: elements of the finite-dimensional $\mathbb{C}$-vector space $\underline{H}(\mathcal{T},\mathbb{C})^{\Gamma}$ of Γ-invariant harmonic cochains ($\Gamma =$ any congruence subgroup of $\Gamma(1)$) on the oriented edges of $\mathcal{T}$ (see [10] for the general framework, [8] for a simplified description in the present case $A = \mathbb{F}_q[T]$, and [9] for a study of the graphs $\Gamma \setminus \mathcal{T}$ in question). There is a natural Hecke action on $\underline{H}(\mathcal{T},\mathbb{C})^{\Gamma}$, whose eigenspace decomposition (e.g. in the case where $\Gamma = \Gamma_0(n)$) corresponds to the splitting into irreducibles of the Jacobian $J_0(n)$ of the Drinfeld modular curve $X_0(n)$. Tables along with some theoretical considerations for the case of $\Gamma = \Gamma_0(n)$ with $n \in A$ of degree 3 (the case when $\deg(n)$ is less than 3 is trivial) and $q \leq 16$ are given in [5]. (There are some - almost self-correcting - misprints in the tables.)

Complete these tables! Are there generic (independent of q) splitting patterns (beyond the obvious ones, which come from symmetries of the conductor $n\cdot\infty$, and which are discussed in [5])? What is the "average" splitting behavior of $\underline{H}(\mathcal{T},\mathbb{C})^{\Gamma}$? Calculations of Hecke operators can be performed as in [5] or, more simply but at the cost of loosing some geometric information, via modular symbols [14]. Such an algorithm has been implemented in the Diploma Thesis [3] of R. Butenuth.

PROBLEM 6: "Cremona tables" over K.

Dimension one components in the splitting (of the new part) of $\underline{H}(\mathcal{T},\mathbb{C})^{\Gamma}$, where $\Gamma = \Gamma_0(n)$ (see Problem 5) correspond to isogeny classes of elliptic curves E over K with conductor $n\cdot\infty$ and split multiplicative reduction at the place at infinity ([5], [8]). Both for theoretical reasons and as examples on which one can check standard (e.g. Birch/Swinnerton-Dyer) and not so standard (e.g., statistical behavior of invariants) conjectures, it is desirable to dispose of an analogue of Cremona's tables [4] for elliptic curves E/K with the above properties.

These tables should, for a reasonable number of small q's and conductors n up to a certain degree depending on q, describe:

- all the isogeny classes of E/K with conductor $n\cdot\infty$, along with their structure (the classes are infinite due to the existence of the global Frobenius isogeny), the strong Weil curve inside the class and the degree of the strong Weil uniformization ([8], [13]);
- the torsion subgroup, rank, and a basis (mod torsion) of the Mordell-Weil group of each of the E/K that appears (in each isogeny class, only a finite number of E/K is needed);
- the height pairing matrix w.r.t. the given basis;

- the Tate-Shafarevich group;
- the zeta function of E/K (a polynomial in q^{-s} with integral coefficients), along with its critical leading value at the BSD-point $s = 1$;

...

References

(These are certainly incomplete; hundreds of papers could and should be cited. For the present purpose I have a natural bias towards those I know best, i.e., my own ones. I apologize.)

[1] C. Armana: Torsion rationnelle des modules de Drinfeld. Thesis Université Paris VII, 2008

[2] V. Bosser - F. Pellarin: Hyperdifferential properties of Drinfeld quasi-modular forms. Int. Math.Res. Not. IMRN 11 (2008)

[3] R. Butenuth: Ein Algorithmus zum Berechnen von Hecke-Operatoren auf Drinfeldschen Modulformen. Diploma thesis Universität Duisburg-Essen 2007

[4] J.E. Cremona: Algorithms for modular elliptic curves. Cambridge University Press 1992

[5] E.-U. Gekeler: Automorphe Formen über $\mathbb{F}_q[T]$ mit kleinem Führer. Abh. Math. Sem. Univ. Hamburg 55, 111-146 (1985)

[6] E.-U. Gekeler: On the coefficients of Drinfeld modular forms. Invent. Math. 93, 667-700 (1988)

[7] E.-U. Gekeler: On regularity of small primes in function fields. J. Number Theory 34, 114-127 (1990)

[8] E.-U. Gekeler: Analytical construction of Weil curves over function fields, J. Th. Nombres Bordeaux 7, 27-49 (1995)

[9] E.-U. Gekeler - U. Nonnengardt: Fundamental domains of some arithmetic groups over function fields. Int. J. Math. 5, 689-708 (1995)

[10] E.-U. Gekeler - M. Reversat: Jacobians of Drinfeld modular curves. J. reine angew. Math. 476, 27-93 (1996)

[11] D. Goss: Basic structures of function field arithmetic. Springer 1996

[12] B. Lopez: A non-standard Fourier expansion for the Drinfeld discriminant function. To appear in Arch. Math.

[13] M. Papikian: Analogue of the degree conjecture over function fields. Trans. AMS 359, 3483-3503 (2007)

[14] J. Teitelbaum: Modular symbols for $\mathbb{F}_q[T]$. Duke Math. J. 68, 271-295 (1992)