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Jacobians of Drinfeld modular curves

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In an attempt to prove the Langlands conjectures for GL(2) in characteristic $p > 0$, V. G. Drinfeld introduced in 1973 what now is called a Drinfeld module [6].

Drinfeld A-modules are group schemes locally isomorphic with the additive group scheme $\mathbb{G}_a$, and provided with a certain A-action, where A is the affine ring of a smooth projective algebraic curve $\mathscr{C}/\mathbb{F}_q$ minus one closed point ∞. They should be considered as the motivic K-counterparts ($K =$ quotient field of A) of the $\mathbb{Q}$-motives given by the multiplicative group, an elliptic curve, or more generally, a semi-abelian scheme.

Drinfeld was then able to prove K-analogues of classical theorems like the Kronecker-Weber theorem, the main theorem of complex multiplication, and the theorem of Eichler-Shimura.

The basic idea underlying Drinfeld's construction is best understood in considering the Weierstraß uniformization of an elliptic curve $E/\mathbb{C}$ by means of a lattice Λ in $\mathbb{C}$. Weierstraß' $\wp$-function and its functional equation yield a purely algebraic description of Λ; conversely, as each $E/\mathbb{C}$ comes from a lattice, the complex points of the moduli scheme for elliptic curves (+ perhaps some extra structure) are described as $\Gamma\backslash H$, where H is the complex upper half-plane and Γ some subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Now replace $\mathbb{C}$ by C, the completed algebraic closure of $K_\infty =$ completion of K at ∞. With each A-lattice Λ in C, Drinfeld associates some entire function $e_\Lambda : C \to C$ and a new A-action Φ^Λ on $C = \mathbb{G}_a(C)$ derived from e_Λ. If $r = \mathrm{rank}_A(\Lambda)$, Φ^Λ will be a Drinfeld A-module of rank r over C. Clearly, the case $r = 2$ (the only one to be treated in this article) corresponds most closely to the elliptic curve case mentioned above. In particular, the C-valued points $M^2(n)(C)$ of the moduli scheme for rank-two Drinfeld A-modules with a level-n structure (for definitions, see below) can be described as a finite union of quotients $\Gamma\backslash\Omega =: M_\Gamma(C)$, where

$$\Omega = C - K_\infty$$

is "Drinfeld's upper half-plane" and $\Gamma \subset \mathrm{GL}(2,K)$ is a subgroup commensurable with $\mathrm{GL}(2,A)$.

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Let $\bar{M}_\Gamma$ be the smooth projective model of the algebraic curve M_Γ/C, which is defined over a finite abelian extension of K. Drinfeld's theory involves the study of the l-adic cohomology $H^1(\bar{M}_\Gamma, \mathbb{Q}_l)$ both as a Galois module and as a module under the Hecke algebra. He arrives at a description of $H^1(\bar{M}_\Gamma, \mathbb{Q}_l)$ as a space of automorphic forms on $\mathrm{GL}(2)$ and at the same time establishes a one-to-one correspondence between certain l-adic Galois representations of K and the automorphic representations occurring in $H^1(\varprojlim \bar{M}^2(n), \mathbb{Q}_l)$ ([6], sect. 10, see also (4.13)).

From the above it is evident that many of the deeper results on arithmetic properties of K are encoded in the moduli schemes $M^r(n)/A$ for Drinfeld A-modules of rank r with a level-n structure. As a matter of fact, the case $r = 1$ essentially yields the abelian class field theory of K (see [6], Thm. 1 and [23], where the class fields totally split at ∞ are constructed). As in the classical case of elliptic moduli schemes, the main tool in their study is the theory of (Drinfeld) modular forms, which for $r = 2$ are rigid holomorphic functions $f: \Omega \to C$ with a prescribed transformation behavior under the natural action of $\Gamma \subset \mathrm{GL}(2, K)$ on Ω and certain regularity conditions "at infinity". The first attempts towards results about these modular forms were made in [19], [20] and later in [8]. These were completed in [10], where the study of the behavior of modular forms around cusps was continued and consequences about the geometry of the modular curves $\bar{M}_\Gamma$ were derived.

Now let for the moment $\Gamma \subset \mathrm{GL}(2, K_\infty)$ be a *Schottky subgroup*, i.e., a finitely generated subgroup consisting of hyperbolic elements, and $\Omega = \mathbb{P}^1(C)$ − closure of {limit points of Γ}. As follows from the work of Manin-Drinfeld, Mumford, and Gerritzen-van der Put [29], [32], [17], the C-analytic space $\Gamma\backslash\Omega$ is the "analytification" of a smooth projective curve M_Γ/C, a so-called *Mumford curve*. Moreover, the Jacobian J_Γ of M_Γ, the space of holomorphic differentials, the Néron model ... of J_Γ may be very explicitly described by means of theta functions for Γ (see [17]).

The aim of the present work is to give a similar description of these data in the case of an *arithmetic subgroup* Γ, i.e., a subgroup Γ of $\mathrm{GL}(2, K)$ commensurable with $\mathrm{GL}(2, A)$, and leading to a Drinfeld modular curve M_Γ. Apart from the superficial analogy between Mumford curves and Drinfeld modular curves (we used this analogy and especially the treatment in [17] as a heuristic guide-line), there are substantial differences between the two concepts. While Mumford curves are "compact" (projective) by construction, Drinfeld modular curves are affine and have to be compactified, which causes some technical but also principal difficulties. Secondly, while Schottky groups are always free (so in particular torsion free), arithmetic groups of the type considered may contain elements of finite order prime to p (which lead to "elliptic points" on the modular curve) and *always* contain unipotent elements of p-order ($p = \mathrm{char}(\mathbb{F}_q)$). Hence many arguments and constructions of e.g. [17] are not applicable in our case. Finally, and most importantly, Drinfeld modular curves are richer in that they carry an amount of supplementary arithmetic structure (rational and integral structure, action of Hecke operators) that should be reflected in the description of $J_\Gamma = \mathrm{Jac}(\bar{M}_\Gamma)$.

As follows from Drinfeld's work (although it is nowhere explicitly stated), the analogue of the Shimura-Taniyama-Weil conjecture on the uniformization of elliptic curves through modular curves holds in our case:

STW/K. Each elliptic curve E/K with split multiplicative reduction at ∞ is the quotient of a suitable Drinfeld modular curve $\bar{M}_\Gamma$, or equivalently, appears up to isogeny in the Jacobian J_Γ.

While this had been known to some experts since many years, no formal reference exists in the literature (with the exception of a few remarks in [9], sect. 9 and [10], VIII). In section 8 of the present paper, we try to fill this gap, i.e., we explain how STW/K may be derived from the results of [6] and prior work of Grothendieck and Jacquet-Langlands.

However, there are several important problems left open by Drinfeld's work.

(A) The above result STW/K is a sheer existence statement. Of course, one would like to dispose of a construction that, given E/K, produces a "Weil uniformization" $p_E : \bar{M}_\Gamma \to E$. Equivalently, one would like to construct E (or some curve isogeneous with E) out of the Hecke newform φ_E, and to understand how properties of E are reflected in φ_E and vice versa. (Precise definitions are given in sections 4 and 8; for the moment it suffices to regard φ_E as a substitute for the newform f_E of weight two for $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$ that hypothetically corresponds to an elliptic curve $E/\mathbb{Q}$ with conductor N.)

(B) In the Drinfeld modular curve context, there are two different concepts that generalize classical modular forms, viz, *automorphic forms*, which are $\mathbb{C}$- or $\mathbb{Q}_l$- or $\mathbb{Q}$-valued functions on some adele groups, and *Drinfeld modular forms*, which are C-valued holomorphic functions on Ω. Both of these are needed for a full understanding of the curves $\bar{M}_\Gamma$, so the question of their relationship arises.

It will turn out that these problems are closely related with our main result, the description (given in sect. 7) of J_Γ as a torus divided by some lattice. We will give satisfactory answers to both questions: See sect. 9, notably (9.6.1) for (A) (we construct an elliptic curve E from its newform φ_E by specifying the Tate period) and sect. 6, notably (6.5) for (B) (roughly speaking, Drinfeld modular forms of a certain type "are" the reductions (mod p) of $\mathbb{Z}$-valued automorphic forms).

Here a word of warning is in order. Due to the loss of information caused by reduction (mod p), Drinfeld modular forms (together with their Hecke operators) are less rigid objects than their number-theoretic counterparts. Many "non-classical" phenomena may occur:

- eigenforms of different weights may share the same set of eigenvalues;
- "multiplicity one" fails to hold even for eigenforms of weight two;
- the Hecke action on modular forms may fail to be semi-simple.

An example based on the numerical results of [9] is worked out in (9.7.4).

The contents of the paper are as follows.

In section one, we give a detailed discussion of the Drinfeld upper half-plane Ω, its standard covering, the associated reduction, and its relation with the Bruhat-Tits tree $\mathcal{T}$

of $PGL(2, K_\infty)$. We carefully describe the group actions on various associated structures, since this is crucial for our approach and there is some confusion in the literature about notation and signs.

In section 2, largely following [10], we browse through Drinfeld modules and modular curves. As far as is needed later on, and without proofs, we explain their description over the field C, their compactification, and their relationship to modular forms.

Section 3 deals with harmonic cochains for arithmetic groups Γ and their connection with $\bar{\Gamma} = \Gamma^{ab}/\mathrm{tor}(\Gamma^{ab})$ and with the modular curve $\bar{M}_\Gamma$.

In section 4 we introduce those concepts from the theory of automorphic forms that are needed to formulate Drinfeld's reciprocity law ([6], Thm. 2) and STW/K, and for our construction (9.5) of the strong Weil curve. Of course we cannot recall even the most important facts of the theory; so we assume the reader to be familiar with automorphic forms on GL(2) as presented e.g. in [14], and limit ourselves to indicate the special features that appear when considering automorphic forms $/K$ "of Drinfeld type". For a more detailed discussion of these questions (which unfortunately does not include Hecke operators and the Petersson product), we refer to the article of Deligne and Husemöller [5].

In section 5, theta functions u for arithmetic groups $\Gamma \subset GL(2, K)$ are studied. Their logarithmic derivatives u'/u are double cuspidal modular forms of weight two and type one, and thus give rise to holomorphic differential forms on M_Γ. The basic theta functions $\theta(\omega, \eta, z)$ and $u_\alpha(z)$ $(\alpha \in \Gamma)$ have first been introduced, in the case of Schottky groups Γ, by Manin and Drinfeld [29], and have subsequently been used by Gerritzen and van der Put in their uniformization theory of Mumford curves. By reasons explained above, the theory is different for our arithmetic groups. In the case of the special arithmetic groups $\Gamma = GL(2, A)$, these functions have first been studied by Radtke [38], to whom a part of Theorem 5.4.1 is due. The most important steps towards the construction of J_Γ are given by Theorems 5.6.1 and 5.7.1, which relate theta functions with automorphic data. There results in particular the positive definiteness of the theta pairing (α, β) on $\bar{\Gamma}$ (Cor. 5.7.5). Let us remark here our feeling that from a conceptual point of view, theta functions should be defined simultaneously on all the components of the moduli scheme, which would require an adelic treatment. We abstained from this since for our purposes, the additional technicalities would not be compensated by essential advantages. For other questions, however, it might very well be worthwhile to develop such an adelic theory of theta functions.

In section 6 we compare theta functions for different arithmetic groups $\Delta \subset \Gamma$. We obtain the principal result (6.5.1) and its consequences Theorems 6.5.3 and 6.5.4, which state that essentially all holomorphic differentials on $\bar{M}_\Gamma$ are obtained from the u_α by logarithmic derivation.

Having collected enough material about theta functions, it is quite straightforward to construct the Jacobian J_Γ as the torus C^{*g} divided by the period lattice of theta functions. This is done in section 7 (Thm. 7.4.1), where also some consequences about the groups of all holomorphic resp. meromorphic theta functions are derived. E.g., each holomorphic theta function u for Γ whose multiplier is trivial on tor (Γ^{ab}) equals const. $\times\, u_\alpha$ for some $\alpha \in \Gamma$.

Finally, in sections 8 and 9 we discuss how STW/K "abstractly" follows from Drinfeld's reciprocity law, and how a Weil uniformization actually is obtained from an automorphic eigenform. For this we have to define a Hecke action on $\bar{\Gamma}$ and to show that the basic diagrams (6.5.1) and (7.3.3) are compatible with Hecke operators. We conclude with some examples for strong Weil curves (where our results suffice to predict the Néron type) and with the discussion of open problems.

Some of the results of the present article have been announced in [13]. But note that we have made important changes in notation, signs (e.g. definition of theta functions) and some more conceptual matters, mainly the definition of $\underline{H}_{!!}(\mathcal{T}, B)^{\Gamma}$, and our present (6.4.4), which replaces (4.5) of *loc. cit.* In fact, we are not sure whether the latter holds in full generality: Our intended proof works (besides the case of p'-torsion free groups) only if the base ring A is a polynomial ring [34]. Anyway, (6.4.4) suffices for everything we have in mind.

Throughout the paper, we make free use of the well-known "GAGA" relationship between "algebraic" and "analytic" properties of varieties defined over complete valued fields. If misconceptions are unlikely, we do not distinguish between a variety, its associated analytic space, and its set of closed points over an algebraically closed field.

Also, in order to save notation, we use one single symbol "T_v" for Hecke operators acting on various objects (moduli schemes, modular forms, automorphic forms, harmonic cochains ...).

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0. Preliminaries

(0.1) Let $\mathscr{C}$ be a geometrically connected nonsingular projective curve over the finite field $\mathbb{F}_q$ of characteristic p. Once for all, we fix a point ∞ of $\mathscr{C}$ of degree δ, i.e., the residue class field $k(\infty)$ has degree δ over $\mathbb{F}_q$. We denote by A the ring of regular functions on $\mathscr{C} - \{\infty\}$, by K its quotient field.

(0.2) We will not distinguish between prime divisors of $\mathscr{C}$, places (= normalized absolute values) of K, and prime ideals of associated coordinate rings, and label them briefly as *places* or *primes*. Thus a prime v is either equal to ∞ or to a maximal ideal $\mathfrak{p}$ of A, in which case it is called *finite*. Similarly, each divisor of $\mathscr{C}$ (written multiplicatively) is the product of a finite divisor, i.e., of a fractional ideal $\mathfrak{n}$ of A, and a power of ∞.

(0.3) Given a place v of K, K_v, O_v, π_v, q_v denote the completion at v, its ring of integers, a local parameter, the size of the residue class field $k(v)$, respectively. The absolute value $|.| = |.|_\infty$ and the degree map $\deg : A \to \mathbb{N}_0 \cup \{-\infty\}$ at ∞ are normalized such that

$$|\pi_\infty| = q_\infty^{-1} = q^{-\delta} \quad \text{and} \quad q^{\deg a} = |a|. \tag{0.3.1}$$

Throughout, C will be the completion of an algebraic closure of K_∞, which again is algebraically closed. Note that the value groups of K_∞ and C are $q_\infty^{\mathbb{Z}}$ and $q_\infty^{\mathbb{Q}} \hookrightarrow \mathbb{R}^*$, respectively. If not stated otherwise, "log" will be the logarithm to base q_∞. Most of the features of the general set-up can be seen in the following basic example.

(0.4) Example. $\mathscr{C} = \mathbb{P}^1 =$ projective line over $\mathbb{F}_q$ and $\infty =$ usual place at infinity. Then A is the polynomial ring $\mathbb{F}_q[T]$ with quotient field $\mathbb{F}_q(T)$, deg is the usual degree, and $K_\infty = \mathbb{F}_q((\pi_\infty))$, the field of formal Laurent series in $\pi_\infty = T^{-1}$.

(0.5) For general facts in rigid geometry, we refer to [1], [7], and [17]. If X is a rigid analytic space over K_∞, $C, \ldots$ provided with a *pure covering* $\mathscr{U} = (U_i)$, $R_{\mathscr{U}} : X \to \tilde{X}_{\mathscr{U}}$ denotes the corresponding analytic reduction. We usually omit the subscript $\mathscr{U}$. Recall that $\tilde{X}$ is a scheme over the residue field of K_∞, C, ... with structure sheaf

(0.5.1) $$\mathcal{O}_{\tilde{X}} = R_* \tilde{\mathcal{O}}_X\,,$$

where

(0.5.2) $$\tilde{\mathcal{O}}_X = \mathcal{O}_X^0 / \mathcal{O}_X^{00}\,,$$

and the subsheaves $\mathcal{O}_X^{00} \subset \mathcal{O}_X^0 \subset \mathcal{O}_X$ are given on parts of $\mathscr{U}$ by

(0.5.3) $$\mathcal{O}_X^0(U) = \{f \in \mathcal{O}_X(U) \mid \|f\|_U^{\mathrm{sp}} \leqq 1\}\,,$$
$$\mathcal{O}_X^{00}(U) = \{f \in \mathcal{O}_X(U) \mid \|f\|_U^{\mathrm{sp}} < 1\}\,,$$

respectively. Here $\|\,.\,\|_U^{\mathrm{sp}}$ is the *spectral norm* on U. Our basic references for "GAGA"-type theorems are [25], [26], and [27].

(0.6) For all definitions and properties concerning graphs, we refer to the book [45] of J.-P. Serre. In particular, $X(T)$ and $Y(T)$ are the sets of vertices and of oriented edges, respectively, of the graph T, and $o(e)$, $t(e)$, $\bar{e}$ the origin, the terminus, the inverse edge of $e \in Y(T)$.

(0.7) For a field K, K^{sep} (resp. $\bar{K}$) denotes the separable (resp. algebraic) closure. R^* is the unit group of the ring R, $\#(S)$ the cardinality of the set S. If the group Γ acts on X, Γ_x, Γx, $\Gamma\backslash X$ are the stabilizer of $x \in X$, its orbit, the set of all orbits, respectively. Γ^{ab} is the group Γ made abelian.

Furthermore, in the largest part of the paper, G will denote the group scheme GL(2) with the center Z of scalar matrices.

1. The Drinfeld upper half-pane

(1.1) Set-theoretically, the Drinfeld upper half-plane is

(1.1.1) $$\Omega = \mathbb{P}^1(C) - \mathbb{P}^1(K_\infty) = C - K_\infty\,.$$

On Ω, we dispose of the two positive real functions $z \mapsto |z|$ and $z \mapsto |z|_i$, where

(1.1.2) $$|z|_i = \inf\{|z-x| \mid x \in K_\infty\}$$

is the *imaginary absolute value* of $z \in \Omega$. Among its properties, let us mention:

(1.1.3) if $|z| \notin |K_\infty^*| = q_\infty^{\mathbb{Z}}$ then $|z|_i = |z|$,

and:

(1.1.4) if $|z| = 1$ with class $\tilde{z}$ *in the residue field* $\overline{k(\infty)}$ of C then $|z|_i = 1$ if and only if $\tilde{z} \notin k(\infty)$.

Slightly less obvious is the following

(1.1.5) Lemma. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma$ *be an element of* $\mathrm{GL}(2, K_\infty)$ *and* $z \in \Omega$. *Then*

$$\left|\frac{az+b}{cz+d}\right|_i = |\det\gamma|\,|cz+d|^{-2}|z|_i\,.$$

Proof. As is easily seen, the assertion holds for $\gamma \cdot \gamma'$ provided it is true for γ and γ'. Since it is obvious for upper triangular matrices, we are reduced to showing it for $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e., $|z^{-1}|_i = |z|^{-2}|z|_i$.

If $|z|_i = |z|$ then $|z^{-1}|_i \leqq |z^{-1}| = |z|^{-2}|z|_i$. If $|z|_i = |z-x| < |z|$ then $|z| = |x|$ and $|z^{-1}|_i \leqq |z^{-1} - x^{-1}| = |z-x|/|zx| = |z|^{-2}|z|_i$. On the other hand,

$$|z|_i = |1/z^{-1}|_i \leqq |z|^2|z^{-1}|_i. \quad \square$$

(1.2) Analytic structure on Ω. As $\mathbb{P}^1(K_\infty)$ is compact, Ω has a natural structure of rigid analytic space ([7], I, 7). We now recall this structure and describe a pure covering of Ω (see also [7], V, [2], I, and [40]). Fix a uniformizer $\pi = \pi_\infty$.

(1.2.1) For $n \in \mathbb{Z}$, we set D_n for the subset of $z \in C$ that satisfy

(i) $$|\pi|^{n+1} \leqq |z| \leqq |\pi|^n$$

and

(ii) $$|z - c\pi^n| \geqq |\pi|^n,\ |z - c\pi^{n+1}| \geqq |\pi|^{n+1} \quad \text{for all } c \in k(\infty)^* \hookrightarrow K_\infty^*\,.$$

The last condition may be replaced by $|z|_i = |z|$, which shows that D_n is contained in Ω, and is independent of the choice of π. It is an affinoid space over K_∞ with ring of holomorphic functions

(1.2.2) $$A_n = K_\infty\langle \pi^{-n}z, \pi^{n+1}z^{-1}, (\pi^{-n}z - c)^{-1}, (\pi^{-n-1}z - c)^{-1} \mid c \in k(\infty)^*\rangle\,,$$

the algebra of "strictly convergent power series" in $\pi^{-n}z, \ldots$ (see [1], VI, 1 for the notation). It follows that the canonical reduction $\tilde{D}_n$ of D_n ([17], [7]) is isomorphic with a union of two projective lines M, M' over $k(\infty)$ meeting transversally in a $k(\infty)$-rational point, and the other rational points deleted:

$$(1.2.3) \qquad \tilde{D}_n = M \cup M' - \big(M(k(\infty)) \cup M'(k(\infty))\big) \cup (M \cap M').$$

We further put

$$(1.2.4) \qquad D_{(n,z)} = x + D_n \quad (x \in K_\infty).$$

Note that

$$(1.2.5) \qquad D_{(n,x)} = D_{(n',x')} \Leftrightarrow n = n' \quad \text{and} \quad |x - x'| \leqq |\pi|^{n+1}.$$

Let $I = \{(n, x) \mid n \in \mathbb{Z}, x \in K_\infty / \pi^{n+1} O_\infty\}$, where for each n, x runs through a set of representatives modulo $\pi^{n+1} O_\infty$. Then

$$(1.2.6) \qquad \Omega = \bigcup D_i \quad (i \in I).$$

Given i, there are only finitely many $i \neq i' \in I$ such that $D_i \cap D_{i'}$ is non-empty, in which case the intersection is isomorphic with a ball with q_∞ holes $= \mathbb{P}^1(C)$ with $q_\infty + 1$ holes. If e.g. $i = (n, x)$ and $i' = (n-1, x)$ then

$$(1.2.7) \qquad \begin{aligned} D_i \cap D_{i'} &= \{z \in C \mid |z - x - c\pi^n| = |\pi|^n \ \forall c \in k(\infty)\} \\ &= \{z \in C \mid |z - x|_i = |z - x| = |\pi|^n\}. \end{aligned}$$

Its reduction is

$$(1.2.8) \qquad (D_i \cap D_{i'})^\sim = M - M(k(\infty)),$$

where M is a projective line over $k(\infty)$. The analytic structure of Ω is obtained by glueing those of the D_i, $i \in I$. With (1.2.3) and (1.2.8) we see that $(D_i)_{i \in I}$ is a pure covering (*loc. cit.*) of Ω. Let

$$(1.2.9) \qquad R: \Omega \to \tilde{\Omega}$$

be the associated analytic reduction. Then $\tilde{\Omega}$ is a scheme over $k(\infty)$, locally of finite type. Each irreducible component M of $\tilde{\Omega}$ is isomorphic with $\mathbb{P}^1/k(\infty)$ and meets exactly $q_\infty + 1$ other components M'. The intersections are ordinary double points which are rational over $k(\infty)$. Conversely, each $k(\infty)$-rational point s of M determines a component M' such that $M \cap M' = \{s\}$. M and M' as above are called *adjacent* or *neighbors*. Putting

$$(1.2.10) \qquad M^* = M - M(k(\infty))$$

and for adjacent M, M'

$$(M \cup M')^* = M \cup M' - \big(M(k(\infty)) \cup M'(k(\infty))\big) \cup (M \cap M'),$$

there exist $i, i' \in I$ such that

$$(1.2.11) \qquad R^{-1}(M^*) = D_i \cap D_{i'} \quad \text{and} \quad R^{-1}((M \cup M')^*) = D_i\,.$$

The *intersection graph* of $\tilde{\Omega}$ is the combinatorial graph T with vertices

$$(1.2.12) \qquad X(T) = \{\text{components } M \text{ of } \tilde{\Omega}\}$$

and oriented edges

$$Y(T) = \{(M, M') \mid M, M' \text{ adjacent components of } \tilde{\Omega}\}\,.$$

It is a $(q_\infty + 1)$-regular tree. The reduction map R induces via (1.2.11) a bijection

$$(1.2.13) \qquad Y(T) \text{ (mod. orientation)} \xrightarrow{\;\simeq\;} I\,.$$

(1.2.14) The group $\mathrm{GL}(2, K_\infty)$ acts on Ω by fractional linear transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$. We easily see that it acts in fact on the covering $(D_i)_{i \in I}$, hence also on the tree T.

(1.3) The Bruhat-Tits tree of PGL(2, K_∞) ([45], [5]).

(1.3.1) An O_∞-*lattice* in K_∞^2 is a rank-two O_∞-submodule of K_∞^2. Two lattices L and L' are *equivalent* if there exists $x \in K_\infty^*$ such that $L' = xL$. We write $[L]$ for the equivalence class of L. Two classes $[L]$ and $[L']$ are *adjacent* or *neighbors* if there exists $L'' \in [L']$ such that $L'' \subset L$ and L/L'' has length one as an O_∞-module. Let $\mathcal{T}$ be the combinatorial graph whose vertices are the classes $[L]$, two vertices being connected by an edge if and only if the corresponding classes are neighbors. The graph $\mathcal{T}$ is a $(q_\infty + 1)$-regular tree, the *Bruhat-Tits tree* of $\mathrm{PGL}(2, K_\infty)$. In (1.5.4), we shall canonically identify $\mathcal{T}$ with T.

(1.3.2) Let now G be the group scheme $\mathrm{GL}(2)$ with center Z. $G(K_\infty)$ acts as a matrix group from the *right* on the set of lattices L in K_∞^2, thus also on $X(\mathcal{T}) = \{\text{lattice classes}\}$. We consider the associated *left action* $(\gamma, L) \mapsto \gamma_* L := L\gamma^{-1}$. Clearly, this action is compatible with the simplicial structure on $X(\mathcal{T})$, so there results a left action of $G(K_\infty)$ on $\mathcal{T}$. The stabilizer of the standard vertex $[L_0] = [O_\infty^2]$ is $\mathcal{K} \cdot Z(K_\infty)$, where $\mathcal{K} = G(O_\infty)$. This yields the description

$$(1.3.3) \qquad G(K_\infty)/\mathcal{K} \cdot Z(K_\infty) \xrightarrow{\;\simeq\;} X(\mathcal{T})\,, \qquad \gamma \longmapsto \gamma_*[L_0]\,.$$

Similarly,

$$(1.3.4) \qquad G(K_\infty)/\mathcal{I} \cdot Z(K_\infty) \xrightarrow{\;\simeq\;} Y(\mathcal{T})\,, \qquad \gamma \longmapsto \gamma_*([L_1], [L_0])\,,$$

where $\mathcal{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K} \mid c \equiv 0 \,(\mathrm{mod}\, \infty) \right\}$ is the *Iwahori group* and $([L_1], [L_0])$ the standard edge, $L_1 = \pi O_\infty \oplus O_\infty$. Having made these identifications, the obvious map

$Y(\mathscr{T}) \to X(\mathscr{T})$ induced by $\mathscr{I} \hookrightarrow \mathscr{K}$ associates with each oriented edge e its terminus $t(e) \in X(\mathscr{T})$.

(1.3.5) Let now L be a lattice in K_∞^2. The edges emanating from the vertex $[L]$ correspond to the points of $\mathbb{P}(L/\pi L) \cong \mathbb{P}^1(k(\infty))$, the paths (without backtracking) of length l to the points of the projective line $\mathbb{P}(L/\pi^l L) \cong \mathbb{P}^1(O_\infty/(\pi^l))$ over the finite ring $O_\infty/(\pi^l)$. A *half-line* in $\mathscr{T}$ is a subgraph isomorphic with •—•—•—•—•—..., an *end* of $\mathscr{T}$ an equivalence class of half-lines, where two half-lines are equivalent if they differ in a finite graph. Letting $l \to \infty$ yields

$$\left\{\begin{matrix}\text{half-lines emanating}\\ \text{from } [L]\end{matrix}\right\} \xrightarrow{\ \cong\ } \varprojlim \mathbb{P}(L/\pi^l L) = \mathbb{P}^1(K_\infty),$$

and in fact a bijection

$$\{\text{ends of } \mathscr{T}\} \xrightarrow{\ \cong\ } \mathbb{P}^1(K_\infty), \tag{1.3.6}$$

which is independent of the choice of L. To the point $(0:1) \in \mathbb{P}^1(K_\infty)$ there corresponds the end represented by $[L_l]_{l \geqq 0}$, $L_l = \pi^l O_\infty \oplus O_\infty$. In (1.6) we will see, however, that it is more natural to use a different labelling of the ends.

(1.4) Norms on K_∞^2. A non-archimedean norm on a K_∞-vector space V is a map $v: V \to \mathbb{R}$ that satisfies

$$\begin{aligned} v(v) &\geqq 0, \quad v(v) = 0 \Leftrightarrow v = 0,\\ v(xv) &= |x|\, v(x),\\ v(v+w) &\leqq \sup\{v(v), v(w)\} \end{aligned}$$

for $v, w \in V$, $x \in K_\infty$. Two norms are *similar* if they differ by a real constant. Let the linear group $\mathrm{GL}(V)$ of V act from the right on V. This yields a left action

$$(\gamma v)(v) = v(v\gamma) \tag{1.4.2}$$

on the set of norms v and also on the set of similarity classes $[v]$.

Recall that the *realization* $\mathscr{T}(\mathbb{R})$ of $\mathscr{T}$ is a topological space that consists of a real unit interval for every non-oriented edge of $\mathscr{T}$, glued together at their extremities according to the incidence relations of $\mathscr{T}$.

The next theorem is but a special case of a general result valid for all dimensions.

(1.4.3) Theorem (Goldman-Iwahori). *There is a canonical $G(K_\infty)$-equivariant bijection b between $\mathscr{T}(\mathbb{R})$ and the set of similarity classes of non-archimedean norms on $V = K_\infty^2$. (The $G(K_\infty)$-actions are given by* (1.3.2) *and* (1.4.2), *respectively.)*

Instead of the full proof ([18], [5]), we merely give the construction of b, and verify the compatibility with the group actions. To a vertex $[L] \in X(\mathscr{T}) = \mathscr{T}(\mathbb{Z})$, we associate $b([L]) = \mathrm{class}[v_L]$ of the norm v_L with unit ball L:

(1.4.4) $$\nu_L(v) := \inf\{|x| \mid x \in K_\infty, v \in xL\}.$$

If $P \in \mathcal{T}(\mathbb{R})$ belongs to the edge $([L], [L'])$ with $\pi L' \subset L \subset L'$, say

$$P = (1-t)[L] + t[L'] \quad (0 \leqq t \leqq 1),$$

then $b(P)$ is the class of the norm ν_P defined by

(1.4.5) $$\nu_P(v) = \sup\{\nu_L(v), q_\infty^t \nu_{L'}(v)\},$$

which degenerates to ν_L, $\nu_{L'}$ for $t = 0, 1$, respectively. It suffices to verify the $G(K_\infty)$-equivariance on vertices of $\mathcal{T}$. For this we have to show: If $\gamma \in G(K_\infty)$ and L is an O_∞-lattice in K_∞^2, then

(1.4.6) $$b(\gamma_*[L]) = [\nu_{\gamma_* L}] = [\nu_{L\gamma^{-1}}] \quad \text{equals} \quad \gamma(b([L])) = [\gamma \nu_L] = [\nu_L((.)\gamma)],$$

which is clear.

(1.5) The building map. Let $\lambda : \Omega \to \mathcal{T}(\mathbb{R})$ be the *building map*, which to $z \in \Omega$ associates the similarity class $[\nu_z]$ of the norm ν_z on K_∞^2:

(1.5.1) $$\nu_z((u, v)) := |uz + v|.$$

Since the value group of $|.|$ is $q_\infty^{\mathbb{Q}}$, the image of λ is contained in $\mathcal{T}(\mathbb{Q})$. In fact,

(1.5.2) $$\lambda(\Omega) = \mathcal{T}(\mathbb{Q}),$$

as immediately results from the construction (1.4.5).

(1.5.3) Proposition. *The building map is $G(K_\infty)$-equivariant.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(K_\infty)$, $z \in \Omega$, $(u, v) \in K_\infty^2$. Then $\gamma z = \dfrac{az+b}{cz+d}$, thus

$$\nu_{\gamma z}((u, v)) = \left| \frac{az+b}{cz+d} u + v \right| = |cz+d|^{-1} |(az+b)u + (cz+d)v| = |cz+d|^{-1} \nu_z((u, v)\gamma)$$

and $\lambda(\gamma z) = [\nu_{\gamma z}] = [\nu_z((.)\gamma)] = [\gamma \nu_z] = \gamma \lambda(z)$. □

(1.5.4) Using λ, we may now canonically identify the graphs T and $\mathcal{T}$, which in the sequel will not be distinguished. Let $M \in X(T)$ be an irreducible component of $\tilde{\Omega}$. Then there exists a uniquely determined $[L] \in X(\mathcal{T}) = \mathcal{T}(\mathbb{Z})$ such that $\lambda^{-1}([L]) = R^{-1}(M^*)$. The reader may easily (?) verify that $M \mapsto [L]$ is well-defined, bijective, and compatible with the actions of $G(K_\infty)$ defined in (1.2.14) and (1.3.2). It therefore identifies the combinatorial graphs T and $\mathcal{T}$ provided with their $G(K_\infty)$-actions.

Let now $i = (n, x)$ and D_i be as in (1.2),

$$\tilde{D}_i = (M \cup M')^*, \; e = (M', M) = ([L'], [L]) \in Y(\mathcal{T})$$

an associated oriented edge, $e(\mathbb{R})$ and $e^0(\mathbb{R}) = e(\mathbb{R}) - \{M, M'\}$ the corresponding closed and open edges, respectively, of $\mathcal{T}(\mathbb{R})$. Put D_i^0 for the subset of D_i defined by *strict* inequalities (see (1.2.1)):

$$(1.5.5) \qquad D_i^0 = \{z \in C \mid |\pi|^{n+1} < |z-x|_i = |z-x| < |\pi|^n\} = \{z \in C \mid |\pi|^{n+1} < |z-x| < |\pi|^n\}.$$

Let further

$$(1.5.6) \qquad C_i = \{z \in C \mid |z-x|_i = |z-x| = |\pi|^n\}.$$

As results from (1.2.11)–(1.2.13) and the above,

$$(1.5.7) \qquad \lambda^{-1}(e(\mathbb{R})) = D_i = R^{-1}((M \cup M')^*),$$

$$(1.5.8) \qquad \lambda^{-1}(e^0(\mathbb{R})) = D_i^0 = R^{-1}(M \cap M').$$

Furthermore, depending on the orientation of e, either one of the cases (α), (β) holds:

(1.5.9) (α) $\lambda^{-1}([L]) = C_{(n,x)} = R^{-1}(M^*)$, $\lambda^{-1}([L']) = C_{(n+1,x)} = R^{-1}(M'^*)$;

(β): as in (α), but roles of $([L], M)$ and of $([L'], M')$ reversed.

(1.5.10) Remark. The $\mathbb{R}$-valued functions $|.|$ and $|.|_i$ on Ω factor through λ, since $|z| = v_z(1,0)$, $|z|_i =$ distance of $(1,0)$ to $(0, K_\infty)$ w.r.t. v_z. Considered as functions on $\mathcal{T}(\mathbb{Q})$, $\log|.|$ and $\log|.|_i$ are linear on edges.

(1.6) Coordinates. We now provide the set of ends of $\mathcal{T}$ with coordinates compatible with those on Ω. Let s be the end represented by the sequence $[L_n]$ of vertices of $\mathcal{T}$, where $L_0 \supset L_1 \supset \cdots$ are O_∞-lattices with length $(L_n/L_{n+1}) = 1$. Let further $z_n \in \Omega$ be such that $\lambda(z_n) = [L_n]$. Suppose that s corresponds under (1.3.6) to $(u:v) \in \mathbb{P}^1(K_\infty)$, i.e.,

$$K_\infty(\bigcap L_n) = K_\infty(u, v).$$

Now

$$(1.6.1) \qquad |z_n|_i \to \infty \quad \text{if and only if} \quad (u:v) = (0:1),$$

as immediately results from (1.5.10). If $\gamma \in G(K_\infty)$ then we see with (1.4.6) and (1.5.3) that

(1.6.2) $(\lambda(\gamma z_n))_{n \geqq 0}$ determines the end corresponding to

$$\gamma_*(u:v) = (u:v)\gamma^{-1} = {}^t\gamma^{-1}(u:v).$$

In view of $w^{-1}\gamma w = {}^t\gamma^{-1} \bmod Z(K_\infty)$ (where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and ${}^t(.) =$ transpose), the involution $(u:v) \mapsto (-v:u)$ of $\mathbb{P}^1(K_\infty)$ interchanges the two actions

$$(\gamma, (u:v)) \mapsto \gamma(u:v), \ \gamma_*(u:v)$$

of $G(K_\infty)$. This is why from now on, we make the following

(1.6.3) Convention. Let $(u:v) \in \mathbb{P}^1(K_\infty)$. The end $s = s(u:v)$ labelled by $(u:v)$ (we briefly write $s = (u:v)$) is the equivalence class of half-lines that under (1.3.6) corresponds to $(-v:u) \in \mathbb{P}^1(K_\infty)$.

Letting now $s = (u:v)$, (1.6.1) and (1.6.2) translate to

(1.6.4) $$|z_n|_i \to \infty \Leftrightarrow s = \infty := (1:0) \in \mathbb{P}^1(K_\infty),$$

and

(1.6.5) $$(\lambda(\gamma z_n))_{n \geq 0} \text{ determines the end } \gamma s = \gamma(u:v).$$

As a consequence, we have

(1.6.6) Proposition. *Let $(z_n)_{n\geq 0}$ be as above. If $(\lambda(z_n))_{n\geq 0}$ determines the end $(u:v)$ then z_n converges to $(u:v)$ with respect to the strong topology on $\mathbb{P}^1(C)$.*

Proof. By (1.6.5) we may assume that $(u:v) = (1:0)$, in which case the assertion follows from (1.6.4). □

(1.6.7) Remark. The above may be used for constructions similar to the one of [47], 1.3, i.e., to construct a topological space $\Omega^* = \Omega \cup \{\text{cusps of } \Gamma\}$, which after dividing out the action of the arithmetic group Γ yields the "compactification" of the modular curve $\Gamma\backslash\Omega$ (see also (2.4.6)).

(1.6.8) Remark. Since $\mathscr{T}$ is a tree, the end $\infty = (1:0)$ of $\mathscr{T}$ defines an orientation on $\mathscr{T}$, in that precisely one edge of each pair $\{e, \bar{e}\}$ points to ∞. In particular, our domains D_i are canonically oriented. Also, $\log|z|_i$ (which only depends on $\lambda(z)$) increases by one for each step of length one towards ∞.

(1.7) Invertible functions and harmonic cochains.

(1.7.1) Definition. Let B be an abelian group. A harmonic cochain (= "currency" in [36], [37]) on $\mathscr{T}$ with values in B is a map $\varphi: Y(\mathscr{T}) \to B$ that satisfies

$$\varphi(e) + \varphi(\bar{e}) = 0 \quad (e \in Y(\mathscr{T}))$$

and

$$\sum_{\substack{e \in Y(\mathscr{T}) \\ t(e) = v}} \varphi(e) = 0 \qquad (v \in X(\mathscr{T})).$$

We denote by $\underline{H}(\mathscr{T}, B)$ the group of B-valued harmonic cochains.

(1.7.2) Theorem (van der Put). *There is a canonical exact sequence of $G(K_\infty)$-modules*

$$0 \to C^* \to \mathcal{O}_\Omega(\Omega)^* \xrightarrow{r} \underline{H}(\mathscr{T}, \mathbb{Z}) \to 0.$$

For a complete proof, see [36], Prop. 1.1 or [7], I, 8.9. We restrict ourselves to describing the map r, which by construction will be compatible with the $G(K_\infty)$-actions.

(1.7.3) The map *r*. Let $e \in Y(\mathscr{T})$ correspond to the double point $M \cap M'$ in $\tilde{\Omega}$, where M' and M are the irreducible components associated to $o(e)$ and $t(e)$, respectively. There exists $i = (n, x) \in I$ such that $R^{-1}((M \cup M')^*) = D_i$. Let $f \in \mathcal{O}_\Omega(\Omega)^*$. Since f has no zeroes on Ω, there exist $c \in C^*$, $k \in \mathbb{Z}$, and $g \in \mathcal{O}_\Omega(\Omega)^*$ such that

$$f(z) = c(z-x)^k g(z) \tag{1.7.4}$$

and $|g(z)|$ is constant equal to 1 on D_i ([7], I, 8.5). Now $r(f)(e)$ is defined by

$$\begin{aligned} r(f)(e) &= \log(\|f\|^{\mathrm{sp}}_{R^{-1}(M^*)}/\|f\|^{\mathrm{sp}}_{R^{-1}(M'^*)}) \\ &= k \qquad \text{in case } (\alpha) \\ &= -k \qquad \text{in case } (\beta) \text{ of (1.5.9)}. \end{aligned} \tag{1.7.5}$$

(Recall that "log" $= \log_{q_\infty}$.) Suppose for instance that we are in case (α). The reduction

$$\mathcal{O}^0_\Omega(C_{(n,x)}) \to \mathcal{O}_\Omega(C_{(n,x)})^\sim = \mathcal{O}_{\tilde{\Omega}}(M^*)$$

maps $f/(c\pi^{kn})$ to a rational function h on M, which has a zero of order k at the point $M \cap M'$. Since the divisor of h on M has degree zero, $r(f)$ satisfies the sum condition at the vertex $v = M$ of $\mathscr{T}$.

(1.8) Residues. Next we define the residue map attached to a global holomorphic differential form ω on Ω ([17], III, 1, [7], I, 3). Let $e \in Y(\mathscr{T})$, M, M', $D_i^0 \subset D_i$ as in (1.5.5). With $s := \pi^{-n}(z-x)$, $t := \pi^{n+1}(z-x)^{-1}$, ω restricted to D_i^0 may be written as

$$\omega = \sum_{k\in\mathbb{Z}} a_k s^k ds = \sum_{k\in\mathbb{Z}} b_k t^k dt, \quad a_k, b_k \in C. \tag{1.8.1}$$

We set

$$\begin{aligned} \operatorname{res}(\omega, e) &= a_{-1} \quad \text{in case } (\alpha) \\ &= b_{-1} \quad \text{in case } (\beta) \text{ of (1.5.9)}. \end{aligned} \tag{1.8.2}$$

Note that we have $a_{-1} = -b_{-1}$, which means that

$$\operatorname{res}(\omega, e) = -\operatorname{res}(\omega, \bar{e}). \tag{1.8.3}$$

As follows from the residue theorem in rigid geometry ([7], I, 3), $\operatorname{res}(\omega, e)$ is independent of the choice of coordinates s, t on D_i and satisfies

$$\sum_{\substack{e\in Y(\mathscr{T}) \\ t(e)=v}} \operatorname{res}(\omega, e) = 0 \quad (v \in X(\mathscr{T})). \tag{1.8.4}$$

Therefore:

(1.8.5) Proposition. *Let ω be a holomorphic differential form on Ω and*

$$\mathrm{res}(\omega) : e \mapsto \mathrm{res}(\omega, e)$$

its residue map. Then $\mathrm{res}(\omega) \in \underline{H}(\mathcal{T}, C)$.

Finally, we compare the two maps r and $\mathrm{res} : \omega \mapsto \mathrm{res}(\omega)$. Let f be a nowhere vanishing holomorphic function on Ω. Then $\omega := df/f$ is a holomorphic differential form, and one easily verifies

$$r(f)(e) \equiv \mathrm{res}(\omega, e) \,(\mathrm{mod}\, p). \tag{1.8.6}$$

In other words: The diagram

$$\begin{array}{ccc} \mathcal{O}_\Omega(\Omega)^* & \xrightarrow{r} & \underline{H}(\mathcal{T}, \mathbb{Z}) \\ {\scriptstyle \mathrm{dlog}}\downarrow & & {\scriptstyle \mathrm{red}}\downarrow \\ \left\{ \begin{array}{c} \text{holomorphic} \\ \text{differentials on } \Omega \end{array} \right\} & \xrightarrow{\mathrm{res}} & \underline{H}(\mathcal{T}, C) \end{array} \tag{1.8.7}$$

is commutative, where the vertical maps are $\mathrm{dlog} : f \mapsto df/f$ and reduction $\mathrm{mod}\, p$, respectively.

2. Modular curves and forms

Here and in the sequel, G is the group scheme $\mathrm{GL}(2)$ with center Z.

(2.1) Arithmetic subgroups of $G(K)$. Let Y be an A-lattice (projective A-submodule of rank two) in K^2 and

$$\Gamma(Y) := \mathrm{GL}(Y) = \{\gamma \in G(K) \mid Y\gamma = Y\}. \tag{2.1.1}$$

For an ideal $\mathfrak{a}$ of A, $\Gamma(Y, \mathfrak{a}) = \ker(\mathrm{GL}(Y) \to \mathrm{GL}(Y/\mathfrak{a}Y))$ is the $\mathfrak{a}$-th congruence subgroup of $\Gamma(Y)$. An *arithmetic subgroup* Γ of $G(K)$ is an intermediate group $\Gamma(Y, \mathfrak{a}) \subset \Gamma \subset \Gamma(Y)$ for suitable Y and $\mathfrak{a}$. Note that all arithmetic subgroups of $G(K)$ are commensurable.

(2.2) Fix Γ as above. As usual, $G(K_\infty)$ acts on Ω through $z \mapsto (az+b)/(cz+d)$. It is easy to see that the stabilizer in Γ is finite for every $z \in \Omega$, which implies that the quotient $\Gamma \backslash \Omega$ exists as an analytic space over C and even over K_∞. In fact, $\Gamma \backslash \Omega$ is smooth of dimension one, i.e., an "open Riemann surface".

Moreover, it is algebraic:

(2.2.1) Theorem (Drinfeld). *There exists a smooth irreducible affine algebraic curve* M_Γ / C *such that* $\Gamma \backslash \Omega$ *and the underlying analytic space* M_Γ^{an} *of* M_Γ *are canonically isomorphic as analytic spaces over* C.

The curves M_Γ will be referred to as *Drinfeld modular curves.* Since M_Γ is unique up to isomorphism, we often abuse notation in not distinguishing between M_Γ and $\Gamma \backslash \Omega = M_\Gamma(C)$.

(2.2.2) Remark. M_Γ is actually defined over a finite abelian extension $K' \hookrightarrow K_\infty$ of K that depends on Γ. The proof of (2.2.1) ([6], sect. 6) yields naturally that the isomorphism of $\Gamma\backslash\Omega$ with M_Γ^{an} holds over the common field of definition K_∞.

(2.3) Drinfeld modules. In order to describe the algebraic structure on $\Gamma\backslash\Omega$, we have to briefly review the notion of Drinfeld A-module. Let Λ be a discrete projective A-submodule of C (we call this an *A-lattice in C*) of rank r, and put

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right). \tag{2.3.1}$$

Since Λ intersects with each ball in a finite number of points, the product converges and defines an entire, $\mathbb{F}_q$-linear, surjective, and Λ-periodic function $e_\Lambda : C \to C$. It provides the additive group scheme

$$\mathbb{G}_a / C = C \xleftarrow{\cong} C/\ker(e_\Lambda) = C/\Lambda \tag{2.3.2}$$

with a new structure Φ^Λ of A-module. Recall that the endomorphisms $f \in \mathrm{End}_L(\mathbb{G}_a)$ of $\mathbb{G}_a/L$ (L any field of characteristic p) are the "polynomials" f in the Frobenius endomorphism $\tau_p = (x \mapsto x^p)$ with coefficients in L. Thus

$$\mathrm{End}_L(\mathbb{G}_a) = L\{\tau_p\} = \{\textstyle\sum a_i \tau_p^i \mid a_i \in L\} \tag{2.3.3}$$

with the commutator rule $\tau_p a = a^p \tau_p$ for $a \in L$. Now $\Phi^\Lambda : A \to \mathrm{End}_C(\mathbb{G}_a)$ turns out to satisfy:

$$\begin{aligned} \Phi^\Lambda : A &\to \mathrm{End}_C(\mathbb{G}_a) = C\{\tau_p\}, \\ a &\mapsto \Phi_a^\Lambda \end{aligned} \tag{2.3.4}$$

is a ring homomorphism with values in $C\{\tau\}$, where $\tau = \tau_q = \tau_p^e$ ($q = p^e$) and

(i) $$\mathrm{Lie}(\Phi_a^\Lambda) = a,$$

(ii) $$\deg_\tau(\Phi_a^\Lambda) = r \cdot \deg a.$$

Here (i) states that $\Phi_a^\Lambda = a +$ higher terms in τ, and $\deg_\tau$ is the well-defined "degree" in τ. Note that (2.3.4) does not involve the analytic structure on C. It therefore makes sense over arbitrary field extensions L of $K = \mathrm{Quot}(A)$. Thus a *Drinfeld module* $(\mathbb{G}_a, \Phi)$ of rank r over $\mathbb{G}_a/L$ will be given by a ring homomorphism $\Phi : A \to L\{\tau\}$ that satisfies (i) and (ii) above. There are natural notions of morphism and isomorphism of Drinfeld modules, for which one proves ([6], [5], [10]):

(2.3.5) Theorem (Drinfeld). *The association $\Lambda \mapsto \Phi^\Lambda$ defines an equivalence of categories between the category of A-lattices in C of rank r (morphisms are multiplications with elements of C) and the category of Drinfeld A-modules of rank r over C.*

(2.4) Moduli schemes. As follows from (2.3.4), the A-submodule

(2.4.1) $\quad ({}_a\Phi)(L) = \{x \in L \mid \Phi_a(x) = 0\} \quad (a \in A \text{ non-constant})$

of *a-torsion points* of $(\mathbb{G}_a/L, \Phi)$ is free of rank r over the finite ring $A/(a)$, provided that L is algebraically closed. This enables to define level-a structures (and even level-$\mathfrak{a}$ structures, $\mathfrak{a}$ a not necessarily principal ideal) on the Drinfeld module $\Phi = (\mathbb{G}_a/L, \Phi)$: A level-$\mathfrak{a}$ structure is an isomorphism of the A-module $(\mathfrak{a}^{-1}/A)^r$ with $({}_\mathfrak{a}\Phi)(L)$. Moreover, all these notions may be extended to work over arbitrary A-schemes S (instead of $S = \operatorname{Spec} L$, L/K field extension). There results a moduli functor

(2.4.2)
$$\mathfrak{M}^r(\mathfrak{a}) : S \mapsto \left\{ \begin{array}{l} \text{set of isomorphism classes of} \\ \text{rank-}r \text{ Drinfeld } A\text{-modules over} \\ S \text{ with level-}\mathfrak{a} \text{ structure} \end{array} \right\}$$

on the category Sch_A of A-schemes. Under the "admissibility" condition that $\mathfrak{a}$ is divisible by at least two different primes, $\mathfrak{M}^r(\mathfrak{a})$ is representable by an affine flat A-scheme $M^r(\mathfrak{a})$, which is smooth of finite type over $\mathbb{F}_q$ and has dimension $r-1$ over $\operatorname{Spec} A$. Further, the fibers of $M^r(\mathfrak{a})$ over $\operatorname{Spec} A$ are smooth away from $\mathfrak{a}$ ([6], sect. 5). Taking quotients by finite groups if necessary, one constructs for arbitrary $\mathfrak{a}$ (including the trivial ideal $(1) = A$) a scheme $M^r(\mathfrak{a})$ which in any case is a coarse moduli scheme for $\mathfrak{M}^r(\mathfrak{a})$. In particular, we will have

(2.4.3) $$\mathfrak{M}^r(\mathfrak{a})(L) = M^r(\mathfrak{a})(L),$$

if L is an algebraically closed field, e.g. $L = C$. (For a more complete discussion of this, see [6] and [5].)

(2.5) Analytic description of moduli schemes. Let now $\mathscr{P}_A^2$ be the set of isomorphism classes of projective A-modules of rank 2. By elementary lattice theory, $\mathscr{P}_A^2$ is finite and may be represented by A-lattices Y in K^2. We let $i_z : K^2 \to C$ be the K-linear map given by $(1,0) \mapsto z$ and $(0,1) \mapsto 1$. Up to multiplication by elements of C^*, each A-lattice Λ in C of rank two has the form $i_z(Y)$, where Y is as above and $z \in \Omega = C - K_\infty$ is uniquely determined up to the action of $\Gamma(Y)$. Therefore, (2.3.5) yields

$$\coprod_{Y \in \mathscr{P}_A^2} \Gamma(Y)\backslash\Omega \xrightarrow{\cong} \mathfrak{M}^2(1)(C),$$

and combined with (2.4.3),

(2.5.1) $$\coprod_{Y \in \mathscr{P}_A^2} \Gamma(Y)\backslash\Omega \xrightarrow{\cong} M^2(1)(C),$$

both set-theoretically and analytically. Note that

(2.5.2) $$\det : \mathscr{P}_A^2 \to \mathscr{P}_A^1 = \operatorname{Pic}(A) \quad \text{is bijective}.$$

Hence $M^2(1) \times_A C$ has $h := \#(\operatorname{Pic}(A))$ irreducible components. This corresponds to the fact ([10], VII, 1.9) that $M^2(1) \times_A K$ as an algebraic curve is defined over the Hilbert class field H of (K, A). (By definition, H is the maximal unramified abelian extension of K contained in K_∞. Its Galois group over K is canonically identified with $\operatorname{Pic}(A)$.)

There are similar isomorphisms for moduli schemes "with level". This explains the statement of (2.2.1), since for every arithmetic group Γ, $\Gamma\backslash\Omega$ appears as a component of a convenient moduli scheme (see also (4.11)).

(2.6) Compactification. For the moment, we discard the modular interpretation of $M^2(\mathfrak{a})$ and restrict to studying the components $M_\Gamma = \Gamma\backslash\Omega$. The only fact used is that M_Γ is a smooth affine curve over C, which implies the existence of a smooth projective model $\bar{M}_\Gamma$. So our next task is to describe $\bar{M}_\Gamma$. This being rather technical, we merely outline the main ideas and refer to [10] for details. Since the building map $\lambda: \Omega \to \mathcal{T}$ is $G(K_\infty)$-equivariant, there results for each arithmetic group $\Gamma \subset G(K)$ a map

$$\lambda_\Gamma : \Gamma\backslash\Omega \to \Gamma\backslash\mathcal{T}, \tag{2.6.1}$$

which may be used to describe an admissible covering of $\Gamma\backslash\Omega$. Now $\Gamma\backslash\mathcal{T}$ is the edge-disjoint union

$$\Gamma\backslash\mathcal{T} = (\Gamma\backslash\mathcal{T})^0 \cup \bigcup h_i \tag{2.6.2}$$

of a finite graph $(\Gamma\backslash\mathcal{T})^0$ with a finite number of half-lines h_i ([45], II, Thm. 9). An end s of $\mathcal{T}$ gives rise to an end of $\Gamma\backslash\mathcal{T}$ if and only if s is K-rational. Thus the ends (= equivalence classes of half-lines) appearing in $\Gamma\backslash\mathcal{T}$ correspond bijectively to the finite set

$$\operatorname{cusp}(\Gamma) := \Gamma\backslash\mathbb{P}^1(K) \tag{2.6.3}$$

of *cusps* of Γ. As explained below, the inverse image $\lambda_\Gamma^{-1}(s)$ of an end s of $\Gamma\backslash\mathcal{T}$ consists of a nested sequence of annuli with decreasing radii, whose union is a pointed disk. "Compactifying" $\Gamma\backslash\Omega$ therefore just means filling in the missing points

$$\bar{M}_\Gamma - M_\Gamma \xrightarrow{\cong} \{\text{ends of } \Gamma\backslash\mathcal{T}\} \xrightarrow{\cong} \operatorname{cusp}(\Gamma). \tag{2.6.4}$$

This will be achieved by specifying a uniformizer for $\bar{M}_\Gamma$ at every cusp s of Γ. Set-theoretically, $\bar{M}_\Gamma$ will then be the quotient of $\Omega \cup \mathbb{P}^1(K) \hookrightarrow \mathbb{P}^1(C)$ by the arithmetic group Γ.

(2.7) Uniformizers. Let first s be the cusp represented by $\infty \in \mathbb{P}^1(K)$ and Γ_∞ the stabilizer of ∞ in Γ. It contains a maximal subgroup Γ_∞^u of matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, where b runs through a subgroup $\mathfrak{b}$ of K commensurable with a fractional A-ideal. Γ_∞^u acts by shifts $z \mapsto z + b$ on Ω and stabilizes

$$\Omega_c := \{z \in \Omega \mid |z|_i \geqq c\}. \tag{2.7.1}$$

We form

$$e_{\mathfrak{b}}(z) = z \prod_{0 \neq b \in \mathfrak{b}} \left(1 - \frac{z}{b}\right), \tag{2.7.2}$$

which has properties similar to those stated in (2.3.1). In particular, it is $\mathfrak{b}$-invariant and therefore a (non-vanishing) function on $\mathfrak{b}\backslash\Omega = \Gamma_\infty^u\backslash\Omega$. Easy estimates show that for c sufficiently large, (2.7.3) $$t = t(\Gamma, \infty) := e_{\mathfrak{b}}^{-1}$$

identifies $\mathfrak{b}\backslash\Omega_c$ with a small pointed ball $B_r(0) - \{0\} = \{z \in C \mid 0 < |z| \leqq r\}$ of radius $r = r(c)$ in C. (We have to suppose that c belongs to the value group $q_\infty^{\mathbb{Q}}$ of "$|.|$", which suffices.) If $\Gamma_\infty = \Gamma_\infty^{\mathrm{u}}$, t is a uniformizer at $s = \infty$. In general, $\Gamma_\infty^{\mathrm{u}}$ is strictly contained in Γ_∞, which may also contain transformations of type $z \mapsto az$ $(a \in \mathbb{F}_q^*)$. Let $w = w(\Gamma, \infty)$ be the order of the cyclic group of these transformations. Then we use t^w as uniformizer. We need the following result, which is an easy consequence of (1.1.5).

(2.7.4) Proposition. *There exists a constant* $c_0 = c_0(\Gamma, \infty) > 0$ *such that for* $c \geqq c_0$ *and* $\gamma \in \Gamma$, $\Omega_c \cap \gamma(\Omega_c) \neq \emptyset$ *implies* $\gamma \in \Gamma_\infty$.

Now for c as above,

(2.7.5) $$\begin{array}{ccc} \Gamma_\infty\backslash\Omega_c & \hookrightarrow & \Gamma\backslash\Omega \\ \cong \downarrow & z \mapsto t^w(z) & \\ B_{r'}(0) - \{0\} & & (r' = r^w) \end{array}$$

is an open immersion of analytic spaces. Hence a function f on $\Gamma\backslash\Omega$ is holomorphic at the cusp ∞ if and only if it has a power series expansion

(2.7.6) $$f(z) = \sum_{i \geqq 0} a_i t^{wi}(z)$$

with a strictly positive radius of convergence. Similarly, we express meromorphy and the order of vanishing of f at ∞ through t^w.

(2.7.7) Finally, let $s \in \mathbb{P}^1(K)$ be arbitrary and $v \in G(K)$ with $v(\infty) = s$. If f is meromorphic and Γ-invariant on Ω, $f_v := f \circ v$ will be invariant under the arithmetic group $\Gamma^v = v^{-1}\Gamma v$, and holomorphy properties of f at s will be the corresponding properties of f_v at ∞. Here we have to be aware of some subtleties. In order to describe the behavior of f at a cusp in $\Gamma\backslash\mathbb{P}^1(K)$, we have to choose (a) $s \in \mathbb{P}^1(K)$ in its class mod Γ, (b) $v \in G(K)$ with $v(\infty) = s$. It is easy to see that holomorphy and the order of vanishing of f_v at ∞ do not depend on these choices, but the coefficients of the expansion of f_v w.r.t. $t(\Gamma^v, \infty)$ do.

(2.7.8) Remark. As results from the preceding discussion, the curve $\bar{M}_\Gamma$ (which in fact is defined over K_∞) is a totally split curve over K_∞ in the sense of [17], IV, 3. The required pure covering of $\bar{M}_\Gamma^{\mathrm{an}}$ is essentially obtained from the covering $(D_i)_{i \in I}$ of (1.2.6) by dividing out the action of Γ. The associated reduction R yields a scheme $(\bar{M}_\Gamma^{\mathrm{an}})^\sim$ over $k(\infty)$, which is a finite union of projective lines intersecting with normal crossings in $k(\infty)$-rational points. Its intersection graph is the "finite part" $(\Gamma\backslash\mathcal{T})^0$ of $\Gamma\backslash\mathcal{T}$, i.e., the graph obtained by collapsing each end of $\Gamma\backslash\mathcal{T}$ to a point.

(2.8) Modular forms. As for classical elliptic modular curves, the geometry of $\bar{M}_\Gamma$ will be studied through (Drinfeld) modular forms. Let

(2.8.1) $$d(\Gamma) = \#(\det(\Gamma))$$

be the order of $\det(\Gamma) \hookrightarrow \mathbb{F}_q^*$. For the most important groups Γ considered later, $d(\Gamma)$ will equal $q-1$.

A *Drinfeld modular form* for Γ of weight k and type m (where k is a non-negative integer and m a class modulo $d(\Gamma)$) is a function $f: \Omega \to C$ that satisfies

(2.8.2) (i) $f(\gamma z) = (\det\gamma)^{-m}(cz+d)^k f(z) \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right)$;

(ii) f is rigid holomorphic;

(iii) f is holomorphic at the cusps.

We briefly explain the last condition. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(K)$, put

(2.8.3) $$f_{[\gamma]_{k,m}}(z) := (\det\gamma)^m (cz+d)^{-k} f(\gamma z),$$

which defines a right action of $G(K)$. If f satisfies (i), i.e., $f_{[\gamma]_{k,m}} = f$ for $\gamma \in \Gamma$, then (notation as in (2.7.7)) $f_{[v]_{k,m}}$ will be fixed by Γ^v. Now condition (iii) above means that for every $v \in G(K)$, $f_{[v]_{k,m}}$ has a power series expansion with respect to $t(\Gamma^v, \infty)$.

(2.8.4) Due to the presence of non-trivial determinants in Γ, condition (i) of (2.8.2) implies that a modular form f is usually not invariant under Γ_∞ but only under Γ_∞^u. Hence there is no t^w-expansion but only a t-expansion for f. We therefore define the *order of f at the cusp* ∞ (and similarly at the other cusps) by means of its t-expansion. A form f will be called *cuspidal* (*i times cuspidal*) if it vanishes (vanishes i times) at all the cusps of Γ.

(2.8.5) We set $M_{k,m}(\Gamma)$ for the C-vector space of modular forms of weight k and type m for Γ and $M_{k,m}^i(\Gamma)$ for the subspace of i-cuspidal forms. As we will see below (2.10.2), $M_{2,1}^2(\Gamma)$ may be identified with the space of holomorphic differentials on $\bar{M}_\Gamma$, which by a standard argument implies that all the $M_{k,m}(\Gamma)$ are finite-dimensional. Their dimensions, i.e., the Hilbert function of the graded ring

(2.8.6) $$M(\Gamma) = \bigoplus_{k,m} M_{k,m}(\Gamma)$$

of modular forms may be calculated explicitly [10].

Note also that

(2.8.7) $$M_{k,m}(\Gamma) = 0,$$

whenever k is incongruent to $2m$ modulo $z(\Gamma) := \#(\Gamma \cap Z(K))$.

(2.9) Examples of modular forms ([19], [10]). For simplicity, let $\Gamma = \Gamma(Y)$ be a maximal arithmetic subgroup, where $Y \hookrightarrow K^2$ is a rank-two A-lattice. The most typical example of modular form for Γ is

(2.9.1) $$l_i(a): z \mapsto l_i(a, z),$$

where $a \in A$ and $\sum_i l_i(a, z)\tau^i$ is the operator Φ_a^Λ for the Drinfeld module Φ^Λ associated with the lattice $\Lambda = i_z(Y)$ in C. Then $l_i(a)$ is modular of weight $q^i - 1$ and type 0. It is analogous with the elliptic modular forms g_2, g_3, and is called a *coefficient form*. The sum

(2.9.2) $$E^{(k)}(z) = \sum_{0 \neq y \in Y} i_z(y)^{-k}$$

converges for $z \in \Omega$, $k > 0$ and defines for $k \equiv 0 \pmod{q-1}$ a non-zero element $E^{(k)}$ of $M_{k,0}(\Gamma)$. It is called an *Eisenstein series* [19]. Modular forms for non-zero types may be constructed as *Poincaré series* [11]. However, most important for our purposes are the logarithmic derivatives of the theta functions u_α defined in section 5. They are elements of $M_{2,1}^2(\Gamma)$.

(2.10) Modular forms and differentials. Let $f \in M_{2,1}(\Gamma)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Since $d(\gamma z)/dz = \det\gamma\,(cz+d)^{-2}$, the differential form $f(z)dz$ on Ω is Γ-invariant. Let us investigate when it comes from a holomorphic differential on $\bar{M}_\Gamma$. First, if $z \in \Omega$ is unramified in $p_\Gamma: \Omega \to \Gamma\backslash\Omega$, i.e., if z is fixed by no non-trivial element of $\tilde{\Gamma} = \Gamma/\Gamma \cap Z(K)$, $f(z)dz$ descends in a neighborhood of $p_\Gamma(z)$ to $\Gamma\backslash\Omega = M_\Gamma$. Thus assume that z is a fixed point of $\tilde{\Gamma}$ (such points are called *elliptic*), and let $\tilde{\Gamma}_z$ be its stabilizer. It is known ([10], p. 50) that $\tilde{\Gamma}_z$ is a cyclic group of order e a divisor of $q+1$. In particular, e is prime to $p = \mathrm{char}(C)$. Choose local uniformizers x and y of Ω and $\Gamma\backslash\Omega$, respectively, with $x^e = y$, and write

$$f(z)dz = \sum_{i \geq 0} a_i x^i dx = e^{-1} \sum a_i x^{i+1-e} dy .$$

Since $f(z)dz$ is $\tilde{\Gamma}_z$-invariant, $a_i \neq 0$ implies $i \equiv -1 \pmod e$. Hence

$$f(z)dz = e^{-1}(a_{e-1} + a_{2e-1}y + \cdots)dy$$

is holomorphic on $\Gamma\backslash\Omega$, too.

Finally we discuss what happens at the cusps. The function $e_b(z)$ of (2.7.2) has constant derivative 1, so

(2.10.1) $$dz = -t^{-2}dt$$

for $t = e_b^{-1}$. Hence $f(z)dz$ is holomorphic at ∞ (and at the other cusps) if and only if $f(z)$ is double cuspidal, i.e., belongs to $M_{2,1}^2(\Gamma)$. Similarly, as is easily verified by a small computation, $f(z)dz$ has at most simple poles at the cusps of $\bar{M}_\Gamma$ if and only if $f \in M_{2,1}^1(\Gamma)$. Since by GAGA, we may identify "algebraic" and "analytic" differential forms on the projective curve $\bar{M}_\Gamma$, we have shown:

(2.10.2) Proposition. *The map $f \mapsto f(z)dz$ identifies $M_{2,1}^2(\Gamma)$ with $H^0(\bar{M}_\Gamma, \Omega^1)$ and $M_{2,1}^1(\Gamma)$ with $H^0(\bar{M}_\Gamma, \Omega^1(\text{cusps}))$.*

Here $H^0(\bar{M}_\Gamma, \Omega^1)$ and $H^0(\bar{M}_\Gamma, \Omega^1(\text{cusps}))$ are the spaces of holomorphic differential forms on M_Γ, of forms with at most simple poles at the cusps, respectively. Note that Riemann-Roch yields g and $g+c-1$ for the respective dimensions of these spaces, where

(2.10.3) $$g = g(\Gamma) = \text{genus of } \bar{M}_\Gamma, \quad c = c(\Gamma) = \#(\text{cusp}(\Gamma)).$$

Similar isomorphisms exist between spaces of higher weight modular forms and spaces of multi-differentials. Special consideration has to be given to the behavior of modular forms at elliptic points and at cusps.

3. Harmonic cochains for Γ

In the whole section, Γ is a fixed arithmetic subgroup of $G(K)$ and $\tilde{\Gamma}$ its quotient by the finite group $\Gamma \cap Z(K)$ of order $z(\Gamma)$.

(3.1) Recall that $\underline{H}(\mathcal{T}, B)$ is the group of functions $\varphi: Y(\mathcal{T}) \to B =$ any abelian group that satisfy (1.7.1). Let $\underline{H}(\mathcal{T}, B)^\Gamma$ be the subgroup of invariants, i.e., of φ with

(3.1.1) $$\varphi(\gamma e) = \varphi(e) \quad (\gamma \in \Gamma, e \in Y(\mathcal{T})).$$

We shall consider $\underline{H}(\mathcal{T}, B)^\Gamma$ as a space of functions on the quotient graph $\Gamma \backslash \mathcal{T}$ as follows: Let $v \in X(\mathcal{T})$ with image $\tilde{v} \in X(\Gamma \backslash \mathcal{T})$. The stabilizer group Γ_v acts on the set

$$\{e \in Y(\mathcal{T}) \mid t(e) = v\},$$

and the orbits correspond to $\{\tilde{e} \in Y(\Gamma \backslash \mathcal{T}) \mid t(\tilde{e}) = \tilde{v}\}$. Let $m(e) := [\Gamma_v : \Gamma_e]$ be the length of the orbit corresponding to e, which only depends on its image $\tilde{e}$ in $\Gamma \backslash \mathcal{T}$. Considering $\varphi \in \underline{H}(\mathcal{T}, B)^\Gamma$ as a function on $Y(\Gamma \backslash \mathcal{T})$, the sum condition of (1.7.1) translates to

(3.1.2) $$\sum_{\substack{\tilde{e} \in Y(\Gamma \backslash \mathcal{T}) \\ t(\tilde{e}) = \tilde{v}}} m(\tilde{e}) \varphi(\tilde{e}) = 0 \quad (\tilde{v} \in X(\Gamma \backslash \mathcal{T})).$$

Note that $\sum_{t(\tilde{e}) = \tilde{v}} m(\tilde{e})$ equals $q_\infty + 1$. We finally put

(3.1.3) $$\underline{H}_!(\mathcal{T}, B)^\Gamma = \{\varphi \in \underline{H}(\mathcal{T}, B)^\Gamma \mid \varphi \text{ has compact support modulo } \Gamma\}$$
$=:$ group of B-valued *cuspidal harmonic cochains* for Γ.

Let now $\Gamma \backslash \mathcal{T} = (\Gamma \backslash \mathcal{T})^0 \cup \bigcup h_i$ be decomposed as in (2.6.2). The following is obvious:

(3.1.4) Proposition. *If B is torsion free, any $\varphi \in \underline{H}_!(\mathcal{T}, B)^\Gamma$ vanishes on edges belonging to one of the h_i.*

(3.2) As will be explained below, the groups $\underline{H}_!(\mathcal{T}, B)^\Gamma$ for $B = \mathbb{Z}$ and $B = C$ are closely related to the modular curve $\bar{M}_\Gamma$. In particular, we will have:

(3.2.1) $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is a free abelian group of rank g, where

(a) $g =$ abelian rank $\dim_\mathbb{Q}(\Gamma^{\text{ab}} \otimes \mathbb{Q})$ of Γ,

and

(b) $g = g(\Gamma) =$ genus of $\bar{M}_\Gamma$.

Statement (b) will be described in detail in the next section (see (4.7.6) and (4.13.1)). We explain (a).

Let Γ_f be the normal subgroup of Γ generated by the elements of finite order. Then $\Gamma^* := \Gamma/\Gamma_f$ is canonically identified with the fundamental group of $\Gamma \backslash \mathcal{T}$ ([45], I, Thm. 13, Cor. 1). In particular, Γ^* is free and

$$(3.2.2) \qquad (\Gamma^*)^{\text{ab}} \xrightarrow{\cong} H_1(\Gamma \backslash \mathcal{T}, \mathbb{Z}).$$

Let

$$(3.2.3) \qquad \bar{\Gamma} := \Gamma^{\text{ab}} / \text{tor}(\Gamma^{\text{ab}})$$

be the factor commutator group modulo torsion. Since $(\Gamma^*)^{\text{ab}}$ is torsion free, the canonical map from Γ^{ab} to $(\Gamma^*)^{\text{ab}}$ induces an isomorphism

$$(3.2.4) \qquad \bar{\Gamma} \xrightarrow{\cong} (\Gamma^*)^{\text{ab}}.$$

Finally, as results from (3.1.2), the map

$$(3.2.5) \qquad H_1(\Gamma \backslash \mathcal{T}, \mathbb{Z}) \to \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma, \quad \varphi \mapsto \varphi^*$$

defined by $\varphi^*(e) = n(e)\,\varphi(\tilde{e})$ with

$$(3.2.6) \qquad n(e) := z(\Gamma)^{-1} \# (\Gamma_e) = \# (\tilde{\Gamma}_e)$$

is well-defined, injective, and becomes bijective after tensoring with $\mathbb{Q}$. Together, we have an injection with finite cokernel

$$(3.2.7) \qquad \bar{\Gamma} \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma,$$

which will turn out to be bijective in important cases.

(3.3) We now give a more explicit description of this map. For any two vertices v, w of $\mathcal{T}$, let $c(v, w)$ be the unique geodesic (= path without backtracking) from v to w. Choose a base vertex v. For $e \in Y(\mathcal{T})$ and $\alpha, \gamma \in \Gamma$ we put

$$(3.3.1) \qquad i(e, \alpha, \gamma, v) = 1, -1, 0,$$

if $\gamma(e)$ belongs to $c(v, \alpha(v))$, $c(\alpha(v), v)$, neither one, respectively. As a function of γ it has finite support. Since Γ acts on $\mathscr{T}$ through $\bar{\Gamma}$, the function

$$(3.3.2) \qquad \varphi_{\alpha,v}(e) := z(\Gamma)^{-1} \sum_{\gamma \in \Gamma} i(e, \alpha, \gamma, v)$$

on $Y(\mathscr{T})$ is well-defined and $\mathbb{Z}$-valued.

(3.3.3) Lemma. *$\varphi_{\alpha,v}$ has the following properties:*

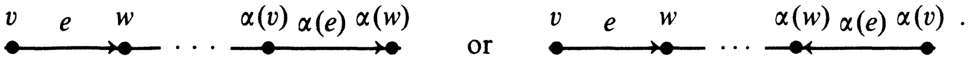
(i) $\varphi_{\alpha,v} \in H_!(\mathscr{T}, \mathbb{Z})^\Gamma$;

(ii) *$\varphi_\alpha := \varphi_{\alpha,v}$ is independent of the choice of v;*

(iii) *$\varphi_{\alpha,\beta} = \varphi_\alpha + \varphi_\beta$, thus $\alpha \mapsto \varphi_\alpha$ induces a homomorphism $j: \bar{\Gamma} \to \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$;*

(iv) *j is injective with finite cokernel.*

Proof. It is obvious from the definition that $\varphi_{\alpha,v}$ belongs to $\underline{H}(\mathscr{T}, \mathbb{Z})^\Gamma$. Since its support in $\Gamma \backslash \mathscr{T}$ is the cycle defined by $c(v, \alpha(v))$, we have (i). Let v and w be adjacent vertices and $\alpha \in \Gamma$. After possibly exchanging v and w, the subgraph spanned by the two geodesics $c(v, \alpha(v))$ and $c(w, \alpha(w))$ looks like



This is, $c(v, \alpha(v)) \cup \{\alpha(e)\} = \{e\} \cup c(w, \alpha(w))$ or $c(v, \alpha(v)) = \{e\} \cup c(w, \alpha(w)) \cup \{\overline{\alpha(e)}\}$. The same number of $\gamma \in \Gamma$ carries a given $e' \in Y(\mathscr{T})$ to e and to $\alpha(e)$, which in the first case implies $\varphi_{\alpha,v} = \varphi_{\alpha,w}$. In the second case, the contributions of e and $\alpha(e)$ in $\varphi_{\alpha,v}$ cancel. We thus have always $\varphi_{\alpha,v} = \varphi_{\alpha,w}$, i.e., (ii), since $\mathscr{T}$ is connected. The formal identity

$$\varphi_{\alpha\beta,v} = \varphi_{\alpha,\beta(v)} + \varphi_{\beta,v}$$

combined with the above yields (iii). Both groups $\bar{\Gamma}$ and $\underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$ are free abelian of the same rank. It therefore suffices for (iv) to show the surjectivity of $j \otimes \mathbb{Q}$. Let T be a maximal tree in $\Gamma \backslash \mathscr{T}$, and let $\{\tilde{e}, \ldots, \tilde{e}_g\}$ be a set of representatives modulo orientation of $Y(\Gamma \backslash \mathscr{T}) - Y(T)$. For $i = 1, 2, \ldots, g$, let $\tilde{v}_i = o(\tilde{e}_i)$, $\tilde{w}_i = t(\tilde{e}_i)$. There exists a unique geodesic $\tilde{c}_i'$ in T that connects $\tilde{w}_i$ to $\tilde{v}_i$. Let $\tilde{c}_i$ be the closed path around $\tilde{v}_i$ obtained by composing $\tilde{e}_i$ with $\tilde{c}_i'$. Define $\varphi_i: Y(\Gamma \backslash \mathscr{T}) \to \mathbb{Z}$ as follows:

$$(3.3.4) \qquad \varphi_i(\tilde{e}) = \left\{ \begin{array}{lll} n(e), & \text{if } \tilde{e} \\ -n(e), & \text{if } \bar{\tilde{e}} \\ 0, & \text{if neither } \tilde{e} \text{ nor } \bar{\tilde{e}} \end{array} \right\} \text{ appears in } \tilde{c}_i.$$

Here $e \in Y(\mathscr{T})$ is a lift of $\tilde{e}$ and $n(e)$ is given by (3.2.6). From (3.1.2) we see that

$$\varphi_i \in \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma.$$

Since $\varphi_i(\tilde{e}_j) = \delta_{i,j} n(e_i)$, they are independent. If $\varphi \in \underline{H}_!(\mathscr{T}, \mathbb{Q})^\Gamma$, its projection

$$\varphi - \sum_i \frac{\varphi(\tilde{e}_i)}{n(e_i)} \varphi_i$$

vanishes outside of the maximal tree T, thus vanishes identically, and $\{\varphi_1, \ldots, \varphi_g\}$ is a basis of $\underline{H}_!(\mathscr{T}, \mathbb{Q})^\Gamma$. Let c_i be a lift of $\tilde{c}_i$ to $\mathscr{T}$ with origin v_i and terminus v_i'. There exists $\alpha_i \in \Gamma$ with $\alpha_i(v_i) = v_i'$, and a closer look shows that in fact $\varphi_i = \varphi_{\alpha_i}$, which ends the proof. □

Going through the constructions, the reader may verify that j agrees with the map of (3.2.7).

(3.4) Some group G is called *p'-torsion free* if every torsion element of G has order a power of p. The following is easy to see:

(3.4.1) Every arithmetic group Γ contains a normal subgroup Δ of finite index which is p'-torsion free. Take e.g. the intersection of $\Gamma \hookrightarrow \Gamma(Y)$ with a congruence subgroup $\Gamma(Y, \mathfrak{a})$, where $\mathfrak{a} \neq A$.

Assume now that $\tilde{\Gamma}$ is p'-torsion free.

(3.4.2) A vertex v (resp. an edge e) of $\mathscr{T}$ is called *stable* with respect to Γ if its stabilizer $\tilde{\Gamma}_v$ (resp. $\tilde{\Gamma}_e$) is trivial, and unstable if not (cf. [45], pp. 132–134). Let $\mathscr{T}_\infty$ be the subgraph of $\mathscr{T}$ consisting of the Γ-unstable vertices and edges. The stabilizer of an unstable vertex v or edge e is a finite p-group that necessarily fixes a unique K-rational end $s(v)$ or $s(e)$, respectively, of $\mathscr{T}$. Moreover (*loc. cit.*, proof of Lemma 13):

(3.4.3) Proposition. *The map $e \mapsto s(e)$ induces a bijection of the set of connected components of $\mathscr{T}_\infty$ with $\mathbb{P}^1(K) =$ set of rational ends of $\mathscr{T}$ (see* (1.6.3)).

Let $\mathscr{T}(s)$ be the connected component of $\mathscr{T}_\infty$ with end $s \in \mathbb{P}^1(K)$. An unstable edge e belongs to $\mathscr{T}(s)$ if and only if $s(e) = s$. Further, $\gamma(\mathscr{T}(s)) = \mathscr{T}(\gamma(s))$ for $\gamma \in \Gamma$, which implies

(3.4.4) $\tilde{\Gamma}_s \backslash \mathscr{T}(s)$ embeds into the quotient graph $\Gamma \backslash \mathscr{T}$, and

$$\Gamma \backslash \mathscr{T}_\infty = \coprod_{s \in \Gamma \backslash \mathbb{P}^1(K)} \tilde{\Gamma}_s \backslash \mathscr{T}(s).$$

Since $\tilde{\Gamma}$ has no p'-torsion, the stabilizers $\tilde{\Gamma}_s$ are elementary abelian p-groups of infinite ranks. In particular, $\tilde{\Gamma}_s \backslash \mathscr{T}(s)$ has no cycles, and is therefore an infinite tree with precisely one end. The graph $\Gamma \backslash \mathscr{T}$ is the union of a finite graph all of whose edges are stable, and a finite number of trees $\Gamma_s \backslash \mathscr{T}(s)$.

(3.4.5) Proposition. *The map $j: \bar{\Gamma} \to \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$ is an isomorphism whenever $\tilde{\Gamma}$ is p'-torsion free.*

Proof. Let T be a maximal tree in $\Gamma \backslash \mathscr{T}$ such that $Y(\Gamma \backslash \mathscr{T}) - Y(T)$ consists of the classes of stable edges of $\mathscr{T}$. Such a T exists by the considerations above. Forming the φ_i

as in (3.3.4), they constitute a $\mathbb{Z}$-basis of $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ since $\varphi_i(\tilde{e}_i) = n(e_i) = \#(\tilde{\Gamma}_{e_i}) = 1$. But φ_i is in the image of j, so j is surjective. □

(3.5) Let B be an abelian group with $pB = 0$ and Γ an arithmetic subgroup of $G(K)$.

(3.5.1) Proposition (see [51], Prop. 3). (i) *If $\tilde{\Gamma}$ is p'-torsion free, each $\varphi \in \underline{H}(\mathcal{T}, \bar{B})^\Gamma$ vanishes on edges in $\mathcal{T}_\infty$.*

(ii) *In the general case, $\underline{H}(\mathcal{T}, B)^\Gamma = \underline{H}_!(\mathcal{T}, B)^\Gamma$. More precisely, $\varphi \in \underline{H}(\mathcal{T}, B)^\Gamma$ vanishes on $e \in Y(\mathcal{T})$ if $\#(\Gamma_e) \equiv 0 \pmod{p}$.*

Proof. (i) By (3.4.3), $\mathcal{T}_\infty = \coprod \mathcal{T}(s)$ $(s \in \mathbb{P}^1(K))$. On $\mathcal{T}(s)$ there is a canonical orientation: An edge e is *positive* if it points to the end s, and *negative* otherwise. Let $e \in Y^+(\mathcal{T}(s))$ and $\{e'\}$ the q_∞ edges of $\mathcal{T}$ with $t(e') = o(e)$. $\varphi(e)$ is determined by the $\varphi(e')$. Since any path in $\mathcal{T}(s)$ without backtracking and that emanates from e into the negative direction is finite, it suffices to show

$$\sum_{e' \in Y(e)} \varphi(e') = 0\,, \tag{3.5.2}$$

where $Y(e) = \{e' \in Y(\mathcal{T}) \mid e' \text{ stable and } t(e') = o(e)\}$. Now since $Y(e)$ consists of stable edges, $\tilde{\Gamma}_e$ acts on $Y(e)$ with orbits of length $\#(\tilde{\Gamma}_e) \equiv 0 \pmod{p}$. Hence the contribution to (3.5.2) of each orbit cancels.

(ii) Let Δ be a normal subgroup of finite index of Γ which is p'-torsion free. Then $\underline{H}(\mathcal{T}, B)^\Gamma = \left(\underline{H}(\mathcal{T}, B)^\Delta\right)^{\Gamma/\Delta}$, and similarly for $\underline{H}_!$. Thus the first assertion on Γ follows from that on Δ. If $e \in Y(\mathcal{T})$ has a stabilizer Γ_e whose order is divisible by p, we can find Δ as above which retains this property, so $\varphi \in \underline{H}(\mathcal{T}, B)^\Gamma \hookrightarrow \underline{H}(\mathcal{T}, B)^\Delta$ vanishes on e. □

(3.6) Recall that the map res of (1.8.5) associates to each holomorphic differential from ω on Ω the harmonic C-valued cochain $\operatorname{res}(\omega) \in \underline{H}(\mathcal{T}, C)$. For a holomorphic function f on Ω, we put $\operatorname{res}(f) = \operatorname{res}(f(z)\,dz)$. If now f is a modular form of weight 2 and type 1, $f(z)\,dz$ is Γ-invariant and so $\operatorname{res}(f) \in \underline{H}(\mathcal{T}, C)^\Gamma = \underline{H}_!(\mathcal{T}, C)^\Gamma$. The next assertion is a generalization of a special case of [51], Theorem 16 (see also [52], Theorem 1).

(3.6.1) Theorem. *Let Γ be an arithmetic subgroup of $G(K)$. The map* $\operatorname{res} : f \mapsto \operatorname{res}(f)$ *is an isomorphism between $M^1_{2,1}(\Gamma)$ and $\underline{H}_!(\mathcal{T}, C)^\Gamma$.*

The proof in *loc. cit.* is given for principal congruence subgroups Γ only, but may easily be adapted to work for general Γ. We leave details to the reader.

(3.6.2) Remarks. It is clear that the formation of $\underline{H}(\mathcal{T}, B)^\Gamma$ and of $\underline{H}_!(\mathcal{T}, B)^\Gamma$ commutes with *flat* ring extensions B'/B. Hence the theorem yields a canonical $\mathbb{F}_p$-structure $M^1_{2,1}(\Gamma, \mathbb{F}_p)$ on the C-vector space $M^1_{2,1}(\Gamma)$. For an arbitrary commutative ring B, the canonical map $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \otimes B \to \underline{H}_!(\mathcal{T}, B)^\Gamma$ is still injective but in general fails to be surjective. We define $\underline{H}_{!!}(\mathcal{T}, B)^\Gamma$ to be its image. By construction, it commutes with arbitrary ring extensions B'/B. Later on (6.5.1), we shall see that res identifies $M^2_{2,1}(\Gamma)$ with $\underline{H}_{!!}(\mathcal{T}, C)^\Gamma$.

4. Automorphic forms of Drinfeld type

In the present section, we explain how harmonic cochains for Γ are related to automorphic forms on $G = \mathrm{GL}(2)$. The basic references are [24], [14], and [55].

(4.1) In what follows, $\mathfrak{A} = \mathfrak{A}_K$ will denote the adele ring of K with ring of integers $\mathcal{O} = \mathcal{O}_K$ and idele group $I = I_K$ (see e.g. [54]). Having fixed the place ∞ of K, they decompose into "finite" and "infinite" parts

$$(4.1.1) \qquad \mathfrak{A} = \mathfrak{A}_f \times K_\infty, \quad \mathcal{O} = \mathcal{O}_f \times O_\infty, \quad I = I_f \times K_\infty^*.$$

The natural inclusions between K, K_∞, $\mathcal{O}_f$, $\mathcal{O}$, $\mathfrak{A}_f$, $\mathfrak{A}$ give rise to inclusions between the groups of points of G over these rings. We refrain from labelling these inclusions and hope that the context will always make clear whether e.g. K will be considered as a subring of K_∞, $\mathfrak{A}_f$, or $\mathfrak{A}$. We recall the principal facts about the above, which are well known but dispersed in the literature.

(4.1.2) $\mathfrak{A}$ is a locally compact ring that contains K as a discrete and cocompact subring (i.e., $K\backslash\mathfrak{A}$ is compact).

(4.1.3) $\mathcal{O}$ is a maximal compact subring of $\mathfrak{A}$, moreover, $\mathcal{O}_f$ is the completion of $A = K \cap \mathcal{O}_f$ with respect to the ideal topology (here the intersection is taken in $\mathfrak{A}_f$).

(4.1.4) The quotient $K^*\backslash I_f/\mathcal{O}_f^*$ is canonically isomorphic with the class group of A, and in particular, is finite.

As in (1.3.2), $G = \mathrm{GL}(2)$ acts as a matrix group from the right on the affine plane $\mathbb{A}^2$. For $\underline{g} \in G(\mathfrak{A}_f)$, let $Y = Y(\underline{g})$ be the A-lattice in K^2 uniquely determined by its span in $\mathcal{O}_f^2$:

$$(4.1.5) \qquad Y \cdot \mathcal{O}_f = \mathcal{O}_f^2 \cdot \underline{g}^{-1}.$$

(Here and in the sequel, we denote adelic data by underlined letters.) Then $\underline{g} \mapsto Y(\underline{g})$ is a bijection of the double coset space $G(K)\backslash G(\mathfrak{A}_f)/G(\mathcal{O}_f)$ with $\mathcal{P}_A^2$ (see (2.5)).

Now let $\mathcal{K}_f$ be an open subgroup of $G(\mathcal{O}_f)$. Since the latter is compact, $\mathcal{K}_f$ has finite index and $G(K)\backslash G(\mathfrak{A}_f)/\mathcal{K}_f$ is finite. As results from the strong approximation theorem for $\mathrm{SL}(2)$, the determinant induces a bijection

$$(4.1.6) \qquad G(K)\backslash G(\mathfrak{A}_f)/\mathcal{K}_f \xrightarrow{\cong} K^*\backslash I_f/\det \mathcal{K}_f.$$

(4.1.7) Finally, we note that $G(K)$ as a subgroup of $G(\mathfrak{A})$ is discrete and "close to cocompact". Viz, choose a Haar measure μ on the locally compact group $G(\mathfrak{A})/Z(\mathfrak{A})$. (In (4.8.1), we will specify μ.) Then the quotient $G(K)\backslash G(\mathfrak{A})/Z(\mathfrak{A})$ has finite volume. Here as usual, $Z \cong \mathbb{G}_m$ is the center of G. Since

$$Z(\mathfrak{A}) \cong I \quad \text{and} \quad K^*\backslash I/\mathcal{O}_f^* \times K_\infty^* \xrightarrow{\cong} K^*\backslash I_f/\mathcal{O}_f^*$$

is finite and $\mathcal{O}_f^*$ is compact, even the quotient $G(K)\backslash G(\mathfrak{A})/Z(K_\infty)$ has finite volume with respect to a Haar measure on $G(\mathfrak{A})/Z(K_\infty)$.

(4.2) Automorphic representations. Let $L^2(G(K)\backslash G(\mathfrak{A})/Z(K_\infty))$ be the Hilbert space of square integrable complex-valued functions on $G(K)\backslash G(\mathfrak{A})/Z(K_\infty)$. It is a unitary $G(\mathfrak{A})$-module, where $G(\mathfrak{A})$ acts through right translations. Let $d\underline{x}$ be a Haar measure on $K\backslash\mathfrak{A}$, and put $L^2_!(\dots)$ for the $G(\mathfrak{A})$-stable subspace of functions $\varphi \in L^2(\dots)$ that satisfy the *cusp condition*

$$\int_{K\backslash\mathfrak{A}} \varphi\left(\begin{pmatrix} 1 & \underline{x} \\ 0 & 1 \end{pmatrix} \underline{g}\right) d\underline{x} = 0 \qquad (4.2.1)$$

for almost all $\underline{g} \in G(\mathfrak{A})$. The space $L^2_!$ decomposes discretely

$$L^2_! = \widehat{\bigoplus} L^2_!(\psi) \qquad (4.2.2)$$

according to characters ψ of the compact group

$$Z(K)\backslash Z(\mathfrak{A})/Z(K_\infty) \xrightarrow{\cong} K^*\backslash I/K_\infty^* .$$

Finally, each $L^2_!(\psi)$ is a Hilbert sum

$$L^2_!(\psi) = \widehat{\bigoplus} V_\varrho , \qquad (4.2.3)$$

where the V_ϱ are irreducible unitary $G(\mathfrak{A})$-submodules occurring with multiplicity one in $L^2_!(\psi)$. All of this is discussed in more detail in [14], sections 3C, 5A. The $G(\mathfrak{A})$-module V_ϱ or rather the underlying unitary representation ϱ will be called a cuspidal *automorphic representation*. Among the automorphic representations considered in [24], they are distinguished through the fact:

(4.2.4) The central character ψ of ϱ has a trivial ∞-component.

(4.3) Newforms. For a positive divisor $\mathfrak{n}$ of K, we define the open subgroups

$$(4.3.1)\qquad \begin{aligned} \mathcal{K}_0(\mathfrak{n}) &= \left\{\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \mid \underline{c} \equiv 0 \,(\operatorname{mod} \mathfrak{n})\right\}, \\ \mathcal{K}(\mathfrak{n}) &= \left\{\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \mid \underline{b} \equiv \underline{c} \equiv 0,\ \underline{a} \equiv \underline{d} \equiv 1 \,(\operatorname{mod} \mathfrak{n})\right\} \end{aligned}$$

of $G(\mathcal{O})$. They are called the *Hecke* and the *principal* congruence subgroup with conductor $\mathfrak{n}$, respectively. Similarly, if $\mathfrak{n}$ is coprime with ∞, we define the open subgroups $\mathcal{K}_{f,0}(\mathfrak{n})$ and $\mathcal{K}_f(\mathfrak{n})$ of $G(\mathcal{O}_f)$. Now for each V_ϱ of the type considered above, with central character ψ, there exists a positive divisor $\mathfrak{n} = \mathfrak{n}_\varrho$ of K, the *conductor of* ϱ, and a distinguished function $0 \neq \varphi = \varphi_\varrho \in V_\varrho$, well-defined up to scalars, the *newform* of ϱ, characterized by

$$\varphi(\underline{g}\underline{k}) = \varphi(\underline{g})\psi(\underline{k}) \quad (\underline{g} \in G(\mathfrak{A}),\ \underline{k} \in \mathcal{K}_0(\mathfrak{n})), \qquad (4.3.2)$$

where $\mathfrak{n}$ is minimal with this property. Here we have extended ψ to a character of $\mathcal{K}_0(\mathfrak{n})$ by putting

(4.3.3) $$\psi(\underline{k}) = \prod_{v \mid \mathfrak{n}} \psi_v(a_v) \quad \text{for } \underline{k} = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}, \quad \underline{a} = (a_v),$$

v running through the places of K (see [14], Thm. 4.24 for details). Thus, considered as a function on $G(\mathfrak{A})$, φ is left $G(K)$-invariant and transforms according to ψ under $Z(\mathfrak{A})$ and $\mathcal{K}_0(\mathfrak{n})$, and may therefore be considered as a function with some extra properties on the double coset space $G(K)\backslash G(\mathfrak{A})/\mathcal{K}(\mathfrak{n})Z(K_\infty)$. Note that $\mathfrak{n}_\varrho$ is a multiple of the ramification divisor of ψ.

(4.4) Automorphic forms. We define an *automorphic cusp form* for an open subgroup $\mathcal{K}$ of $G(\mathcal{O})$ to be a $\mathbb{C}$-valued function φ on

(4.4.1) $$Y(\mathcal{K}) := G(K)\backslash G(\mathfrak{A})/\mathcal{K} \cdot Z(K_\infty)$$

that satisfies (4.2.1). Since $\mathcal{K}$ is open, $Y(\mathcal{K})$ is a discret set. Our definition therefore agrees more or less with (a special case of) the one given in *loc. cit.*, 3.17. (Conditions (iv) and (v) given there are obsolete since we have no archimedean places, (iii) is satisfied since $\mathcal{K}$ has finite index in $G(\mathcal{O})$, and (ii) results from the $Z(K_\infty)$-invariance.) As is easy to see, the argument $\begin{pmatrix} 1 & \underline{x} \\ 0 & 1 \end{pmatrix} \underline{g}$ in (4.2.1) takes on only finitely many values in $Y(\mathcal{K})$. Hence the integral is in fact a finite sum. A closer look shows that automorphic cusp forms have their support in a finite subset of $Y(\mathcal{K})$ ([22], 1.2.3). They are therefore square-integrable, and:

(4.4.2) The *space $W(\mathcal{K})$ of automorphic cusp forms* for $\mathcal{K}$ has finite dimension. Conversely, the newform of an automorphic representation V_ϱ of conductor $\mathfrak{n}$ is an automorphic cusp form for $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ (and even for the subgroup of $G(\mathcal{O})$ of matrices $\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}$ with $c \equiv 0$, $a \equiv 1 \pmod{\mathfrak{n}}$).

(4.4.3) Remark. As results from the absence of archimedean places of K and the discreteness of $Y(\mathcal{K})$, the topology on the coefficient field $\mathbb{C}$ of $W(\mathcal{K})$ plays no role. Furthermore, writing (4.2.1) as a finite linear relation for values of φ, the coefficients are volumes of double cosets, whose ratios are rational numbers. There results a canonical $\mathbb{Q}$-structure $W(\mathcal{K}, \mathbb{Q})$ of $W(\mathcal{K})$. If F is any field of characteristic zero, we put

(4.4.4) $$W(\mathcal{K}, F) = W(\mathcal{K}, \mathbb{Q}) \otimes F$$

and call it the *space of F-valued automorphic cusp forms for $\mathcal{K}$.*

(4.5) Suppose now that the open subgroup $\mathcal{K}$ of $G(\mathcal{O})$ decomposes as a product

(4.5.1) $$\mathcal{K} = \mathcal{K}_f \times \mathcal{K}_\infty$$

with subgroups $\mathcal{K}_f$ of $G(\mathcal{O}_f)$ and $\mathcal{K}_\infty$ of $G(O_\infty)$, respectively. Choose a system S of representatives for the finite set $G(K)\backslash G(\mathfrak{A}_f)/\mathcal{K}_f$ (see (4.1.6)). Each $\underline{g} \in G(\mathfrak{A})$ may be written as

(4.5.2) $$\underline{g} = \gamma \underline{x} \underline{k} g_\infty$$

with $\gamma \in G(K)$, $\underline{k} \in \mathscr{K}_f$, $g_\infty \in G(K_\infty)$, and some uniquely determined $\underline{x} \in S$. (Here $G(K), \mathscr{K}_f \hookrightarrow G(\mathfrak{A}_f)$, and $G(K_\infty)$ are considered as subgroups of $G(\mathfrak{A})$.) Let

(4.5.3) $$\Gamma_{\underline{x}} := G(K) \cap \underline{x} \mathscr{K}_f \underline{x}^{-1}$$

be the intersection in $G(\mathfrak{A}_f)$. It is an arithmetic subgroup of $\Gamma(Y(\underline{x}))$.

(4.5.4) Lemma. *The map on $Y(\mathscr{K})$ that to the double class of $\underline{g} \in G(\mathfrak{A})$ associates the double class of g_∞ in $\Gamma_{\underline{x}} \backslash G(K_\infty)/\mathscr{K}_\infty \cdot Z(K_\infty)$ (see 4.5.2)) is well-defined and identifies $Y(\mathscr{K})$ with the disjoint union $\coprod_{\underline{x} \in S} \Gamma_{\underline{x}} \backslash G(K_\infty)/\mathscr{K}_\infty \cdot Z(K_\infty)$.*

Proof. Straightforward.

The case of interest to us is when $\mathscr{K}_\infty$ is the Iwahori group $\mathscr{I}_\infty = \mathscr{I}$ defined in (1.3.4). In this case,

(4.5.5) $$\begin{aligned} Y(\mathscr{K}) \xrightarrow{\cong} & \coprod_{\underline{x} \in S} \Gamma_{\underline{x}} \backslash G(K_\infty)/Z(K_\infty) \cdot \mathscr{I}_\infty \\ = & \coprod_{\underline{x} \in S} Y(\Gamma_{\underline{x}} \backslash \mathscr{T}). \end{aligned}$$

(4.6) Our next task will be to describe which kind of automorphic forms on

$$\mathscr{K} = \mathscr{K}_f \times \mathscr{I}_\infty$$

correspond under the above isomorphism to harmonic cochains. Let (ϱ, V_ϱ) be an automorphic respresentation of $G(\mathfrak{A})$. As explained in [14], 4C, it decomposes into an infinite tensor product

(4.6.1) $$V_\varrho = \hat{\bigotimes} V_{\varrho,v},$$

where v runs through the set of places of K and $V_{\varrho,v}$ is the space of an irreducible unitary $G(K_v)$-representation ϱ_v. The type of ϱ_v is uniquely determined by ϱ, whose conductor $\mathfrak{n}_\varrho$ splits into the product of local conductors. Further, the newform φ_ϱ of V_ϱ may be represented as the tensor product of local newvectors $\varphi_{\varrho,v} \in V_{\varrho,v}$ with a local property analogous with (4.3.2).

(4.7) The special representation of $G(K_\infty)$. Let F be a field of characteristic zero. $G(K_\infty)$ acts from the left on the projective line $\mathbb{P}^1(K_\infty)$ and from the right on the space of locally constant F-valued functions on $\mathbb{P}^1(K_\infty)$. Dividing out the constants, the resulting *special representation* $\varrho_{\mathrm{sp},F}$ on

(4.7.1) $$V_{\mathrm{sp},F} := \{f \colon \mathbb{P}^1(K_\infty) \to F \mid f \text{ locally constant}\}/F$$

is irreducible. Clearly,

(4.7.2) $$\varrho_{\mathrm{sp},F'} = \varrho_{\mathrm{sp},F} \otimes F'$$

for field extensions F'/F, which justifies omitting the subscript F in $(\varrho_{\mathrm{sp}}, V_{\mathrm{sp}})$. Furthermore, for $F = \mathbb{C}$ the field of complex numbers, there is a (unique up to scaling) $G(K_\infty)$-invariant scalar product on V_{sp} such that

(4.7.3) ϱ_{sp} extends to an irreducible unitary representation $\hat{\varrho}_{\mathrm{sp}}$ on the completion $\hat{V}_{\mathrm{sp}}$ of V_{sp}.

We then have:

(4.7.4) The conductor of ϱ_{sp} or $\hat{\varrho}_{\mathrm{sp}}$ is the prime divisor ∞.

In view of (4.3) this means that its subspace of $\mathcal{I}_\infty$-invariants has dimension one. (For more details, see [14], 4B.)

(4.7.5) An F-valued automorphic cusp form φ for $\mathcal{K}$ will be said to *transform like* ϱ_{sp} if its right $G(K_\infty)$-translates generate a module isomorphic with a finite number of copies of ϱ_{sp}. If the coefficient field F equals $\mathbb{C}$, this is equivalent to saying that φ is right $G(O_\infty)$-finite, and belongs to the direct sum $\bigoplus V_\varrho \hookrightarrow L_!^2$ (see (4.2.2) and (4.2.3)) over a finite number of automorphic representations ϱ all of whose ∞-components are isomorphic with $\hat{\varrho}_{\mathrm{sp}}$.

We put $W_{\mathrm{sp}}(\mathcal{K}, F) \subset W(\mathcal{K}, F)$ for the space of φ that transform like ϱ_{sp}. The next result gives an answer to the question raised in (4.6). It is merely part of [6], 10.3, but for the sake of transparence, we state it separately.

(4.7.6) Theorem (Drinfeld). *Let $\mathcal{K} = \mathcal{K}_f \times \mathcal{I}_\infty$ with an open subgroup $\mathcal{K}_f$ of $G(\mathcal{O}_f)$ and F a field of characteristic zero. Under the bijection* (4.5.5), *the space $\bigoplus_{\underline{x} \in S} \underline{H}_!(\mathcal{T}, F)^{\Gamma_{\underline{x}}}$ of F-valued harmonic cochains corresponds to $W_{\mathrm{sp}}(\mathcal{K}, F)$.*

Note that the cusp condition (4.2.1) forces $\varphi \in W_{\mathrm{sp}}$ to have compact (= finite) support as a function on $\coprod_{\underline{x} \in S} Y(\Gamma_{\underline{x}} \backslash \mathcal{T})$.

(4.8) Petersson product. Let $\underline{g} \in G(\mathfrak{A})$ be written $\underline{g} = \gamma \underline{x} \underline{k} g_\infty$ as in (4.5.2). With its class $\mathrm{cl}(\underline{g})$ in $Y(\mathcal{K})$, we associate

(4.8.1) $$\mu(\mathrm{cl}(\underline{g})) := \frac{z(\Gamma_{\underline{x}})}{\#(\Gamma_{\underline{x}} \cap g_\infty \mathcal{I}_\infty g_\infty^{-1})}.$$

Recall that $z(\Gamma_{\underline{x}}) = \#(\Gamma_{\underline{x}} \cap Z(K))$ and $\Gamma_{\underline{x}} \cap g_\infty \mathcal{I}_\infty g_\infty^{-1}$ is the stabilizer of the edge e of $\mathcal{T}$ determined by g_∞. Thus with (3.2.6), $\mu(\mathrm{cl}(\underline{g})) = n(e)^{-1}$. Clearly, this number depends only on $\mathrm{cl}(\underline{g})$ but not on the choice of representatives S of $G(K) \backslash G(\mathfrak{A}_f)/\mathcal{K}_f$. It is the volume of $\mathrm{cl}(\underline{g})$ with respect to a conveniently normalized Haar measure μ on $G(\mathfrak{A})/Z(K_\infty)$, which we regard as a measure on $Y(\mathcal{K})$. As follows from the considerations of (4.1.7), $\mathrm{vol}_\mu(Y(\mathcal{K}))$ is finite. For more details, see [24], section 10.

There results a scalar product

$$(4.8.2) \qquad (\varphi_1, \varphi_2)_\mu := \int_{Y(\mathscr{K})} \varphi_1(\underline{g})\,\varphi_2(\underline{g})\,d\mu(\underline{g})$$

on the space of $\mathbb{Q}$-valued functions with compact support on $Y(\mathscr{K})$, and notably, on $W_{\text{sp}}(\mathscr{K}, \mathbb{Q})$. Since φ_1 and φ_2 have compact support, the integral is in fact a finite sum. This product, or its real euclidean, or complex hermitian extension, will be called the *Petersson product*. It is the restriction of the scalar product on $L^2(G(K)\backslash G(\mathfrak{A})/Z(K_\infty), \mu)$ to its subspace $W_{\text{sp}}(\mathscr{K}, \mathbb{C})$.

(4.9) Hecke operators. We merely give a special case of the general construction and refer to [14], 5 B for an exhaustive treatment of Hecke operators. Let $\mathfrak{n}$ be the *conductor of* $\mathscr{K}_f$, i.e., the least positive divisor $\mathfrak{n}$ coprime with ∞ such that $\mathscr{K}_f(\mathfrak{n})$ is contained in $\mathscr{K}_f$. If v is a finite place coprime with $\mathfrak{n}$ (such v are called *unramified* for $\mathscr{K}_f$, or for $\mathfrak{n}$), $\mathscr{K}_v := G(O_v)$ embeds into $\mathscr{K}_f$. Let π_v be a local uniformizer and τ_v the matrix

$$\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} \in G(K_v).$$

The group $\mathscr{H}_v := \mathscr{K}_v \cap \tau_v \mathscr{K}_v \tau_v^{-1}$ has index $q_v + 1$ in $\mathscr{K}_v$. Let dk_v be the Haar measure on $\mathscr{K}_v$ normalized such that

$$(4.9.1) \qquad \operatorname{vol}_{dk_v}(\mathscr{H}_v) = 1.$$

For a function φ on $Y(\mathscr{K})$ put

$$(4.9.2) \qquad (T_v\varphi)(\underline{g}) := \int_{\mathscr{K}_v} \varphi(\underline{g}k_v\tau_v)\,dk_v = \sum_{k_v \in \mathscr{K}_v/\mathscr{H}_v} \varphi(\underline{g}k_v\tau_v),$$

where k_v runs through a set of representatives for $\mathscr{K}_v/\mathscr{H}_v$. The *Hecke operator* $T_v : \varphi \mapsto T_v\varphi$ has the following properties:

(4.9.3) $T_v\varphi$ is again a function of $Y(\mathscr{K})$, and T_v maps $W_{\text{sp}} = W_{\text{sp}}(\mathscr{K}, \mathbb{C})$ into itself. We therefore regard T_v as an operator on W_{sp} only.

(4.9.4) Any two T_v, T_w commute. Together with their adjoints T_v^* with respect to the Petersson product, they generate a commutative algebra of normal operators in W_{sp}.

By the above, there exists a basis $\{\varphi\}$ of W_{sp} that consists of simultaneous eigenforms for all the T_v. Let $\lambda(\varphi, v)$ be the corresponding eigenvalue. Then

$$(4.9.5) \qquad \lambda(\varphi, v) \text{ is an algebraic integer},$$

and

$$(4.9.6) \qquad |\lambda(\varphi, v)| \leqq 2q_v^{1/2}$$

for every embedding of $\mathbb{Q}(\lambda(\varphi, v))$ into $\mathbb{C}$.

Among these statements, (4.9.3) and (4.9.4) are straightforward from the definitions, and (4.9.5) results since T_v comes from a correspondence on $Y(\mathscr{K})$ with integral coefficients. In contrast, (4.9.6), the analogue of Ramanujan's conjecture, is deep. It has been proven by Drinfeld [6], who reduced it to Weil's conjectures, i.e., to properties of Frobenius eigenvalues on the cohomology of modular curves of the type considered in section 2. See also (4.13.1).

(4.9.7) Remark. Our space $W_{\mathrm{sp}}(\mathscr{K}, \mathbb{C})$ splits

$$W_{\mathrm{sp}} = \bigoplus_{\psi} W_{\mathrm{sp}}(\psi)$$

according to characters ψ of the finite group $Z(K)\backslash Z(\mathfrak{A}_f)/Z(\mathfrak{A}_f) \cap \mathscr{K}_f$. Clearly, the operators T_v respect the decomposition. An easy change of variables in (4.9.2) now shows the formula

$$T_v^* \mid W_{\mathrm{sp}}(\psi) = \psi^{-1}(\pi_v)\, T_v \mid W_{\mathrm{sp}}(\psi) \qquad (4.9.8)$$

for the adjoint T_v^* of T_v restricted to $W_{\mathrm{sp}}(\psi)$. Since v is unramified, the value $\psi(\pi_v)$ does not depend on the choice of π_v but only on the place v. In particular, T_v is hermitian if restricted to the component $W_{\mathrm{sp}}(1)$ of the trivial character $\psi = 1$.

(4.9.9) Remark. We introduced Hecke operators T_v for prime divisors v only. For some purposes, it is useful to extend the definition to arbitrary positive finite divisors $\mathfrak{m}$ coprime with $\mathfrak{n}$, using a formula similar to (4.9.2). The resulting operators $T_{\mathfrak{m}}$ will satisfy $T_{\mathfrak{m}\cdot\mathfrak{m}'} = T_{\mathfrak{m}} \cdot T_{\mathfrak{m}'}$ $((\mathfrak{m}, \mathfrak{m}') = 1)$ and $T_{v^n} =$ polynomial in T_v with integral coefficients. In particular, $T_{\mathfrak{m}}$ lies in the $\mathbb{Z}$-algebra generated by the T_v. We refer to [47], 3.2, 3.3 or [44], VII, 5, where analogous cases are treated.

(4.10) The special Galois representation. Let K_∞^{ur} be the maximal unramified extension of K_∞. The Galois group $\mathrm{Gal}(K_\infty^{\mathrm{ur}}/K_\infty)$ is isomorphic with the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$, where the canonical generator corresponds to the Frobenius automorphism F_∞ of K_∞^{ur} over K_∞. Let l be a prime different from $p = \mathrm{char}(\mathbb{F}_q)$, and let $E_l/K_\infty^{\mathrm{ur}}$ be the field extension obtained by adjoining all the l^r-th roots ($r \in \mathbb{N}$) of the uniformizer π_∞ to K_∞^{ur}. Then $\mathrm{Gal}(E_l/K_\infty^{\mathrm{ur}}) = \mathbb{Z}_l(1) = \varprojlim \mu(l^r) \approx \mathbb{Z}_l$. Moreover, E_l is galois over K_∞, and its group is a semi-direct product

$$\begin{aligned} \mathrm{Gal}(E_l/K_\infty) &= \mathrm{Gal}(K_\infty^{\mathrm{ur}}/K_\infty) \ltimes \mathrm{Gal}(E_l/K_\infty^{\mathrm{ur}}) \\ &= \hat{\mathbb{Z}} \ltimes \mathbb{Z}_l(1), \end{aligned} \qquad (4.10.1)$$

where the action of $F_\infty = 1 \in \hat{\mathbb{Z}}$ on $\mathbb{Z}_l(1)$ is given by

$$F_\infty u F_\infty^{-1} = u^{q_\infty} \quad (u \in \mathbb{Z}_l(1)). \qquad (4.10.2)$$

Choose an isomorphism of $\mathbb{Z}_l(1)$ with $\mathbb{Z}_l$. The two-dimensional l-adic *special representation* $\mathrm{sp} = \mathrm{sp}_l$ of the local Galois group $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ is the representation on $\mathbb{Q}_l^2$ given by

(4.10.3) $\mathrm{sp}\colon \mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty) \to \mathrm{Gal}(E_l/K_\infty) \xrightarrow{\cong} \hat{\mathbb{Z}} \ltimes \mathbb{Z}_l \to \mathrm{Gl}(2, \mathbb{Q}_l)$,

where the right hand arrow maps (1,0) to $\begin{pmatrix} 1 & 0 \\ 0 & q_\infty^{-1} \end{pmatrix}$ and (0,1) to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For a more detailed discussion, see [3], 3.1.

(4.11) Let now $\mathscr{K}_f$ be any open subgroup of $G(\mathcal{O}_f)$. To $\mathscr{K}_f$ there corresponds a moduli scheme $M_{\mathscr{K}_f}$, which may be described as follows: If $\mathscr{K}_f(\mathfrak{n}) \subset \mathscr{K}_f$, then $M_{\mathscr{K}_f}$ = quotient of $M^2(\mathfrak{n})$ by the finite group $\mathscr{K}_f/\mathscr{K}_f(\mathfrak{n})$. Here $\mathfrak{n}$ is any positive admissible (i.e., divisible by at least two primes) finite divisor of K. (Recall that $M^2(\mathfrak{n})$ represents the functor "Drinfeld A-modules of rank two with a level-$\mathfrak{n}$ structure", so there is a natural action of $G(\mathcal{O}_f)/\mathscr{K}_f(\mathfrak{n})$ on $M^2(\mathfrak{n})$, see (2.4).) $M_{\mathscr{K}_f}$ is a coarse moduli scheme for a moduli problem that depends on $\mathscr{K}_f$.

The main examples are $\mathscr{K}_f = \mathscr{K}_f(\mathfrak{n})$ and $\mathscr{K}_f = \mathscr{K}_{f,0}(\mathfrak{n})$, which lead to $M_{\mathscr{K}_f} = M^2(\mathfrak{n})$ and $M_0^2(\mathfrak{n})$, respectively. $M_0^2(\mathfrak{n})$ is called the *Hecke moduli scheme* of level $\mathfrak{n}$; it classifies "rank-two Drinfeld A-modules with a cyclic A-submodule of exact order $\mathfrak{n}$".

Base extension from A to C of $M_{\mathscr{K}_f}$ yields an affine curve over C, whose associated analytic space has a description similar to the one given in (2.5). More precisely, there is a canonical isomorphism of analytic spaces

$$\text{(4.11.1)} \qquad \coprod_{\underline{x} \in S} \Gamma_{\underline{x}} \backslash \Omega \xrightarrow{\cong} M_{\mathscr{K}_f}(C),$$

where S and $\Gamma_{\underline{x}}$ are as in (4.5), and the components $\Gamma_{\underline{x}} \backslash \Omega = M_{\Gamma_{\underline{x}}}$ are modular curves of the type considered in section 2. Note that (2.5.1) is a special case of (4.11.1).

(4.12) The Hecke correspondence. Next, we let $\bar{M}_{\mathscr{K}_f}$ be the canonical normal compactification of $M_{\mathscr{K}_f}$ ([6], section 9). Using (4.11.1), its space of C-points is the disjoint union $\coprod \bar{M}_{\Gamma_{\underline{x}}}$ of the compactifications of its components. From the interpretation of $M_{\mathscr{K}_f}$ as a moduli scheme there results for every $\mathscr{K}_f$-unramified finite place v of K a Hecke correspondence on $M_{\mathscr{K}_f}$, which uniquely extends to $\bar{M}_{\mathscr{K}_f}$. By abuse of notation, we label this correspondence and everything derived from it (the associated endomorphisms of the Jacobian, and of the l-adic cohomology of $\bar{M}_{\mathscr{K}_f} \times C$) by "$T_v$". As usual, the correspondence T_v associates to every point $x \in M_{\mathscr{K}_f}$ (i.e., to every rank-two Drinfeld module Φ with a level-$\mathscr{K}_f$ structure) the collection of points x' for which there exists a v-isogeny $x \to x'$. It may be described through the action of $G(\mathfrak{A}_f)$ on the projective limit $\varprojlim M^2(\mathfrak{n})$ of schemes $M^2(\mathfrak{n})$ ([6], 5 D, [10], I, 3). After base extension with C, the module Φ is given by an A-lattice Λ in C. Then T_v associates to Λ the collection of lattices Λ' that "contain Λ with index v". For details, see [10], VIII, 1.

(4.13) We are now ready to state (a version appropriate for our purposes of) *Drinfeld's reciprocity law*:

(4.13.1) Theorem (Drinfeld). *Let $\mathscr{K} = \mathscr{K}_f \times \mathscr{I}_\infty$ be as above, $\bar{M} \times C = \bar{M}_{\mathscr{K}_f} \times C$ the corresponding compactified modular curve over C, and $l \neq p$ a prime number. The l-adic cohomology module $H^1(\bar{M} \times C, \mathbb{Q}_l)$ is canonically isomorphic with $W_{\mathrm{sp}}(\mathscr{K}, \mathbb{Q}_l) \otimes \mathrm{sp}_l$, where W_{sp} and sp_l are given by (4.7.5) and (4.10.2). The isomorphism is compatible with the actions of*

(a) *the Hecke operators* T_v (*v an unramified finite place*);

(b) *the local Galois group* $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ (*which acts on* $H^1(\bar{M} \times C, \mathbb{Q}_l)$ *since* $M_{\mathcal{K}_f}$ *and* $\bar{M}_{\mathcal{K}_f}$ *are defined over* K).

(4.13.2) Remarks and comments. The statement just given follows from combining Proposition 10.3 and Theorem 2 of [6]. The assertion given there is about the cohomology of $\varprojlim \bar{M}^2(\mathfrak{n}) \times C$, considered as a representation space for $G(\mathfrak{A}_f)$. (4.13.1) comes out by taking $\mathcal{K}_f$-invariants, since knowledge of the $G(\mathfrak{A}_f)$-action is equivalent with the knowledge of *all* the T_v (v unramified for $\mathcal{K}_f$) for *all* open subgroups $\mathcal{K}_f$ of $G(\mathcal{O}_f)$. Similarly, we see that the isomorphism commutes with "Atkin-Lehner involutions" whenever these are defined, e.g. $\mathcal{K}_f = \mathcal{K}_{f,0}(\mathfrak{n})$.

Now let (V, ϱ) be any $\bar{\mathbb{Q}}_l$-valued automorphic representation of $G(\mathfrak{A}_f)$ whose ∞-component ϱ_∞ is isomorphic with ϱ_{sp}. Its newvector φ_ϱ occurs in $W_{\mathrm{sp}}(\mathcal{K}, \bar{\mathbb{Q}}_l)$ for some $\mathcal{K} = \mathcal{K}_f \times \mathcal{I}_\infty$. By "multiplicity one", the $G(\mathfrak{A}_f) \times \mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$-submodule $V \otimes \mathrm{sp}$ of $H^1(\varprojlim \bar{\mathrm{M}}^2(\mathfrak{n}), \bar{\mathbb{Q}}_l)$ is even Gal (K^{sep}/K)-stable. So the latter splits as a

$$G(\mathfrak{A}_f) \times \mathrm{Gal}(K^{\mathrm{sep}}/K)\text{-module}$$

into a direct sum $\bigoplus_i V_i \otimes s_i$, where (V_i, ϱ_i) is an automorphic representation with $\varrho_{i,\infty} \cong \varrho_{\mathrm{sp}}$ and s_i a representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in $\bar{\mathbb{Q}}_l^2$ whose restriction to $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ is isomorphic with sp. This establishes a one-to-one correspondence between the V_i and the s_i that preserves conveniently normalized L-factors. This correspondence is a special case of the Langlands conjectures for $G = \mathrm{GL}(2)$ over K; it justifies the labelling as a "reciprocity law". A much more detailed discussion of the theorem along with its implications for the Langlands conjectures is given in [5]. We have chosen our more simple-minded version of (4.13.1) since in this form, it is best adapted to our purposes (see sections 8, 9). Note in particular that by (4.7.6), the information about the Hecke action on a modular curve $\bar{M}_{\mathcal{K}_f}$ is encoded in $W_{\mathrm{sp}}(\mathcal{K}, \mathbb{Q}_l) \xrightarrow{\cong} \bigoplus_{\underline{x} \in S} \underline{H}_!(\mathcal{T}, \mathbb{Q}_l)^{\Gamma_{\underline{x}}}$. The latter is a quite accessible object, on which calculations can be carried out.

5. Theta functions

In the whole section, Γ is a fixed arithmetic subgroup of $G(K) = \mathrm{GL}(2, K)$.

(5.1) A *holomorphic theta function* for Γ is a function $u : \Omega \to C$ that for each $\alpha \in \Gamma$ satisfies a functional equation

$$(5.1.1) \qquad u(\alpha z) = c_u(\alpha)\, u(z)$$

with some $c_u(\alpha) \in C^*$ independent of z, and is holomorphic without zeroes on Ω and at the cusps. In view of (5.1.1), u is invariant under $\Gamma_\infty^{\mathrm{u}}$, since this is a p-group and C^* is p-torsion free. Therefore, u has a Laurent expansion w.r.t. $t(\Gamma, \infty)$ (see (2.7.3)). Holomorphy of u at ∞ (and at the other cusps) is then defined as in (2.7.7). For a *meromorphic theta function*, we allow poles and zeroes on Ω (but not at the cusps). We set

$\Theta_h(\Gamma) \hookrightarrow \Theta_m(\Gamma)$ for the multiplicative groups of holomorphic or meromorphic theta functions, respectively. Clearly

$$c_u : \Gamma \to C^*, \quad \alpha \mapsto c_u(\alpha)$$

and

(5.1.2) $$c : \Theta_m(\Gamma) \to \mathrm{Hom}(\Gamma, C^*) = \mathrm{Hom}(\Gamma^{ab}, C^*), \quad u \mapsto c_u$$

are group homomorphisms. Furthermore,

(5.1.3) $$\ker(c) \cap \Theta_h(\Gamma) = C^*,$$

since Ω is a Stein domain [16].

(5.1.4) Lemma. *Let $u(z)$ be a holomorphic theta function for Γ. Then its logarithmic derivative $u'(z)/u(z)$ lies in $M^2_{2,1}(\Gamma)$.*

Proof. The transformation rule for $f(z) = u'(z)/u(z)$ comes directly from (5.1.1), thus $f \in M_{2,1}(\Gamma)$. Let $u = \sum_{i \geq 0} a_i t^i$ be the expansion of u at the cusp ∞ (compare (2.7.6)), where $a_0 \neq 0$. From (2.10.1), $u' = \text{const.} \times t^2 +$ higher terms, thus u'/u has a double zero at ∞, and similarly at the other cusps. □

(5.1.5) Remark. In (7.5.2) we shall see that the divisor on $\bar{M}_\Gamma$ of a theta function has degree zero. The non-vanishing requirement for holomorphic theta functions is therefore superfluous.

(5.2) Let ω, η be fixed elements of Ω, and put

(5.2.1) $$\theta_\Gamma(\omega, \eta, z) = \prod_{\gamma \in \tilde{\Gamma}} \frac{z - \gamma\omega}{z - \gamma\eta}.$$

Note that the product is not over Γ but over $\tilde{\Gamma} = \Gamma/\Gamma \cap Z(K)$. In order to simplify notation, we write elements of $\tilde{\Gamma}$ as matrices, and omit most parantheses in expressions $\gamma\omega = \gamma(\omega)$. The product is said to *converge locally uniformly on Ω* if there exists an admissible covering (U_n) of Ω such that, given U_n and $\varepsilon > 0$, almost all the factors satisfy

(5.2.2) $$\left| \frac{z - \gamma\omega}{z - \gamma\eta} - 1 \right| < \varepsilon \quad \text{uniformly for } z \in U_n.$$

This being the case, the resulting meromorphic function $\theta_\Gamma(\omega, \eta, \cdot)$ will turn out to be a theta function for Γ. As long as Γ is fixed, we omit the subscript Γ.

Products of this type have been introduced by Manin-Drinfeld [29] and studied by Gerritzen-van der Put [17] and Myers [33]. In the case of the special arithmetic groups

$\Gamma = \mathrm{GL}(2, A)$, the basic properties (5.2.3) and (parts of) (5.4.1) are due to W. Radtke ([38], sections 7, 8, see also [39]). We simplify his proofs and complete his results.

(5.2.3) Proposition. *The product for $\theta(\omega, \eta, z)$ converges locally uniformly on Ω. If $\Gamma\omega \neq \Gamma\eta$, $\theta(\omega, \eta, z)$ has a zero (pole) of order $\#(\tilde{\Gamma}_\omega)$ $(\#(\tilde{\Gamma}_\eta))$ at ω, η, respectively, and no other zeroes or poles. If $\Gamma\omega = \Gamma\eta$, $\theta(\omega, \eta, z)$ has neither zeroes nor poles on Ω.*

The proof is contained in (5.3), notably (5.3.8).

(5.3) Estimates. Let $\omega, \eta \in \Omega$ be given. We will do the relevant estimates on

(5.3.1) $$U_n = \{z \in \Omega \,|\, |z| \leqq |\pi|^{-n}, |z|_i \geqq |\pi|^n\}.$$

(Recall that π is a uniformizer at ∞ and $|\pi| = q_\infty^{-1}$.) Clearly, U_n is affinoid and $\Omega = \bigcup_{n \in \mathbb{N}} U_n$ an admissible covering. Let Γ_∞ be the subgroup of upper triangular matrices in Γ. The following is easy:

(5.3.2) Lemma. *Two elements $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ of Γ define the same class in $\Gamma_\infty \backslash \Gamma$ if and only if $u(c, d) = (c', d')$ with some $u \in \mathbb{F}_q^*$.*

(5.3.3) Lemma. *Let two constants c_0, $c_1 > 0$ be given. For almost all the classes in $\Gamma_\infty \backslash \Gamma$ of elements $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ, $|z|_i > c_0$ implies $|cz + d| > c_1$. (By abuse of language, we write "for almost all pairs (c, d)". Of course, the same assertion holds for the first rows (a, b) of $\gamma \in \Gamma$.)*

Proof. Clear in view of $|cz + d| \geqq |c||z|_i$ and the fact that the c, d occurring as coefficients in Γ are contained in a fractional A-ideal. □

(5.3.4) Corollary. *Let $\varepsilon > 0$ be given. For almost all pairs (c, d) such that*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

we have $|\gamma\eta|_i < \varepsilon$.

Proof. Straight from (5.3.3) and (1.1.5). □

(5.3.5) Lemma. *Let (c, d) be fixed and c_2, $c_3 > 0$. For almost all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ in the class determined by (c, d), $|z| < c_2$ implies $|z - \gamma\eta| > c_3$.*

Proof. Applying (5.3.3) to the first row (a, b) of γ and $z = \eta$,

$$|\gamma\eta| = |c\eta + d|^{-1} |a\eta + b| > \sup(c_2, c_3)$$

for almost all (a, b). Then $|z - \gamma\eta| = |\gamma\eta| > c_3$. □

(5.3.6) Lemma. *For almost all* $\gamma \in \Gamma$, $|z - \gamma\eta| \geqq |\pi|^n$ *uniformly on* U_n.

Proof. By (5.3.4), $|\gamma\eta|_i < |\pi|^n$ for almost all pairs (c, d). For these,

$$|z - \gamma\eta| \geqq |z - \gamma\eta|_i \geqq |\pi|^n$$

since $|z|_i \geqq |\pi|^n$. For the remaining pairs (c, d), apply (5.3.5). □

An easy calculation yields

$$\text{(5.3.7)} \qquad \frac{z - \gamma\omega}{z - \gamma\eta} - 1 = \frac{\det\gamma(\eta - \omega)}{(z - \gamma\eta)(c\eta + d)(c\omega + d)} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(5.3.8) Lemma. *Let* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *be given. For almost all* $\gamma \in \Gamma$ *we have*

$$\left| \frac{z - \gamma\omega}{z - \gamma\eta} - 1 \right| < \varepsilon$$

uniformly on U_n. (Note this proves Proposition (5.2.3)!)

Proof. (5.3.7) combined with (5.3.6) and (5.3.3) gives the wanted estimate for almost all pairs (c, d). Fixing (c, d) and applying (5.3.5) shows that it holds for almost all $\gamma \in \Gamma$. □

(5.3.9) Lemma. *Let* $s = u/v \in K$. *The product* $\prod_{\gamma \in \bar{\Gamma}} \frac{s - \gamma\omega}{s - \gamma\eta}$ *converges.*

Proof. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $|s - \gamma\eta| = \dfrac{|(uc - av)\eta + ud - bv|}{|v||c\eta + d|} \geqq c_4 |c\eta + d|^{-1}$, where $c_4 > 0$ is independent of γ. Now (5.3.7) yields $\left| \dfrac{s - \gamma\omega}{s - \gamma\eta} - 1 \right| \leqq c_4^{-1} \dfrac{|\eta - \omega|}{|c\omega + d|}$. As usual, for all but a finite number of pairs (c, d), the above is less than ε for $\varepsilon > 0$ given. We conclude by combining (5.3.7) with (5.3.5). □

(5.3.10) Lemma. *Given* $\varepsilon > 0$, *there exists* $c_5 > 0$ *such that for all* $\gamma \in \Gamma$ *and* $z \in \Omega$ *with* $|z|_i > c_5$ *we have* $\left| \dfrac{z - \gamma\omega}{z - \gamma\eta} - 1 \right| < \varepsilon$.

Proof. From (5.3.7) we see that $\left| \dfrac{z - \gamma\omega}{z - \gamma\eta} - 1 \right| \leqq c_6 |z - \gamma\eta|^{-1}$ with some constant c_6 that depends only on ω, η, and Γ, but not on $z \in \Omega$ and $\gamma \in \Gamma$. Let

$$c_5 \geqq \sup\{c_6/\varepsilon, |\gamma\eta|_i, \gamma \in \Gamma\},$$

which exists e.g. by (5.3.4). Then for $|z|_i > c_5$ and $\gamma \in \Gamma$,

$$|z - \gamma\eta| \geqq |z - \gamma\eta|_i \geqq |z|_i > c_5 \quad \text{and} \quad \left| \frac{z - \gamma\omega}{z - \gamma\eta} - 1 \right| < c_6/c_5 \leqq \varepsilon. \quad \square$$

(5.4) The most important properties of the functions $\theta(\omega, \eta, z)$ are collected in the following result.

(5.4.1) Theorem. *Let $\omega, \eta \in \Omega$ and $\alpha \in \Gamma$ be given.*

(i) *There exists a constant $c(\omega, \eta, \alpha) \in C^*$ such that*

$$\theta(\omega, \eta, \alpha z) = c(\omega, \eta, \alpha)\, \theta(\omega, \eta, z) \tag{5.4.2}$$

independently of z.

(ii) *$c(\omega, \eta, \alpha)$ depends only on the class of α in $\bar{\Gamma} = \Gamma^{\mathrm{ab}}/\mathrm{torsion}$. Thus $\alpha \mapsto c(\omega, \eta, \alpha)$ defines a group homomorphism from Γ to C^* that factors through $\bar{\Gamma}$.*

(iii) *The function $\theta(\omega, \eta, z)$ is holomorphic and non-zero at the cusps of Γ.*

(iv) *The holomorphic function*

$$u_\alpha(z) = \theta(\omega, \alpha\omega, z) \tag{5.4.3}$$

is independent of the choice of $\omega \in \Omega$. It depends only on the class of α in $\bar{\Gamma}$.

(v) *For $\alpha, \beta \in \Gamma$ we have $u_{\alpha\beta} = u_\alpha u_\beta$.*

(vi) *We have*

$$c(\omega, \eta, \alpha) = u_\alpha(\eta)/u_\alpha(\omega)\,. \tag{5.4.4}$$

In particular, $c(\omega, \eta, \alpha)$ is holomorphic in ω and η.

(vii) *Let*

$$c_\alpha(\cdot) = c(\omega, \alpha\omega, \cdot) : \Gamma \to C^* \tag{5.4.5}$$

be the multiplier of u_α. Then $(\alpha, \beta) \mapsto c_\alpha(\beta)$ defines a symmetric bilinear map from $\bar{\Gamma} \times \bar{\Gamma}$ to C^*.

As results from (i) and (iii), the functions $\theta(\omega, \eta, z)$ $(u_\alpha(z))$ are meromorphic (holomorphic) theta functions for Γ. Note also that by (vii),

$$(\alpha, \beta) := \log |c_\alpha(\beta)| \tag{5.4.6}$$

is a $\mathbb{Q}$-valued symmetric bilinear pairing on $\bar{\Gamma}$. Before giving the proof, we note that

$$\frac{\alpha z - \omega}{\alpha z - \eta} = h_\alpha \frac{z - \alpha^{-1}\omega}{z - \alpha^{-1}\eta}, \quad \text{where } h_\alpha = \frac{\alpha\infty - \omega}{\alpha\infty - \eta} \in C^*\,. \tag{5.4.7}$$

(The two rational functions in z have the same divisor. Letting $z=\infty$ yields the formula. Of course, $h_\alpha = 1$ if $\alpha\infty = \infty$.)

Accordingly, we have

$$\frac{\alpha z-\gamma\omega}{\alpha z-\gamma\eta} = h_{\alpha,\gamma}\frac{z-\alpha^{-1}\gamma\omega}{z-\alpha^{-1}\gamma\eta}, \quad h_{\alpha,\gamma} = \frac{\alpha\infty-\gamma\omega}{\alpha\infty-\gamma\eta}. \tag{5.4.8}$$

Proof of (5.4.1). (i) $\theta(\omega,\eta,\alpha z) = \prod_{\gamma\in\bar{\Gamma}}\left(h_{\alpha,\gamma}\frac{z-\alpha^{-1}\gamma\omega}{z-\alpha^{-1}\gamma\eta}\right)$, so we have to show that

$$c(\omega,\eta,\alpha) = \prod_{\gamma\in\bar{\Gamma}} h_{\alpha,\gamma} \tag{5.4.9}$$

converges. For $\alpha\infty=\infty$ this is trivial, for $\alpha\infty = u/v\in K$ it is (5.3.9).

(iv) $\theta(\omega,\alpha\omega,z)\,\theta(\eta,\alpha\eta,z)^{-1} = \prod_\gamma\left(\frac{z-\gamma\omega}{z-\gamma\alpha\omega}\right)\prod_\gamma\left(\frac{z-\gamma\alpha\eta}{z-\gamma\eta}\right)$

$$= \prod_\gamma\left(\frac{z-\gamma\omega}{z-\gamma\eta}\right)\prod_\gamma\left(\frac{z-\gamma\alpha\eta}{z-\gamma\alpha\omega}\right) = \theta(\omega,\eta,z)\,\theta(\eta,\omega,z) = 1\,.$$

(v) $u_{\alpha\beta}(z) = \theta(\omega,\alpha\beta\omega,z) = \theta(\omega,\beta\omega,z)\,\theta(\beta\omega,\alpha\beta\omega,z) = u_\beta(z)\,u_\alpha(z)$.

(vi) The γ-th factor of $c(\omega,\eta,\alpha)$ is

$$\frac{\alpha z-\gamma\omega}{\alpha z-\gamma\eta}\bigg/\frac{z-\gamma\omega}{z-\gamma\eta} = \frac{\gamma\omega-\alpha z}{\gamma\omega-z}\bigg/\frac{\gamma\eta-\alpha z}{\gamma\eta-z} = \frac{\omega-\gamma^{-1}\alpha z}{\omega-\gamma^{-1}z}\bigg/\frac{\eta-\gamma^{-1}\alpha z}{\eta-\gamma^{-1}z}$$

(see (5.4.7)), which is the γ^{-1}-th factor of $u_\alpha(\eta)/u_\alpha(\omega)$.

(ii) Since $\alpha\mapsto c(\omega,\eta,\alpha)$ is a homomorphism into the abelian group C^*, it factors through Γ^{ab}. Let $\alpha\in\Gamma$ have order m in Γ^{ab}. Fixing ω, the holomorphic function $\eta\mapsto c(\omega,\eta,\alpha)$ takes its values in the m-th roots of unity, hence is constant. Inserting $\eta=\omega$ shows it is identically one.

(iii) In view of (2.7.7), it suffices to see that $\theta_\Gamma(\omega,\eta,z)$ is holomorphic and non-zero at ∞ for every arithmetic group Γ in $G(K)$. Now (5.3.10) shows that actually

$$\theta_\Gamma(\omega,\eta,z) = 1 + o(t)\,, \tag{5.4.10}$$

where $t = t(\Gamma,\infty)$ is the uniformizer at ∞.

(iv) (continued) We still have to show that u_α only depends on the class of α in $\bar{\Gamma}$. Let α be such that α^m is a commutator. Then u_α must be constant equal to an m-th root of unity. Evaluating at $z=\infty$ shows $u_\alpha\equiv 1$.

(vii) $c_\alpha(\beta)$ is defined on $\bar{\Gamma} \times \bar{\Gamma}$ and bimultiplicative. By (vi) we have

$$c_\alpha(\beta) = c(\omega, \alpha\omega, \beta) = u_\beta(\alpha\omega)/u_\beta(\omega) = c_\beta(\alpha)\,. \quad \square$$

(5.4.11) Remark. The functions $\theta(\omega, \eta, z)$ may be analytically continued such that ω and η are allowed to be elements of $\mathbb{P}^1(K)$. This amounts to modifying those of the factors of (5.2.1) that correspond to $\gamma\omega = \infty$ or $\gamma\eta = \infty$. From the modified product, one can read off:

(5.4.12) Proposition. *$c_\alpha(\beta) \in K_\infty$ for all $\alpha, \beta \in \Gamma$.*

Since the complete discussion involves complicated case considerations and an extension of the estimates in (5.3), it will be omitted. (5.4.12) may be seen by the following *ad hoc* argument: Let $\omega \in \Omega$ be algebraic over K_∞. Then (5.4.8) and (5.4.9) show that $c_\alpha(\beta) \in K_\infty(\omega)$. Since u_α is independent of the choice of ω, $c_\alpha(\beta)$ is contained in every proper algebraic extension of K_∞, thus in K_∞. $\square$

(5.5) For a holomorphic theta function u, the cochain $r(u) \in \underline{H}(\mathscr{T}, \mathbb{Z})$ is Γ-invariant, as directly results from the construction of the map r (see (1.7.3)). Let now $\alpha \in \Gamma$ be given. Our next aim is to compare $r(u_\alpha)$ to the cochain φ_α of (3.3). We need some preparations.

(5.5.1) Let $D_0 = C_0 \,\dot{\cup}\, D_0^0 \,\dot{\cup}\, C_1$ be the subsets of Ω discussed in (1.5): C_0, D_0^0, C_1 is given as the set of $z \in \Omega$ that satisfy $|z|_i = |z| = 1$, $|\pi| < |z|_i = |z| < 1$, $|z|_i = |z| = |\pi|$, respectively. By e we denote the oriented edge of $\mathscr{T}$ determined by D_0, i.e.,

$$\lambda(C_1) = o(e), \quad \lambda(C_0) = t(e)\,,$$

having chosen the orientation as in case (α) of (1.5.9). The graph $\mathscr{T} - \{e, \bar{e}\}$ has two connected components

$$\mathscr{T} - \{e, \bar{e}\} = \mathscr{T}^+ \textstyle\coprod \mathscr{T}^-\,, \tag{5.5.2}$$

where $o(e) \in \mathscr{T}^-$, $t(e) \in \mathscr{T}^+$. We let

$$\begin{aligned} &\Omega^+ = \lambda^{-1}(\mathscr{T}^+(\mathbb{R}) - t(e)), && \Omega^- = \lambda^{-1}(\mathscr{T}^-(\mathbb{R}) - o(e)), \\ &\bar{\Omega}^+ = \Omega^+ \cup C_0 = \lambda^{-1}(\mathscr{T}^+(\mathbb{R})), && \bar{\Omega}^- = \Omega^- \cup C_1 = \lambda^{-1}(\mathscr{T}^-(\mathbb{R}))\,. \end{aligned} \tag{5.5.3}$$

As is easily verified,

$$\begin{aligned} \Omega^+ &= \{z \in \Omega \,|\, |z| > 1 \text{ or } |z|_i < |z| = 1\}\,, \\ \Omega^- &= \{z \in \Omega \,|\, |z| \leqq |\pi|,\ |z|_i < |\pi|\}\,. \end{aligned} \tag{5.5.4}$$

Therefore,

$$\begin{aligned} |z - \omega| &= |\omega| \quad (z \in D_0, \omega \in \Omega^+) \quad \text{or } (z \in D_0^0,\ \omega \in \bar{\Omega}^+) \\ &= |z| \quad (z \in D_0, \omega \in \Omega^-) \quad \text{or } (z \in D_0^0,\ \omega \in \bar{\Omega}^-)\,. \end{aligned} \tag{5.5.5}$$

This in turn implies the

(5.5.6) Lemma. *For $z \in D_0$ we have*

$$\log\left(\left\|\frac{z-\omega}{z-\omega'}\right\|_{C_0}^{\mathrm{sp}} \Big/ \left\|\frac{z-\omega}{z-\omega'}\right\|_{C_1}^{\mathrm{sp}}\right) = \begin{cases} 1, & \omega \in \Omega^-, \omega' \in \Omega^+, \\ -1, & \omega \in \Omega^+, \omega' \in \Omega^-, \\ 0, & \omega, \omega' \text{ both in } \Omega^+ \text{ or in } \Omega^-. \end{cases}$$

We further note the trivial

(5.5.7) Observation. Let $f \in \mathcal{O}_\Omega(D_0)^*$. Then

$$\|f\|_{C_0}^{\mathrm{sp}} / \|f\|_{C_1}^{\mathrm{sp}} = \lim_{\substack{z \in D_0^0 \\ |z| \to 1}} |f(z)| \Big/ \lim_{\substack{z \in D_0^0 \\ |z| \to |\pi|}} |f(z)|.$$

(5.6) Theta functions and harmonic cochains. Now we can state and prove the wanted relationship.

(5.6.1) Theorem. *Let $\alpha \in \Gamma$ be given. Then $r(u_\alpha) = \varphi_\alpha$ (see* (1.7.3) *and* (3.3) *for definitions).*

Proof. Up to conjugation with elements of $G(K)$, we only need to show that given any arithmetic group Γ and any $\alpha \in \Gamma$, $r(u_\alpha)$ and φ_α agree on the edge e associated to D_0. Let $\omega \in \Omega$ be such that $v := \lambda(\omega) \in X(\mathcal{T})$. Then $u_\alpha(z) = \prod_{\gamma \in \tilde{\Gamma}} \frac{z - \gamma\omega}{z - \gamma\alpha\omega}$, where almost all the factors have absolute value 1 uniformly on D_0. If γ is such that $\{\gamma\omega, \gamma\alpha\omega\} \cap D_0 = \emptyset$, its contribution to $r(u_\alpha)(e)$ is

$$(5.6.2) \qquad \begin{cases} 1, & \text{if } \gamma\omega \in \Omega^-, \gamma\alpha\omega \in \Omega^+, \\ -1, & \text{if } \gamma\omega \in \Omega^+, \gamma\alpha\omega \in \Omega^-, \\ 0, & \text{otherwise}, \end{cases}$$

as we read off from (5.5.6). The product

$$f = \prod \frac{z - \gamma\omega}{z - \gamma\alpha\omega}$$

over those $\gamma \in \tilde{\Gamma}$ such that $\gamma\omega$ or $\gamma\alpha\omega$ lies in D_0 is finite and invertible on D_0, i.e., its zeroes and poles on D_0 cancel. We thus may evaluate $\log(\|f\|_{C_0}^{\mathrm{sp}} / \|f\|_{C_1}^{\mathrm{sp}})$ termwise by means of (5.5.7) and (5.5.5). As $\{\gamma\omega, \gamma\alpha\omega\} \cap D_0^0 = \emptyset$, this yields

$$(5.6.3) \qquad \begin{cases} 1, & \text{if } \gamma\omega \in \bar{\Omega}^-, \gamma\alpha\omega \in \bar{\Omega}^+, \\ -1, & \text{if } \gamma\omega \in \bar{\Omega}^+, \gamma\alpha\omega \in \bar{\Omega}^-, \\ 0, & \text{otherwise} \end{cases}$$

for the contribution of $(z - \gamma\omega)/(z - \gamma\alpha\omega)$ to $r(u_\alpha)(e)$, even if $\gamma\omega$ or $\gamma\alpha\omega$ meets D_0. We therefore have

$$
\begin{aligned}
r(u_\alpha)(e) = \quad & \#\{\gamma \in \tilde{\Gamma} \mid \gamma\omega \in \bar{\Omega}^-, \gamma\alpha\omega \in \bar{\Omega}^+\} - \#\{\gamma \in \tilde{\Gamma} \mid \gamma\omega \in \bar{\Omega}^+, \gamma\alpha\omega \in \bar{\Omega}^-\} \\
= \quad & \#\{\gamma \in \tilde{\Gamma} \mid e \text{ occurs in the path } c(\gamma(v), \gamma\alpha(v)) \text{ on } \mathscr{T}\} \\
& - \#\{\gamma \in \tilde{\Gamma} \mid \bar{e} \text{ occurs in the path } c(\gamma(v), \gamma\alpha(v)) \text{ on } \mathscr{T}\} \\
= \quad & z(\Gamma)^{-1} \sum_{\gamma \in \Gamma} i(e, \alpha, \gamma^{-1}, v) \quad \text{(see (3.3.1))} \\
= \quad & \varphi_\alpha(e)\,. \quad \square
\end{aligned}
$$

(5.6.4) Corollary. $r(u_\alpha) \in \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$ *(i.e., $r(u_\alpha)$ has compact support modulo Γ).*

Of course, this may also be seen directly: It follows from the fact that u_α is finite and non-zero at cusps, so its absolute value becomes constant in the neighborhood of a cusp.

(5.6.5) Corollary. *The homomorphism $\bar{u}: \bar{\Gamma} \to \Theta_h(\Gamma)/C^*$ induced by $\alpha \mapsto u_\alpha$ is injective with finite cokernel, and bijective if $\bar{\Gamma}$ is p'-torsion free.*

Proof. $\alpha \mapsto \varphi_\alpha : \bar{\Gamma} \to \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$ has the same properties. $\square$

(5.7) Recall that the Petersson product $(.,.)_\mu$ on $\underline{H}_!(\mathscr{T}, \mathbb{Q})^\Gamma$ is induced from the measure μ on $Y(\Gamma \backslash \mathscr{T})$ that to each $\tilde{e}$ associates $\mu(\tilde{e}) = n^{-1}(e)$ (see (3.2.6)), where $e \in Y(\mathscr{T})$ lies above $\tilde{e}$. On the other hand, we have the pairing $(.,.)$ on $\bar{\Gamma}$ defined by (5.4.6).

(5.7.1) Theorem. *Let $\alpha, \beta \in \Gamma$. Then $2(\alpha, \beta) = (\varphi_\alpha, \varphi_\beta)_\mu$.*

(Here and in the sequel, we write "$f(\alpha)$" instead of "$f(\bar{\alpha})$" for functions f on $\bar{\Gamma}$ and $\bar{\alpha} =$ class of $\alpha \in \Gamma$.)

Proof. We adapt an argument of van der Put [36] to our situation. Let $P(\mathscr{T}, \mathbb{Q})$be the group of $\mathbb{Q}$-valued potentials on $\mathscr{T}$ (= functions f on $X(\mathscr{T})$ such that df is a harmonic cochain). Then the exact sequence (1.7.2) extends to a commutative diagram of Γ-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^* & \longrightarrow & \mathcal{O}_\Omega(\Omega)^* & \xrightarrow{\;r\;} & \underline{H}(\mathscr{T}, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \log|.|} & & \Big\downarrow{\scriptstyle \log\|.\|^{sp}} & & \Big\downarrow\cap & & \\
0 & \longrightarrow & \mathbb{Q} & \longrightarrow & P(\mathscr{T}, \mathbb{Q}) & \xrightarrow[d]{} & \underline{H}(\mathscr{T}, \mathbb{Q}) & \longrightarrow & 0\,.
\end{array} \tag{5.7.2}
$$

The middle vertical map is derived from the spectral norm, and $df(e) = f(t(e)) - f(o(e))$. The long exact cohomology sequence yields a commutative diagram

$$
\begin{array}{ccc}
H^0(\Gamma, \underline{H}(\mathscr{T}, \mathbb{Z})) = \underline{H}(\mathscr{T}, \mathbb{Z})^\Gamma & \longrightarrow & \mathrm{Hom}(\Gamma, C^*) = H^1(\Gamma, C^*) \\
\Big\downarrow\cap & & \Big\downarrow{\scriptstyle \log|.|} \\
\underline{H}(\mathscr{T}, \mathbb{Q})^\Gamma & \longrightarrow & \mathrm{Hom}(\Gamma, \mathbb{Q})\,.
\end{array}
$$

Now $c_\alpha \in \operatorname{Hom}(\bar{\Gamma}, C^*)$ is the cocycle associated to $\varphi_\alpha = r(u_\alpha) \in \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$, thus

(5.7.3) $(\alpha, \beta) = \log|c_\alpha(\beta)| = P_\alpha(\beta v) - P_\alpha(v)$ $(v \in X(\mathscr{T})$ any base vertex),

where $P_\alpha = \log\|u_\alpha\|^{\mathrm{sp}}$ is the potential characterized up to an additive constant by $dP_\alpha = \varphi_\alpha$. Let $(e_1, \dots, e_r) = c(v, \beta v)$ and $(e'_1, \dots, e'_s) = c(v, \alpha v)$ be the geodesics in $\mathscr{T}$ connecting v to βv and v to αv, respectively. (5.7.3) reads $(\alpha, \beta) = \sum_{1 \le i \le r} \varphi_\alpha(e_i)$. By the very definition of φ_α, $\varphi_\alpha(e) = \sum_{1 \le j \le s} \{e, e'_j\}$, so

$$(\alpha, \beta) = \sum_{1 \le i \le r} \sum_{1 \le j \le s} \{e_i, e'_j\}$$

with

(5.7.4)
$$\{e, e'\} = \begin{cases} n(e), & \Gamma e = \Gamma e', \\ -n(e), & \Gamma \bar{e} = \Gamma e', \\ 0 & \text{otherwise}. \end{cases}$$

On the other hand

$$\begin{aligned} (\varphi_\alpha, \varphi_\beta)_\mu &= \sum_{e \in Y(\Gamma \backslash \mathscr{T})} n(e)^{-1} \varphi_\alpha(e) \varphi_\beta(e) \\ &= \sum_e n(e)^{-1} \sum_i \sum_j \{e, e_i\}\{e, e'_j\}, \end{aligned}$$

having extended the function $\{.,.\}$ to $Y(\Gamma \backslash \mathscr{T})$.

Now $\{e, e_i\}\{e, e'_j\} \neq 0$ if and only if the three edges e, e_i, e'_j agree up to sign in $\Gamma \backslash \mathscr{T}$, in which case it takes the value $n(e)^2$ if $e_i = e'_j$ and $-n(e)^2$ if $\bar{e}_i = e'_j$. Since both e and $\bar{e}$ give the same contribution, we have

$$(\varphi_\alpha, \varphi_\beta)_\mu = 2 \sum_{1 \le i \le r} \sum_{1 \le j \le s} \{e_i, e'_j\} = 2(\alpha, \beta). \quad \square$$

(5.7.5) Corollary. *The form (α, β) on $\bar{\Gamma} \otimes \mathbb{R} \xrightarrow{\cong} \underline{H}_!(\mathscr{T}, \mathbb{R})^\Gamma$ is positive definite.*

(5.7.6) Corollary. *The homomorphism $\bar{c}: \bar{\Gamma} \to \operatorname{Hom}(\bar{\Gamma}, C^*)$ induced by $\alpha \mapsto c_\alpha$ is injective.*

6. Changing levels

In this section, Γ will be an arithmetic group and Δ a congruence subgroup of Γ. We use subscripts Γ, Δ to indicate the level of objects treated, e.g. $j_\Gamma : \bar{\Gamma} \to \underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$, $u_{\Gamma,\alpha} \in \Theta_h(\Gamma)$ etc. Our first aim is to relate the theta functions for Γ and for Δ.

(6.1) First note that Γ and $\tilde{\Gamma} = \Gamma / \Gamma \cap Z(K)$ have the same torsion free abelizations: The canonical map from Γ^{ab} to $\tilde{\Gamma}^{\mathrm{ab}}$ induces an isomorphism

(6.1.1)
$$\bar{\Gamma} = \Gamma^{\mathrm{ab}} / \operatorname{tor}(\Gamma^{\mathrm{ab}}) \xrightarrow{\cong} \tilde{\Gamma}^{\mathrm{ab}} / \operatorname{tor}(\tilde{\Gamma}^{\mathrm{ab}}).$$

Let

$$(6.1.2) \qquad \begin{array}{ll} I\colon \ \tilde{\Delta}^{\mathrm{ab}} \to \tilde{\Gamma}^{\mathrm{ab}}, & \bar{I}\colon \ \bar{\Delta} \to \bar{\Gamma}, \\ V\colon \ \tilde{\Gamma}^{\mathrm{ab}} \to \tilde{\Delta}^{\mathrm{ab}}, & \bar{V}\colon \ \bar{\Gamma} \to \bar{\Delta} \end{array}$$

be the maps induced from the inclusion $\tilde{\Delta} \hookrightarrow \tilde{\Gamma}$ and the transfer maps, respectively. The transfer is defined and discussed e.g. in [43], VII, 7/8 (see also proof of (6.2.1)). We will write $V\alpha$ for V applied to the class of $\alpha \in \Gamma$. Similar notation will be used for I, $\bar{V}$, $\bar{I}$. We have (*loc. cit.*, Prop. 6): If n is the index $[\tilde{\Gamma} : \tilde{\Delta}]$ then

$$(6.1.3) \qquad I \circ V = n = \bar{I} \circ \bar{V},$$

where "n" is multiplication by n on the additively written groups $\tilde{\Gamma}^{\mathrm{ab}}$, $\bar{\Gamma}$, respectively. Since $\bar{\Gamma}$ is torsion free, this implies:

(6.1.4) Proposition. $\bar{V}\colon \bar{\Gamma} \to \bar{\Delta}$ *is injective.*

(6.2) Trivially, any theta function for Γ is a theta function for Δ, i.e., $\Theta_h(\Gamma) \hookrightarrow \Theta_h(\Delta)$. A more precise description is as follows:

(6.2.1) Proposition. *Let $\alpha \in \Gamma$ and $u_{\Gamma,\alpha}$ the attached holomorphic theta function. Then $u_{\Gamma,\alpha} = u_{\Delta,\bar{V}\alpha}$. In other words, the diagram*

$$(6.2.2) \qquad \begin{array}{ccc} \bar{\Gamma} & \xrightarrow{u_\Gamma} & \Theta_h(\Gamma) \\ \Big\downarrow \bar{V} & & \Big\downarrow \\ \bar{\Delta} & \xrightarrow[u_\Delta]{} & \Theta_h(\Delta) \end{array}$$

is commutative, where u_Γ is derived from $\alpha \mapsto u_{\Gamma,\alpha}$.

Proof (compare [17], p. 234/235, where a similar case is treated). Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of representatives in $\tilde{\Gamma}$ for $\tilde{\Delta}\backslash\tilde{\Gamma}$ and σ the permutation of $\{1, 2, \ldots, n\}$ such that the classes of $\alpha_i\alpha$ and $\alpha_{\sigma(i)}$ agree. Let further $\gamma_i = \alpha_i \alpha \alpha_{\sigma(i)}^{-1}$. With some $\omega \in \Omega$,

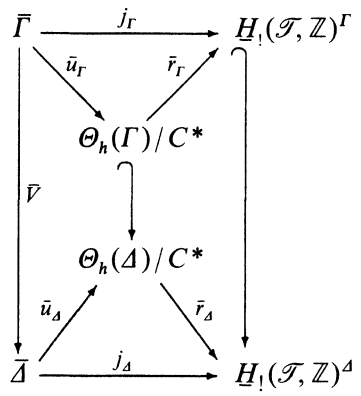
$$\begin{aligned} u_{\Gamma,\alpha}(z) &= \prod_{\gamma \in \tilde{\Gamma}} \left(\frac{z - \gamma\omega}{z - \gamma\alpha\omega} \right) = \prod_{1 \leq i \leq n} \prod_{\gamma \in \tilde{\Delta}} \left(\frac{z - \gamma\alpha_i\omega}{z - \gamma\alpha_i\alpha\omega} \right) \\ &= \prod_{\gamma \in \tilde{\Delta}} \prod_{1 \leq i \leq n} \left(\frac{z - \gamma\alpha_i\omega}{z - \gamma\gamma_i\alpha_{\sigma(i)}\omega} \right) \\ &= \prod_{\gamma} \prod_{i} \left(\frac{z - \gamma\alpha_{\sigma(i)}\omega}{z - \gamma\gamma_i\alpha_{\sigma(i)}\omega} \right) \\ &= \prod_{i} \prod_{\gamma} \cdots = \prod_{i} u_{\Delta,\gamma_i} = u_{\Delta,\beta}, \end{aligned}$$

where the class of $\beta := \prod_i \gamma_i = \prod_i \alpha_i \alpha \alpha_{\sigma(i)}^{-1}$ equals $\bar{V}$ (class of α) ([43], VII, Prop. 7). □

Essentially the same rearranging procedure yields the corresponding result for $\varphi_{\Gamma,\alpha}$, whose proof is left to the reader:

(6.2.3) Proposition. $\varphi_{\Gamma,\alpha} = \varphi_{\Delta,\bar{V}\alpha}$.

Hence the diagram



(6.2.4)

is commutative, where $\bar{r}_\Gamma$ and $\bar{r}_\Delta$ are derived from $r : \mathcal{O}_\Omega(\Omega)^* \to \underline{H}(\mathcal{T}, \mathbb{Z})$. Note also that $\bar{I}$ corresponds to the norm map on harmonic cochains, i.e.,

(6.2.5)
$$\begin{array}{ccc} \bar{\Gamma} & \xrightarrow{j_\Gamma} & \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \\ \uparrow \bar{I} & & \uparrow N \\ \bar{\Delta} & \xrightarrow{j_\Delta} & \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Delta \end{array}$$

commutes, with $(N\varphi)(e) = \sum \varphi(\alpha_i e)$.

(6.3) Next, let f be a (holomorphic or meromorphic) theta function for Δ. With $\{\alpha_i\}$ as above, put

(6.3.1)
$$L_\Delta^\Gamma f(z) := \prod_i f(\alpha_i z).$$

(6.3.2) Proposition. (i) $g := L_\Delta^\Gamma f(z)$ *is a (holomorphic or meromorphic) theta function for* Γ, *which up to constants does not depend on the choice of* $\{\alpha_i\}$.

(ii) *Let* $c_f : \bar{\Delta}^{\mathrm{ab}} \to C^*$ *be the multiplier of* f. *The multiplier of* g *is* $c_g = c_f \circ V$.

Proof. For $\alpha \in \Gamma$, let $\sigma(i)$ and γ_i be as in the proof of (6.2.1). Then

$$g(\alpha z) = \prod_i f(\alpha_i \alpha z) = \prod f(\gamma_i \alpha_{\sigma(i)} z)$$
$$= \prod c_f(\gamma_i) \prod f(\alpha_{\sigma(i)} z) = c_f(V\alpha) g(z),$$

i.e., g transforms under Γ according to $c_f \circ V$. The other assertions are then clear. □

The construction applies notably to our basic theta functions. For these we get

(6.3.3) Proposition. *Let $f(z) = \theta_\Delta(\omega, \eta, z)$. Then $L_\Delta^\Gamma f(z) = \text{const.} \times \theta_\Gamma(\omega, \eta, z)$.*

Proof.

$$\begin{aligned} L_\Delta^\Gamma f(z) &= \prod_i \theta_\Delta(\omega, \eta, \alpha_i z) \\ &= \prod_i \prod_{\gamma \in \tilde{\Delta}} \left(\frac{\alpha_i z - \gamma\omega}{\alpha_i z - \gamma\eta}\right) \\ &= c \cdot \prod_i \prod_{\gamma \in \tilde{\Delta}} \left(\frac{z - \alpha_i^{-1}\gamma\omega}{z - \alpha_i^{-1}\gamma\eta}\right) \quad \text{by (5.4.7)} \\ &= c \cdot \prod_{\gamma \in \tilde{\Gamma}} \left(\frac{z - \gamma\omega}{z - \gamma\eta}\right) \\ &= c\theta_\Gamma(\omega, \eta, z) \end{aligned}$$

with some $c \in C^*$. □

(6.3.4) Corollary. *Let $\alpha \in \Delta$. Then $L_\Delta^\Gamma u_{\Delta,\alpha} = \text{const.} \times u_{\Gamma,\alpha}$.*

(6.4) From now on, we make the **assumption**:

(6.4.1) $\tilde{\Delta}$ is p'-torsion free, and the index $n = [\tilde{\Gamma} : \tilde{\Delta}]$ is not divisible by p.

(6.4.2) Lemma. *Each arithmetic group Γ contains some Δ subject to* (6.4.1).

Proof. Let $\Gamma \hookrightarrow \Gamma(Y)$ be as in (2.1), let $\mathfrak{n} \subsetneqq A$ be an ideal, and π the projection from $\Gamma(Y)$ to $\mathrm{GL}(Y/\mathfrak{n}Y)$ modulo the image of $Z(\mathbb{F}_q) \cong \mathbb{F}_q^*$. Let further S be a p-Sylow subgroup of $\pi(\Gamma)$ and $\Delta := \pi^{-1}(S) \cap \Gamma$. Then $[\tilde{\Gamma} : \tilde{\Delta}] = [\pi(\Gamma) : S] \not\equiv 0 \pmod p$. Since $\Gamma(Y, \mathfrak{n}) = \ker \pi$ has no p'-torsion ([45], p. 130, Ex. 3), the same must hold for Δ. □

(6.4.3) Corollary. *A theta function $f \in \Theta_h(\Delta)$ belongs to $\Theta_h(\Gamma)$ if and only if $r_\Delta(f) \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. The map $\bar{r}_\Gamma : \Theta_h(\Gamma)/C^* \to \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is always bijective.*

Proof. Let $r_\Delta(f) \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. If $\gamma \in \Gamma$, put $f_\gamma(z) := f(\gamma z)$. From $r_\Delta(f) = r_\Delta(f_\gamma)$,

$$f_\gamma = c(\gamma) f$$

for some $c(\gamma) \in C^*$, which means that $f \in \Theta_h(\Gamma)$. The second assertion follows from the bijectivity of $\bar{r}_\Delta$. □

Now the crucial statement is

(6.4.4) Proposition. *For each arithmetic subgroup Γ of $G(K)$, the cokernel of*

$$j_\Gamma : \bar\Gamma \to \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$$

is finite with order not divisible by p.

Proof. We already know that j_Γ is injective with finite cokernel. Consider (6.2.4) with some $\Delta \subset \Gamma$ that satisfies (6.4.1). Put $H := j_\Delta^{-1}(\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma)$ and consider $\bar V$ as an embedding. Then $\bar\Gamma \subset H$. Since $\bar I \circ \bar V = n$ on $\bar\Gamma$, $\bar I$ restricted to H is again multiplication by n and maps H into $\bar\Gamma$. Thus $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma / j_\Gamma(\bar\Gamma) \cong H/\bar\Gamma$ has an order that divides

$$\#(H/nH) = n^{g(\Gamma)} \not\equiv 0 \,(\mathrm{mod}\, p). \qquad \square$$

(6.4.5) Remark. We suspect that in fact j_Γ is always bijective. This is at least true in the cases

(a) $\tilde\Gamma$ is p'-torsion free;

(b) K is a rational function field and ∞ is the usual place at infinity, i.e., $A = \mathbb{F}_q[T]$.

Case (a) has been treated in (3.4.5). Following the same lines, the result for (b) comes out by explicit determination of the quotient graph $\Gamma \backslash \mathcal{T}$ [34]. However, this method seems not to be applicable in the general case. For another approach, see [41].

(6.5) A large part of our results so far may be summarized in the following commutative diagram (explanations below) with surjective vertical maps:

$$\begin{array}{ccccc} & & \bar\Gamma & & \\ & \swarrow^{\bar u} & & \searrow^{j} & \\ \Theta_h(\Gamma)/C^* & & \xrightarrow[\cong]{\bar r} & & \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \\ \Big\downarrow {\scriptstyle u \mapsto u'/u} & & & & \Big\downarrow {\scriptstyle \text{reduction } \mathrm{mod}\, p} \\ M^2_{2,1}(\Gamma, \mathbb{F}_p) & & \xrightarrow[\text{res}]{\cong} & & \underline{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma \end{array} \tag{6.5.1}$$

Here $j: \alpha \mapsto \varphi_\alpha$ and $\bar u$ is derived from $\alpha \mapsto u_\alpha$. For the upper triangle, see (5.6.1) and (6.4.3). By (1.8.7), (3.6.1), and (5.1.4), we have a commutative diagram (6.5.1)′ similar to the above, but with $\underline{H}_!(\mathcal{T}, \mathbb{F}_p)^\Gamma$ instead of $\underline{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma$. (Clearly, the res mapping in the bottom row of (6.5.1)′ will not be surjective, since $\underline{H}_!$ in general is strictly larger than $\underline{H}_{!!}$, see (2.10.2).) Now by definition (3.6.2) of $\underline{H}_{!!}$, reduction mod$p$ maps $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ surjectively onto $\underline{H}_{!!} \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{F}_p)^\Gamma$. In view of (6.4.4), $j(\bar\Gamma)$ has the same image $\underline{H}_{!!}$ in $\underline{H}_!(\mathcal{T}, \mathbb{F}_p)^\Gamma$ as $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. Recall that

(6.5.2) $$\dim_{\mathbb{F}_p} M^2_{2,1}(\Gamma, \mathbb{F}_p) = \dim_C M^2_{2,1}(\Gamma) = g \quad \text{and}$$
$$\dim_{\mathbb{F}_p} \underline{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma = \operatorname{rank} \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma = \operatorname{rank} \bar{\Gamma} = g\,.$$

The commutativity of (6.5.1)′ therefore forces the map $\alpha \mapsto u'_\alpha / u_\alpha : \bar{\Gamma} \to M^2_{2,1}(\Gamma, \mathbb{F}_p)$ to be surjective. So res maps $M^2_{2,1}$ in fact into $\underline{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma$, and it is bijective by dimension reasons. □

We point out the next two results, which have been proved in the course of the preceding discussion.

(6.5.3) Theorem. *The residue map* res *identifies* $M^2_{2,1}(\Gamma, \mathbb{F}_p)$ *with* $\underline{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma$ *and* $M^2_{2,1}(\Gamma)$ *with* $\underline{H}_{!!}(\mathcal{T}, C)^\Gamma$.

(6.5.4) Theorem. *Let* $\{\alpha_1, \ldots, \alpha_g\}$ *be elements of* Γ *whose classes in* $\bar{\Gamma}$ *form a* $\mathbb{Z}$-*basis. Put* $u_i = u_{\alpha_i}$. *Then the elements* u'_i / u_i $(i = 1, \ldots, g)$ *form an* $\mathbb{F}_p$-*basis of* $M^2_{2,1}(\Gamma, \mathbb{F}_p)$, *hence a* C-*basis of* $M^2_{2,1}(\Gamma)$.

The intuitive meaning of (6.5.1) is twofold. First recall that by (4.7.6), $\underline{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$ "is" a space of automorphic forms provided with a natural integral structure. Therefore, identifying the left and right hand sides of (6.5.1) by the indicated isomorphisms, we can state:

(6.5.5) "Drinfeld modular forms (double cuspidal, of weight two and type one, and with $\mathbb{F}_p$-residues) are the reductions (mod p) of automorphic forms (of the type described in section 4)."

Secondly,

(6.5.6) a cuspidal $\mathbb{F}_p$-valued harmonic cochain res f for Γ can be lifted to a cuspidal $\mathbb{Z}$-valued harmonic cochain for Γ if and only if f is double cuspidal. In this case, the lifting of res f amounts to integrating $f = u'/u$ to a holomorphic theta function u for Γ.

We finally note that all the groups appearing in (6.5.1) are provided with natural Hecke actions. For $M_{2,1}(\Gamma)$, it is the natural Hecke action on modular forms, described e.g. in [10]. The action on harmonic cochains is as in (4.9), the action on $\bar{\Gamma}$ and on $\Theta_h(\Gamma)$ will be discussed in section 9. It will then turn out that with suitable normalizations, all the maps in (6.5.1) are compatible with Hecke operators.

7. The Jacobian

(7.1) We know from (2.7.8) that $\bar{M}_\Gamma$ is a totally split curve over K_∞. As is shown in [17], this implies that it is also a Mumford curve, i.e., may be described through a Schottky subgroup of $\mathrm{PGL}(2, K_\infty)$. It would be very interesting to make this alternative description of $\bar{M}_\Gamma$ explicit. Recent work of van der Put [37] shows how to find the Schottky group in question. Another consequence of the fact that $\bar{M}_\Gamma$ it totally split is the following ([17], VI):

(7.1.1) Let J_Γ be the Jacobian of $\bar{M}_\Gamma$. Then J_Γ has an analytic uniformization.

This means, there exists a discrete subgroup Λ of $(C^*)^g$ isomorphic with $\mathbb{Z}^g$ such that $J_\Gamma(C)$ is isomorphic as an analytic group variety with $(C^*)^g/\Lambda$. We will construct the lattice Λ by means of theta functions.

(7.2) Abelian varieties und analytic tori (cf. [15]).

(7.2.1) A subgroup Λ of $(C^*)^g$ is called a *lattice* if $\Lambda \cong \mathbb{Z}^g$ and the image of Λ under $\log: (C^*)^g \to \mathbb{R}^g$ is a lattice in $\mathbb{R}^g$. The logarithm map depends on the given coordinates on $(C^*)^g$, i.e., the choice of a basis of its character group, but the lattice property is independent of that choice. Thus we may define lattices in arbitrary tori (= algebraic groups isomorphic with $\mathbb{G}_m^g$), and may do so for tori defined over K_∞ or its extensions in C.

(7.2.2) Let now T be a torus (over C, to fix ideas) of dimension g and Λ a lattice in $T(C)$. Since Λ is discrete, T/Λ exists as an anlytic space: It is an analytic group variety, and is compact in the rigid analytic sense.

(7.2.3) Theorem (Gerritzen [15]). *T/Λ is projective algebraic (i.e., the analytic space associated to an abelian variety) if and only if there exists a homomorphism σ from Λ to the character group $\mathscr{X}(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ of T such that the bilinear map*

$$(\alpha, \beta) \mapsto \sigma(\alpha)(\beta) : \Lambda \times \Lambda \to C^*$$

is symmetric and positive definite. The last condition means that $\log|\sigma(\alpha)(\alpha)| > 0$ whenever $1 \neq \alpha \in \Lambda$.

(7.2.4) Remarks. By GAGA, the abelian variety that corresponds to T/Λ is uniquely determined. As usual, we therefore will not further distinguish the two concepts. The theorem as stated is the synopsis of several assertions of *loc. cit.*, notably Theorem 5. The requirement that the complete valued field C be algebraically closed is not needed. So the corresponding statement holds over each complete subextension of C/K_∞, provided all the ingredients are defined over this field.

(7.3) We now return to our general situation, dealing with modular curves. As a torus, we take

$$T_\Gamma := \mathrm{Hom}(\bar{\Gamma}, \mathbb{G}_m)\,, \tag{7.3.1}$$

which is a split torus defined over K_∞ with character group $\bar{\Gamma}$. As a lattice in $T_\Gamma(C)$, we take $\bar{\Gamma}$, embedded into $T_\Gamma(C)$ via $\bar{c}: \alpha \mapsto c_\alpha$. The requirements of (7.2.3) (with

$$\sigma = \mathrm{id}: \bar{\Gamma} \to \bar{\Gamma} = \mathscr{X}(T_\Gamma))$$

are satisfied due to (5.7.1) and its corollaries. So we obtain

(7.3.2) Proposition. *Let Γ be an arithmetic subgroup of* $\mathrm{GL}(2, K)$ *and*

$$\Lambda = \bar{c}(\bar{\Gamma}) \hookrightarrow \mathrm{Hom}(\bar{\Gamma}, C^*) = T_\Gamma(C)\,.$$

Then Λ is a lattice, and there exists an abelian variety A_Γ over C such that T_Γ/Λ and A_Γ are isomorphic as rigid analytic spaces.

By (5.4.12) and the remarks above, A_Γ is in fact defined over K_∞. Thus it is characterized by the short exact sequence of analytic groups defined over K_∞:

$$(7.3.3) \qquad 1 \to \bar{\Gamma} \to \operatorname{Hom}(\bar{\Gamma}, C^*) \to A_\Gamma(C) \to 0\,, \qquad \alpha \longmapsto c_\alpha\,.$$

(7.4.1) Theorem. *The abelian variety A_Γ defined above is K_∞-isomorphic with the Jacobian J_Γ of the modular curve $\bar{M}_\Gamma$.*

Proof. Let $\omega_0 \in \Omega$ be fixed and $\psi : \Omega \to \operatorname{Hom}(\bar{\Gamma}, C^*) \to A_\Gamma(C^*)$ the map that to $\omega \in \Omega$ assigns the multiplier $c(\omega_0, \omega, \cdot)$ of $\theta(\omega_0, \omega, \cdot)$ followed by the canonical projection. It follows from (5.4.4) that ψ is analytic.

Next consider $\psi(\alpha\omega)$, $\alpha \in \Gamma$. It is represented by the homomorphism $c(\omega_0, \alpha\omega, \cdot)$, whose value on $\beta \in \Gamma$ is $c(\omega_0, \alpha\omega, \beta) = u_\beta(\alpha\omega)/u_\beta(\omega_0)$. Thus

$$\frac{c(\omega_0, \alpha\omega, \beta)}{c(\omega_0, \omega, \beta)} = \frac{u_\beta(\alpha\omega)}{u_\beta(\omega)} = c_\beta(\alpha) = c_\alpha(\beta)\,,$$

i.e., $\psi(\alpha\omega) = \psi(\omega)$, and ψ factors through the projection $p_\Gamma : \Omega \to \Gamma\backslash\Omega = M_\Gamma(C)$. Write $\psi = \psi_\Gamma \circ p_\Gamma$, $\psi_\Gamma : M_\Gamma(C) \to A_\Gamma(C)$. As u_α is holomorphic and non-zero at the cusps, $\psi_\Gamma : \omega \mapsto \text{class} \,(\text{mod}\, \Lambda)$ of $c(\omega_0, \omega, \alpha) = u_\alpha(\omega)/u_\alpha(\omega_0)$ extends to an analytic map $\bar{\psi}_\Gamma : \bar{M}_\Gamma \to A_\Gamma(C)$. By the GAGA theorems (see [27]), $\bar{\psi}_\Gamma$ is a morphism of algebraic varieties.

Let $P_0 = p_\Gamma(\omega_0)$ and $\kappa_\Gamma : \bar{M}_\Gamma \to J_\Gamma$ be the morphism that to each $P \in \bar{M}_\Gamma(C)$ associates the divisor class of $P - P_0$. The universal property of the Jacobian now yields a unique morphism $\varphi_\Gamma : J_\Gamma \to A_\Gamma$ of C-varieties which makes the diagram

$$(7.4.2) \qquad \begin{array}{ccc} \bar{M}_\Gamma & \xrightarrow{\bar{\psi}_\Gamma} & A_\Gamma \\ & {\scriptstyle\kappa_\Gamma}\searrow \quad \nearrow{\scriptstyle\varphi_\Gamma} & \\ & J_\Gamma & \end{array}$$

commutative.

Let $\omega_1, \ldots, \omega_n \in \Omega$, $P_i = p_\Gamma(\omega_i) \in M_\Gamma(C)$, and $[D]$ the class of the divisor $D = P_1 + \cdots + P_n - nP_0$. Suppose that $\varphi_\Gamma([D]) = 0$. This means that there exists $\alpha \in \Gamma$ such that

$$c_\alpha = \prod_{1 \leqq i \leqq n} c(\omega_i, \omega_0, \cdot)\,.$$

The function

$$u_\alpha^{-1}(z) \prod_{1 \leqq i \leqq n} \theta(\omega_i, \omega_0, z)$$

on Ω is invariant under Γ and holomorphic and non-zero at the cusps, thus a meromorphic function on $\bar{M}_\Gamma(C)$, whose divisor equals D. Therefore $[D]=0$, and φ_Γ is injective, since every divisor class of degree zero on $\bar{M}_\Gamma$ can be represented by some D as above.

We are now reduced to showing that φ_Γ is separable. Let $\Omega^1_?$ be the sheaf of 1-differentials on $? = \bar{M}_\Gamma, J_\Gamma, A_\Gamma$, respectively. Then $\kappa^*_\Gamma : H^0(J_\Gamma, \Omega^1) \to H^0(\bar{M}_\Gamma, \Omega^1)$ is bijective ([31], Prop. 5.3). Thus, considering (7.4.2),

$$\begin{aligned} \varphi_\Gamma \text{ is separable } &\Leftrightarrow \varphi^*_\Gamma : H^0(A_\Gamma, \Omega^1) \to H^0(J_\Gamma, \Omega^1) \quad \text{bijective} \\ &\Leftrightarrow \bar{\psi}^*_\Gamma : H^0(A_\Gamma, \Omega^1) \to H^0(\bar{M}_\Gamma, \Omega) \quad \text{bijective}. \end{aligned}$$

Choose $\alpha_1, \ldots, \alpha_g \in \Gamma$ whose classes in $\bar{\Gamma}$ form a basis, and identify $\mathrm{Hom}(\bar{\Gamma}, C^*) \xrightarrow{\cong} (C^*)^g$ by means of the α_i. Then

$$\psi : \Omega \to \mathrm{Hom}(\bar{\Gamma}, C^*) \to A_\Gamma(C) = (C^*)^g / \bar{c}(\bar{\Gamma})$$

is given by $\omega \mapsto$ class of $(c(\omega_0, \omega, \alpha_1), \ldots, c(\omega_0, \omega, \alpha_g))$. Let $w_1, \ldots, w_g$ be the coordinates on $(C^*)^g$. Then $\left\{\frac{dw_i}{w_i} \mid 1 \leqq i \leqq g\right\}$ is a basis of $H^0(A_\Gamma, \Omega^1)$. In view of

$$c(\omega_0, \omega, \alpha) = u_\alpha(\omega) / u_\alpha(\omega_0),$$

we have $\frac{dw_i}{d\omega} = u_i(\omega_0)^{-1} u_i'(\omega)$, having put $u_i := u_{\alpha_i}$ and $u_i' = \frac{du_i}{d\omega}$. This finally yields

$$\psi^*\left(\frac{dw_i}{w_i}\right) = \frac{u_i'(\omega)}{u_i(\omega)} d\omega = \mathrm{dlog}\, u_i \quad (i = 1, \ldots, g),$$

which we view as a holomorphic differential form on $\Gamma \backslash \Omega$ and, in fact, on $\bar{M}_\Gamma = \Gamma \backslash \bar{\Omega}$. Since, by virtue of (6.5.4) and (2.10.2), the dlog u_i form a basis of $H^0(\bar{M}_\Gamma, \Omega^1)$, we have shown: $\bar{\psi}^*_\Gamma$ is bijective, and the proof is finished except for the statement on K_∞-rationality. But all our constructions can be carried out over K_∞ since the $c_\alpha(\beta)$ are contained in K_∞. □

(7.5) We may derive several important corollaries from the theorem and its proof. First, since $J_\Gamma(C) = \mathrm{Hom}(\bar{\Gamma}, C^*) / \bar{c}(\bar{\Gamma})$ is generated by divisor classes of the form $[P - P_0]$, each $\chi \in \mathrm{Hom}(\bar{\Gamma}, C^*)$ may be written as a product $c_\alpha \prod c(\omega_i, \omega_0, \cdot)$ for suitable $\omega_1, \ldots, \omega_n \in \Omega$ and $\alpha \in \Gamma$. Therefore:

(7.5.1) Corollary. *Let $\chi : \bar{\Gamma} \to C^*$ be a homomorphism. There exist*

$$\omega_1, \ldots, \omega_n, \eta_1, \ldots, \eta_n \in \Omega$$

such that χ equals the multiplier c_u of $u(z) = \prod \theta(\omega_i, \eta_i, z)$.

The divisor of a meromorphic theta function u on Ω is Γ-stable and may therefore be regarded as a divisor on $\Gamma \backslash \Omega = M_\Gamma(C)$. Since $p_\Gamma : \Omega \to \Gamma \backslash \Omega$ is ramified in ω with ramification index $\#(\tilde{\Gamma}_\omega)$, the divisor of $\theta(\omega, \eta, z)$ on M_Γ is $P - Q$ if $P = \Gamma\omega \neq \Gamma\eta = Q$ and zero otherwise.

(7.5.2) Corollary. *Let $u \in \Theta_m(\Gamma)$ have divisor D on $\bar{M}_\Gamma$. Then* $\deg D = 0$. *In particular, each u without poles is a holomorphic theta function as defined in* (5.1).

Proof. Upon replacing u by a suitable power, we may assume that the multiplier c_u of u is trivial on $\mathrm{tor}(\Gamma^{\mathrm{ab}})$. (Recall that the non-$p$-part of $\mathrm{tor}(\Gamma^{\mathrm{ab}})$ is finite, [45], p. 130, ex. 2.) Then by (7.5.1), there exists $v(z) = \prod \theta(\omega_i, \eta_i, z)$ such that v/u is a meromorphic function on $\bar{M}_\Gamma$. The divisor of v has degree zero by the above. □

(7.5.3) Corollary. *Let $u \in \Theta_m(\Gamma)$ be such that $c_u = 1$ on* $\mathrm{tor}(\Gamma^{\mathrm{ab}})$. *Then*

$$u(z) = \text{const.} \times \prod \theta(\omega_i, \eta_i, z)$$

for suitable $\omega_i, \eta_i \in \Omega$. Each $u \in \Theta_h(\Gamma)$ with $c_u = 1$ on $\mathrm{tor}(\Gamma^{\mathrm{ab}})$ *has the form* const. $\times\ u_\alpha$, $\alpha \in \Gamma$.

Proof. After dividing $u(z)$ by suitable $\theta(\omega_i, \eta_i, z)$, there results a holomorphic theta function. It therefore suffices to prove the second assertion. Thus let $u \in \Theta_h(\Gamma)$, and let $v(z) = \prod \theta(\omega_i, \eta_i, z)$ have the same multiplier as u (see (7.5.1)). Then v/u is meromorphic on $\bar{M}_\Gamma$, i.e., its divisor $\mathrm{div}(v/u) = \mathrm{div}(v) = \sum P_i - Q_i$ is principal, $P_i = p_\Gamma(\omega_i)$, $Q_i = p_\Gamma(\eta_i)$. But this means that $\prod c(\omega_i, \eta_i, \cdot) = c_\alpha$ for some $\alpha \in \Gamma$. Now u and u_α are both holomorphic with multiplier c_α, thus agree up to a constant. □

8. Application to uniformization of elliptic curves

(8.1) The conjecture of Shimura-Taniyama-Weil[3]). In the classical number theoretic context, the conjecture (e.g. [53]) asserts the existence of a dominant morphism of algebraic curves over $\mathbb{Q}$:

(8.1.1) $$p_E : \bar{X}_0(N) \to E$$

for each elliptic curve over $E/\mathbb{Q}$ with conductor N. Here $\bar{X}_0(N)$ is the compactification of Hecke's modular curve $X_0(N)$ that classifies elliptic curves with a cyclic subgroup of order N. The conductor, which measures the diophantine complexity of E, is defined in [35]. Modulo known facts, the conjecture is equivalent to certain properties (analytic continuation to $\mathbb{C}$, functional equation under $(s, \omega) \mapsto (2 - s, \bar{\omega})$) of the zeta function $\zeta_E(s)$ of E and its twists $L_E(\omega, s)$ by Dirichlet characters ω. If true, it yields bijections between

(8.1.2) (a) the set of isogeny classes of elliptic curves over $\mathbb{Q}$ with conductor N;

(b) the set of one-dimensional factors (up to isogeny) in the $\mathbb{Q}$-decomposition of the new part $J_0^{\mathrm{new}}(N)$ of the Jacobian $J_0(N)$ of $\bar{X}_0(N)$, and

(c) the set of normalized Hecke newforms with rational eigenvalues in $S_2(N)$, the space of cusp forms of weight two for the Hecke congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}(2, \mathbb{Z})$.

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[3]) We refrain from discussing the proper labelling of the conjecture in question.

Among the remarkable consequences of the conjecture, we note that its validity for square-free N implies Fermat's last theorem [42]. It is this special case which has been proven by Wiles [56] and Taylor-Wiles [50].

(8.2) The automorphic representation attached to an elliptic curve. We come back to our usual function field situation. Let E/K be an elliptic curve. The l-adic cohomology modules $H^1(E, \mathbb{Q}_l)$ $(l \neq p)$ give rise to a system of compatible l-adic representations $\pi_E = (\pi_{E,l})$ of the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K. Let $\zeta_E(s)$ be the Hasse-Weil zeta function of E, which is derived from π_E. As is explained in [4], §9, $\zeta_E(s)$ together with its twists $L_E(\chi, s)$ by "grossen-characters" χ (= unitary characters of the idele class group I_K/K^*) has the properties:

(8.2.1) $L_E(\chi, s)$ is a rational function in q^{-s}, and even a polynomial if E is not a twisted constant curve, i.e., if its j-invariant is not contained in $\mathbb{F}_q \hookrightarrow K$.

(8.2.2) The system $L_E(.,.)$ satisfies the functional equation

$$L_E(\chi, s) = \varepsilon_E(\chi, s)\, L_E(\bar{\chi}, 2-s)$$

with some ε-factor $\varepsilon_E(.,.)$ (*loc. cit.*, 9.2–9.5) that depends on π_E, i.e., on E. Recall that L_E and ε_E are products of local factors $L_{E,v}$ and $\varepsilon_{E,v}$, v running through the places of K.

Provided that E/K is not twisted constant, the preceding properties imply ([24], Thm. 11.3) the existence of a uniquely determined cuspidal automorphic representation (ϱ, V_ϱ) in the space $L^2_!(1)$ of (4.2.2) with the same L- and ε-factors as π_E, i.e.,

$$L_{E,v} = L_{\varrho,v} \quad \text{and} \quad \varepsilon_{E,v} = \varepsilon_{\varrho,v} \tag{8.2.3}$$

for all v.

(8.2.4) Remark. The local L- and ε-factors of automorphic representations are defined and calculated in [24], notably, 2.18, 3.5, 3.8, 4.7. The table in [14], Thm. 6.15 might be helpful. Note that there are different normalizations of the Langlands correspondence between Galois representations π and automorphic representations $\varrho = \varrho(\pi)$, which affects the precise form of the functional equation. An exhaustive discussion is given in [3], 3.2. We use what Deligne calls the "Hecke correspondence", since it produces some $\varrho = \varrho_E$ with trivial central character, starting with an elliptic curve E as above.

(8.3) Shimura-Taniyama-Weil: The function field case. Let E/K be an elliptic curve such that

$$E/K_\infty \text{ is a Tate curve}. \tag{8.3.1}$$

Equivalently, the restriction of π_E to a decomposition group $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ is the special representation sp of (4.10.3). Note that $j(E)$ is not integral at ∞ and hence E is not a twisted constant curve. Under the construction described above, the special Galois representation at $v = \infty$ corresponds to the special representation ϱ_{sp} of $G(K_\infty)$. This means that for $\varrho = \varrho_E$

(8.3.2) $$V_\varrho = \bigotimes V_{\varrho,v} \quad \text{with}$$
$$(\varrho_\infty, V_{\varrho,\infty}) \cong (\varrho_{\text{sp}}, V_{\text{sp}}) .$$

As results from (8.3.1), the conductor of E has the form

(8.3.3) $$\operatorname{cond}(E) = \mathfrak{n} \cdot \infty$$

with some finite divisor $\mathfrak{n}$. On the other hand, the conductors of E (i.e., of π_E) and of ϱ_E agree (cf. [3]). Thus by definition (see (4.7.5)) of the space W_{sp}, a newform $\varphi_E \in V_\varrho$ for $\varrho = \varrho_E$ belongs to $W_{\text{sp}}(\mathscr{K}, \mathbb{C})$, where

(8.3.4) $$\mathscr{K} = \mathscr{K}_0(\mathfrak{n} \cdot \infty) = \mathscr{K}_{f,0}(\mathfrak{n}) \times \mathscr{I}_\infty .$$

Now φ_E is an eigenform for all the Hecke operators T_v ($v \nmid \mathfrak{n} \cdot \infty$) with rational eigenvalues λ_v determined by the reduction of E at v. More precisely,

(8.3.5) $$\lambda_v = q_v + 1 - \# E(k(v)) ,$$

where $\# E(k(v))$ is the number of $k(v)$-rational points of the reduction of E at v. Since T_v is defined over $\mathbb{Q}$, we may normalize φ_E such that $\varphi_E \in W_{\text{sp}}(\mathscr{K}, \mathbb{Q})$.

Let $M = M_0^2(\mathfrak{n}) \times K$ be the Hecke moduli scheme of level $\mathfrak{n}$, base extended to K. As an algebraic curve, it is defined over the Hilbert class field H of (K, A). (See (2.5.2); the field of definition of M corresponds by class field theory to the image of

$$\det : \mathscr{K}_{f,0}(\mathfrak{n}) \times \mathscr{I}_\infty \to I_K$$

and therefore agrees with the field of definition H of $M^2(1) \times K$.) We put $J_0(\mathfrak{n})$ for the Jacobian of $\bar{M}$, where the smooth projective model $\bar{M}$ of M is regarded as a (perhaps absolutely reducible) curve over K. Note that we have canonical bijections

(8.3.6) $$S \xrightarrow[\det]{\cong} \operatorname{Pic} A \xrightarrow{\cong} \operatorname{Gal}(H/K) \xrightarrow{\cong} \left\{ \begin{matrix} K\text{-embeddings } \sigma \\ \text{of } H \text{ into } K_\infty \end{matrix} \right\} ,$$

where S is a system of representatives for $G(K) \backslash G(\mathfrak{A}_f) / \mathscr{K}_{f,0}(\mathfrak{n})$ as in (4.5.2). Thus

(8.3.7) $$J_0(\mathfrak{n}) \times K_\infty = \prod_{\sigma : H \hookrightarrow K_\infty} \operatorname{Jac}(\bar{M}) \underset{H,\sigma}{\times} K_\infty$$

and

$$J_0(\mathfrak{n})(C) = \prod_{\underline{x} \in S} J_{\Gamma_{\underline{x}}}(C) ,$$

where $\Gamma_{\underline{x}}$ is given by (4.5.3) and $J_{\Gamma_{\underline{x}}}$ is the Jacobian of the modular curve $\bar{M}_{\Gamma_{\underline{x}}}$ as discussed in the last section. Combining the above with (4.7.6) and (4.13.1) yields

(8.3.8) $$\bigoplus_{\underline{x} \in S} H^1(J_{\Gamma_{\underline{x}}}, \mathbb{Q}_l) = H^1(J_0(\mathfrak{n}) \times C, \mathbb{Q}_l) = H^1(\bar{M} \times C, \mathbb{Q}_l)$$
$$\xrightarrow{\cong} W_{\text{sp}}(\mathscr{K}, \mathbb{Q}_l) \otimes \text{sp}_l \xrightarrow{\cong} \bigoplus_{\underline{x} \in S} \underline{H}_!(\mathscr{T}, \mathbb{Q}_l)^{\Gamma_{\underline{x}}} \otimes \text{sp}_l$$

as modules under both $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and the Hecke algebra $\mathscr{H}$ generated over $\mathbb{Z}$ by the operators T_v ($v \nmid \mathfrak{n} \cdot \infty$). (A priori, only $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ acts on the two right hand modules; by Drinfeld's reciprocity law, the action may be uniquely extended to an action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in the same space such that (8.3.8) holds, see (4.13.2).) Note that neither $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ nor $\mathscr{H}$ respect the decomposition over S given in (8.3.8). While the Galois action permutes components according to (8.3.7) (in fact, $H^1(\bar{M} \times C, \mathbb{Q}_l)$ is the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module induced from a certain $\mathrm{Gal}(K^{\mathrm{sep}}/H)$-module), the Hecke operator T_v maps the component of $\underline{x} \in S$ to the component of $\underline{x}' = \underline{x}\tau_v$ (see (4.9)). Since the class of $\underline{x} \in S$ only depends on the class of $\det \underline{x}$ in $K^*\backslash I_f/\det \mathscr{K}_{0,f}(\mathfrak{n}) = K^*\backslash I_f/\mathcal{O}_f^* \xrightarrow{\cong} \mathrm{Pic}\, A$, we see:

(8.3.9) Let the unramified place v correspond to the prime ideal $\mathfrak{p}$ of A. If $\mathfrak{p}$ is principal, the Hecke operator T_v respects the decompositions in (8.3.7) and (8.3.8).

This holds more generally for Hecke operators $T_\mathfrak{a}$, where $\mathfrak{a}$ is not necessarily prime but is a principal ideal of A coprime with $\mathfrak{n}$. Such operators $T_\mathfrak{a}$ will be called *admissible*.

(8.3.10) Now let $\mathrm{pr}_E \in \mathrm{End}(W_{\mathrm{sp}}(\mathscr{K}, \mathbb{Q}))$ be the projection onto $\mathbb{Q}\varphi_E$. Thus pr_E lies in the $\mathbb{Q}$-subalgebra generated by the Hecke operators and therefore corresponds to a projector also labelled by $\mathrm{pr}_E \in \mathrm{End}(J_0(\mathfrak{n})) \otimes \mathbb{Q}$. Its image is an abelian sub-variety E' of $J_0(\mathfrak{n})$ that (a) has dimension one and (b) is defined over K. Here (a) holds since

$$H^1(E', \mathbb{Q}_l) \xrightarrow{\cong} \mathbb{Q}_l \varphi_E \otimes \mathrm{sp}_l$$

as a $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$-module, and (b) since pr_E is defined over K. Since the Galois representations of E and E' coincide as sub-$\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules of $H^1(\bar{M} \times C, \mathbb{Q}_l)$, E and E' are isogeneous. Thus we have found E as a factor (up to isogeny) of $J_0(\mathfrak{n})$. Applying the machinery of automorphic forms, the above construction yields bijections between

(8.3.11) (a) the set of isogeny classes of elliptic curves over K with conductor $\mathfrak{n} \cdot \infty$, and which become Tate curves over K_∞;

(b) the set of one-dimensional factors (up to isogeny) in the K-decomposition of the new part $J_0^{\mathrm{new}}(\mathfrak{n})$ of $J_0(\mathfrak{n})$; and

(c) the set of normalized Hecke eigenforms with rational eigenvalues in

$$W_{\mathrm{sp}}^{\mathrm{new}}(\mathscr{K}_{f,0}(\mathfrak{n}) \times \mathscr{I}_\infty).$$

Here we have omitted the obvious definitions of the "new parts" $J_0^{\mathrm{new}}(\mathfrak{n})$ and $W_{\mathrm{sp}}^{\mathrm{new}}$, respectively, in (b) and (c). So the established *fact* (8.3.11) is perfectly analogous with the *conjectural* correspondence (8.1.2).

(8.3.12) Remark. The procedure sketched allows one to find up to isogeny all the elliptic curves E over K with (8.3.1) as factors of $J_0(\mathfrak{n})$. The "Tate condition" on E/K_∞ presents no serious restriction: If E is not a twisted constant curve (those E are easily classified), its j-invariant has a pole at some place v of K. Thus E/K_v will become a Tate curve after a finite extension, and is then covered by our theory, since we are free to choose v as our place ∞.

(8.4) Let E/K be as above, and let $\mathrm{pr}_E : J_0(\mathfrak{n}) \to E$ be a non-zero K-morphism. According to (8.3.7), its base extension $\mathrm{pr}_E \times K_\infty$ splits into components

$$\mathrm{pr}_{E,\sigma} : \mathrm{Jac}(\bar{M}) \underset{H,\sigma}{\times} K_\infty \to E \times K_\infty ,$$

which are related by

(8.4.1) $$\mathrm{pr}_{E,\sigma} \circ \tau_* = \mathrm{pr}_{E,\sigma\tau} \quad (\sigma, \tau \in \mathrm{Gal}(H/K)) .$$

Here

$$\tau_* : \mathrm{Jac}(\bar{M}) \underset{H,\sigma\tau}{\times} K_\infty \to \mathrm{Jac}(\bar{M}) \underset{H,\sigma}{\times} K_\infty$$

is the isomorphism derived from τ. This finally yields for every $\underline{x} \in S$ a map

$$\mathrm{pr}_{E,\underline{x}} : J_{\Gamma_{\underline{x}}}(C) \to E(C)$$

of analytic spaces over C (or even over K_∞). In the next section, we will describe $\mathrm{pr}_{E,\underline{x}}$ and the associated *Weil uniformization* $p_{E,\underline{x}} : \bar{M}_{\Gamma_{\underline{x}}}(C) \to E(C)$.

More precisely, we will construct the *strong Weil curve* E in the given isogeny class, which by definition is the curve E maximal in its class with respect to diagrams

$$J_0(\mathfrak{n}) \to E \to E' .$$

The existence and uniqueness of E are easily seen; e.g., E can be characterized as the unique abelian subvariety of $J_0(\mathfrak{n})$ in the given isogeny class. It results from (8.4.1) that this property may be checked locally, i.e., for $\underline{x} \in S$, E is maximal w.r.t. diagrams

(8.4.2) $$\begin{array}{ccc} J_{\Gamma_{\underline{x}}} & \xrightarrow{\mathrm{pr}_{E,\underline{x}}} & E \times C \\ & {\scriptstyle \mathrm{pr}_{E',\underline{x}}} \searrow \quad \swarrow & \\ & E' \times C . & \end{array}$$

As usual, we did not distinguish between algebraic and analytic morphisms, labelling both by the same symbol.

(8.5) Let $\varphi \in W_{\mathrm{sp}}(\mathscr{K}, \mathbb{Q}) \xrightarrow{\cong} \bigoplus_{\underline{x} \in S} \underline{H}_!(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}$ be an eigenform for the Hecke algebrawith rational eigenvalues, e.g. $\varphi = \varphi_E$ for some E/K. If $\varphi = \sum_{\underline{x} \in S} \varphi_{\underline{x}}$ with $\varphi_{\underline{x}} \in \underline{H}_!(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}$ then:

(8.5.1) $\varphi_{\underline{x}}$ is an eigenform for the admissible Hecke operators, as follows from (8.3.9). We further note that projection to the $\underline{x}$-component yields a decomposition

(8.5.2) $$W_{\mathrm{sp}}^{\mathrm{new}}(\mathscr{X}, \mathbb{Q}) \xrightarrow{\cong} \bigoplus_{\underline{x} \in S} \underline{H}_!^{\mathrm{new}}(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}.$$

Now it is well known that the Hecke algebra $\mathscr{H} \otimes \mathbb{Q}$ restricted to $W_{\mathrm{sp}}^{\mathrm{new}}(\mathscr{X}, \mathbb{Q})$ is a semi-simple and maximal commutative subalgebra of $\mathrm{End}(W_{\mathrm{sp}}^{\mathrm{new}}(\mathscr{X}, \mathbb{Q}))$. We put $\mathscr{H}_{\underline{x}}$ for the $\mathbb{Z}$-algebra generated by the admissible $T_{\mathfrak{a}}$ restricted to $\underline{H}_!(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}$. Then $\underline{H}_!^{\mathrm{new}}(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}$ is $\mathscr{H}_{\underline{x}}$-stable, and $\mathscr{H}_{\underline{x}} \otimes \mathbb{Q}$ restricted to $\underline{H}_!^{\mathrm{new}}(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}$ is semi-simple and maximal commutative. This implies

(8.5.3) $\underline{H}_!^{\mathrm{new}}(\mathscr{T}, \mathbb{Q})^{\Gamma_{\underline{x}}}$ is a cyclic $\mathscr{H}_{\underline{x}} \otimes \mathbb{Q}$-module.

9. Construction of the Weil uniformization

(9.1) We keep the set-up of the last section. Since up to (9.6), we will be working only on one fixed component that corresponds to $\underline{x} \in S$, we put Γ for $\Gamma_{\underline{x}}$, φ for $\varphi_{\underline{x}}$, $J = J_{\Gamma_{\underline{x}}}$, and $\mathscr{H}$ for $\mathscr{H}_{\underline{x}}$ (see (8.3) and (8.5)). We assume that $\varphi \in \underline{H}_!(\mathscr{T}, \mathbb{Q})^{\Gamma}$ satisfies:

(9.1.1) $\varphi \in \underline{H}_!^{\mathrm{new}}(\mathscr{T}, \mathbb{Q})^{\Gamma}$;

(9.1.2) φ is an eigenform for $\mathscr{H}$ with rational (hence integral) eigenvalues; and

(9.1.3) φ is *primitive*, i.e., is normalized such that $\varphi \in j(\bar{\Gamma})$ but $\varphi \notin nj(\bar{\Gamma})$ for $n > 1$.

The last condition determines φ up to sign.

Our aim is to construct the strong Weil curve attached to φ as a Tate curve over K_∞.

(9.2) Let $\mathfrak{a} = (a)$ be a principal ideal coprime with the conductor $\mathfrak{n}$ of Γ, and put $\tau = \tau_{\mathfrak{a}}$ for the element $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ of $G(K)$. Let further

(9.2.1) $$\Delta^+ = \Delta_{\mathfrak{a}}^+ = \Gamma \cap \tau^{-1}\Gamma\tau, \qquad \Delta^- = \Delta_{\mathfrak{a}}^- = \Gamma \cap \tau\Gamma\tau^{-1},$$

which are congruence subgroups of Γ. If e.g. $\Gamma = \mathrm{GL}(2, A)$ then

$$\Delta^+ = \Gamma_0(\mathfrak{a}) = \left\{\begin{pmatrix} * & * \\ c & * \end{pmatrix} \mid c \equiv 0 \pmod{\mathfrak{a}}\right\} \quad \text{and} \quad \Delta^- = \left\{\begin{pmatrix} * & b \\ * & * \end{pmatrix} \mid b \equiv 0 \pmod{\mathfrak{a}}\right\}$$

are the standard Hecke congruence subgroups. Note that Δ^+ and Δ^- contain $\Gamma \cap Z(K)$, so there is no need to distinguish between e.g. $\tilde{\Delta}^- \backslash \tilde{\Gamma}$ and $\Delta^- \backslash \Gamma$. If $\mathfrak{a}$ is prime, a simple calculation (based on (4.5.4), and left to the reader) yields the following expression for $T_{\mathfrak{a}}$ restricted to elements φ of $\underline{H}_!(\mathscr{T}, \mathbb{Q})^{\Gamma}$:

(9.2.2) $$(T_{\mathfrak{a}}\varphi)(g) = \sum_{\alpha_i \in \Delta^- \backslash \Gamma} \varphi(\tau^{-1}\alpha_i g).$$

It is analogous to the adelic description (4.9.2), but involves only data at ∞. Recall that φ is a Γ-invariant function on $Y(\mathcal{T}) = G(K_\infty)/\mathcal{I}_\infty \cdot Z(K_\infty)$, so in particular is a function on $G(K_\infty)$. The formula remains true for admissible $T_{\mathfrak{a}}$ with $\mathfrak{a}$ not necessarily prime, as can be verified by expressing $T_{\mathfrak{a}}$ as a polynomial in $T_{\mathfrak{p}}$ with $\mathfrak{p} \mid \mathfrak{a}$ prime.

(9.3) The Hecke action on $\bar{\Gamma}$. We next define for $\mathfrak{a}$ as above (principal, coprime with $\mathfrak{n}$) a Hecke operator $T_{\mathfrak{a}}$ on $\bar{\Gamma}$ and show that it is compatible with $j : \bar{\Gamma} \to \underline{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$. Let $T = T_{\mathfrak{a}} : \bar{\Gamma} \to \bar{\Gamma}$ be the composition $\bar{I} \circ \kappa \circ \bar{V}$ of the transfer map $\bar{V} : \bar{\Gamma} \to \bar{\Delta}^-$, the map $\kappa : \bar{\Delta}^- \xrightarrow{\cong} \bar{\Delta}^+$ induced from conjugation $\gamma \mapsto \tau^{-1}\gamma\tau : \Delta^- \xrightarrow{\cong} \Delta^+$, and the canonical map $\bar{I} : \bar{\Delta}^+ \to \bar{\Gamma}$ induced from the inclusion. Explicitly,

(9.3.1) $$T_{\mathfrak{a}}(\alpha) = \tau^{-1} \prod_{\alpha_i \in \Delta^- \backslash \Gamma} \alpha_i \alpha \alpha_{\sigma(i)}^{-1} \tau,$$

where α and the right hand side are to be read as classes in $\bar{\Gamma}$, α_i runs through a system of representatives of $\Delta^- \backslash \Gamma$, and σ is the permutation of $\{i\}$ such that $\alpha_i \alpha \alpha_{\sigma(i)}^{-1} \in \Delta^-$.

(9.3.2) Lemma. *Let $T = T_{\mathfrak{a}}$, $\alpha \in \Gamma$, and $\varphi_\alpha = j(\alpha) \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. Then $\varphi_{T(\alpha)} = T(\varphi_\alpha)$.*

Proof. In view of $T = \bar{I} \circ \kappa \circ \bar{V}$, it suffices to trace the effect of $\bar{I}$, κ, and $\bar{V}$ on $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. For $\bar{I}$ and $\bar{V}$, it is given by (6.2.4) and (6.2.5), while κ corresponds to

$$\varphi \mapsto \varphi_\tau : \underline{H}_!(\mathcal{T}, \mathbb{Z})^{\Delta^-} \xrightarrow{\cong} \underline{H}_!(\mathcal{T}, \mathbb{Z})^{\Delta^+}$$

with $\varphi_\tau(g) = \varphi(\tau g)$. Together,

$$\varphi_{T(\alpha)}(g) = \sum_{\beta_i \in \Delta^+ \backslash \Gamma} \varphi_\alpha(\tau \beta_i g).$$

Since φ is $Z(K_\infty)$-invariant, this sum is easily verified to agree with the expression for $(T\varphi_\alpha)(g)$ given by (9.2.2). □

(9.3.3) Proposition. *The Hecke operator $T = T_{\mathfrak{a}}$ on $\bar{\Gamma}$ is formally self-adjoint with respect to the bilinear map $(\alpha, \beta) \mapsto c_\alpha(\beta)$.*

Proof. Consider the exact sequence of Γ-modules (thus of Δ^- and Δ^+-modules)

$$0 \to C^* \to \mathcal{O}_\Omega(\Omega)^* \to \underline{H}(\mathcal{T}, \mathbb{Z}) \to 0$$

given by (1.7.2). The long exact cohomology sequences with respect to the actions of Γ, Δ^-, Δ^+ are related by the following maps:

restriction:	$H^*(\Gamma, \cdot) \to H^*(\Delta^+, \cdot)$,
conjugation by τ:	$H^*(\Delta^+, \cdot) \to H^*(\Delta^-, \cdot)$,
corestriction:	$H^*(\Delta^-, \cdot) \to H^*(\Gamma, \cdot)$.

Their composition on the H^0-level yields

$$\begin{array}{ccc} T: H^0(\Gamma, \underline{H}(\mathcal{T}, \mathbb{Z})) & \to & H^0(\Gamma, \underline{H}(\mathcal{T}, \mathbb{Z})) \\ \| & & \| \\ \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma & \to & H(\mathcal{T}, \mathbb{Z})^\Gamma , \end{array}$$

as follows from (9.2.2). On the H^1-level, we obtain

$$\begin{array}{ccc} (T^{\mathrm{ab}})^* : H^1(\Gamma, C^*) & \to & H^1(\Gamma, C^*) \\ \| & & \| \\ \mathrm{Hom}(\Gamma^{\mathrm{ab}}, C^*) & \to & \mathrm{Hom}(\Gamma^{\mathrm{ab}}, C^*) , \\ & c \mapsto c \circ T^{\mathrm{ab}} , & \end{array}$$

where $T^{\mathrm{ab}}: \Gamma^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$ is given by the same formula (9.3.1) as $T: \bar{\Gamma} \to \bar{\Gamma}$, and which induces T on $\bar{\Gamma}$. Combined with the boundary operator d, we have a commutative diagram

$$(9.3.4) \qquad \begin{array}{ccc} \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma & \xrightarrow{d} & \mathrm{Hom}(\Gamma^{\mathrm{ab}}, C^*) \\ T\downarrow & & \downarrow (T^{\mathrm{ab}})^* \\ \underline{H}(\mathcal{T}, \mathbb{Z})^\Gamma & \xrightarrow{d} & \mathrm{Hom}(\Gamma^{\mathrm{ab}}, C^*) . \end{array}$$

Let $\alpha \in \Gamma$. The homomorphism $c_\alpha: \Gamma^{\mathrm{ab}} \to C^*$ factors through $\bar{\Gamma}$ and is given as $d(\varphi_\alpha)$. Therefore, by (9.3.2), $c_{T(\alpha)} = d(\varphi_{T(\alpha)}) = d(T(\varphi_\alpha)) = (T^{\mathrm{ab}})^*(d(\varphi_\alpha)) = (T^{\mathrm{ab}})^*(c_\alpha) = c_\alpha \circ T$.

In other words, for each $\beta \in \Gamma$, $c_{T(\alpha)}(\beta) = c_\alpha(T(\beta))$, as was to be shown. □

(9.4) Next, consider the exact sequence

$$1 \longrightarrow \bar{\Gamma} \xrightarrow{\bar{c}} \mathrm{Hom}(\bar{\Gamma}, C^*) \longrightarrow J_\Gamma(C) \longrightarrow 0$$

of (7.3.3). Each of its three terms is equipped with an action of $\mathcal{H}$, given by (9.3.1) for $\bar{\Gamma}, f \mapsto f \circ T$ for $f \in \mathrm{Hom}(\bar{\Gamma}, C^*)$, $T \in \mathcal{H}$, and the extension of the canonical $\mathcal{H}$-action on $\bar{M}_\Gamma$ to J_Γ, respectively. By the proposition just proved, $\bar{c}$ is compatible with this $\mathcal{H}$-action. The same holds for the projection $\mathrm{Hom}(\bar{\Gamma}, C^*) \to J_\Gamma(C)$, due to the description of J_Γ given in (7.4.1) and its proof. We leave the details to the reader.

(9.5) The strong Weil curve.

(9.5.1) Proposition. *Let $\varphi \in j(\bar{\Gamma}) \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$ be as in* (9.1), *regarded as the class of some element, also labelled by φ, of Γ. Put Λ for the subgroup $\{c_\alpha(\varphi) \mid \alpha \in \Gamma\}$ of C^*. Then there exists $t \in C^*$ such that $|t| < 1$ and $t^{\mathbb{Z}}$ has finite index in Λ.*

Proof. Consider j as an embedding, and let $\bar{\Gamma}^{\mathrm{new}} := \bar{\Gamma} \cap \underline{H}_!^{\mathrm{new}}(\mathcal{T}, \mathbb{Q})^\Gamma$ and

$$\bar{\Gamma}^{\mathrm{old}} := \bar{\Gamma} \cap \underline{H}_!^{\mathrm{old}}(\mathcal{T}, \mathbb{Q})^\Gamma ,$$

respectively. Then $\bar{\Gamma}^{\mathrm{new}} \oplus \bar{\Gamma}^{\mathrm{old}}$ has finite index in $\bar{\Gamma}$. Thus it suffices to show the statement for the subgroup of Λ that corresponds to $\bar{\Gamma}^{\mathrm{new}} \oplus \bar{\Gamma}^{\mathrm{old}}$. We first show that the contribution of $\bar{\Gamma}^{\mathrm{old}}$ to Λ is finite. Let $\alpha \in \Gamma$ be such that its class in $\bar{\Gamma}$ belongs to $\bar{\Gamma}^{\mathrm{old}}$. Now $\bar{\Gamma}^{\mathrm{old}}$ is that part of $\bar{\Gamma}$ which comes from strictly larger arithmetic groups $\Gamma_1 \supsetneqq \Gamma$ defined by weaker congruence conditions (see (8.3.6) and (4.5.3)). After possibly replacing α by some power α^n, we may therefore assume that $\alpha = \bar{V}(\alpha_1)$, where $\bar{V}: \bar{\Gamma}_1 \to \bar{\Gamma}$ is the transfer. Then by (6.2.1), $c_\alpha(\varphi) = c_{\Gamma,\alpha}(\varphi) = c_{\Gamma_1,\alpha_1}(\varphi)$. Since $\varphi \in \bar{\Gamma}$ is orthogonal to $\bar{\Gamma}^{\mathrm{old}}$, it is contained in

$$\ker \bar{I}: \bar{\Gamma} \to \bar{\Gamma}_1,$$

which implies $c_{\Gamma_1,\alpha_1}(\varphi) = 1$.

We have seen that $\{c_\alpha(\varphi) \mid \alpha \in \bar{\Gamma}^{\mathrm{new}}\}$ has finite index in $\Lambda = \{c_\alpha(\varphi) \mid \alpha \in \bar{\Gamma}\}$. Now since $\underline{H}_!^{\mathrm{new}}(\mathscr{T}, \mathbb{Q})^\Gamma$ is a cyclic $\mathscr{H} \otimes \mathbb{Q}$-module, the $\mathbb{Z}$-lattice $\bar{\Gamma}^{\mathrm{new}}$ contains a sublattice Ξ of full rank (= of finite index) which is cyclic under $\mathscr{H}$, i.e., $\Xi = \mathscr{H}\beta \xrightarrow[\text{finite}]{} \bar{\Gamma}^{\mathrm{new}}$ for some $\beta \in \bar{\Gamma}^{\mathrm{new}}$. Let $\alpha \in \Xi$ be written additively

$$\alpha \in \sum n_{\mathfrak{a}} T_{\mathfrak{a}}(\beta), \quad n_{\mathfrak{a}} \in \mathbb{Z},\ T_{\mathfrak{a}} \text{ some admissible Hecke operator}.$$

Then by (9.3.3),

$$c_\alpha(\varphi) = c_\beta\left(\sum n_{\mathfrak{a}} T_{\mathfrak{a}}(\varphi)\right) = c_\beta(n\varphi) = c_\beta(\varphi)^n \quad (\text{some } n \in \mathbb{Z}),$$

since φ is an eigenform for the $T_{\mathfrak{a}}$ with integral eigenvalues. Hence $\{c_\alpha(\varphi) \mid \alpha \in \Xi\} = t^{\mathbb{Z}}$ with $t = c_\beta(\varphi)$. By virtue of $|c_\varphi(\varphi)| > 1$, $|t| \neq 1$, so we may choose $|t| < 1$, which establishes the result. □

Recall that Λ is actually a subgroup of K_∞^*. Let $w := q_\infty - 1$ be the order of the group of roots of unity in K_∞. Taking $|t|$ maximal yields the

(9.5.2) Corollary (see remarks (9.8)). *Let φ be as in* (9.1) *and $\Lambda \subset K_\infty^*$ as in the proposition. There exists a divisor d of w and $t \in K_\infty^*$ with $|t| < 1$ such that $\Lambda = \mu_d \times t^{\mathbb{Z}}$ (μ_d = d-th roots of unity in K_∞^*).*

Note that groups of this type give rise to Tate curves. Viz,

(9.5.3)
$$C^*/\Lambda \xrightarrow{\cong} C^*/t^{d\mathbb{Z}}, \qquad z \longmapsto z^d$$

which says that Λ uniformizes the Tate curve $\mathrm{Tate}(t^d)$ over K_∞ with period t^d. We thus have constructed, starting with an eigenform $\varphi \in \underline{H}_!(\mathscr{T}, \mathbb{Q})^\Gamma$ subject to the conditions (9.1), an elliptic curve over K_∞

(9.5.4)
$$E_\varphi \xrightarrow{\cong} \mathrm{Tate}(t^d)$$

together with a map $\mathrm{pr}_\varphi : J_\Gamma \to E_\varphi$. Here $t \in K_\infty^*$ and the divisor d of $w = q_\infty - 1$ depend on φ as described by (9.5.1) and (9.5.2). The projection pr_φ is given on C-valued points by the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{\Gamma} & \longrightarrow & \mathrm{Hom}(\bar{\Gamma}, C^*) & \longrightarrow & J_\Gamma(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow^{\mathrm{ev}} & & \downarrow^{\mathrm{pr}_\varphi(C)} & & \\ 1 & \longrightarrow & \Lambda & \longrightarrow & C^* & \longrightarrow & E_\varphi(C) & \longrightarrow & 0\,, \end{array} \tag{9.5.5}$$

where the first line is (7.3.3) (having identified A_Γ with J_Γ as in (7.4.1)), ev is the map "evaluation on φ": $f \mapsto f(\varphi)$, and $E_\varphi(C) = C^*/\Lambda$. Due to (9.4), we can state:

(9.5.6) Each admissible Hecke operator T induces an endomorphism on E_φ, which agrees with multiplication by λ on E_φ, where λ = eigenvalue of T on φ.

Hence the isogeny class of E_φ is the one that corresponds to the subspace $\mathbb{Q}_l\varphi \otimes \mathrm{sp}_l$ of $\underline{H}_!(\mathcal{T}, \mathbb{Q}_l)^\Gamma \otimes \mathrm{sp}_l = H^1(J_\Gamma, \mathbb{Q}_l)$. Next, recall that $\bar{M}_\Gamma$ is defined over the Hilbert class field H and the admissible Hecke operators restricted to $J_\Gamma = \mathrm{Jac}(\bar{M}_\Gamma)$ are H-rational endomorphisms. By its very definition, $(E_\varphi, \mathrm{pr}_\varphi)$ is maximal in the sense of (8.4.2), so is H-rational by standard arguments. (See e.g. [46], Thm. 1; E_φ = quotient of J_Γ by the connected H-rational abelian subvariety that corresponds to the orthogonal of pr_φ in $\mathcal{H} \hookrightarrow \mathrm{End}(J_\Gamma)$.)

Finally, choose $\omega_0 \in \Omega$ with class P_0 in $\Gamma \backslash \Omega = M_\Gamma(C)$, let $\kappa_\Gamma : \bar{M}_\Gamma \hookrightarrow J_\Gamma$ be the embedding as in the proof of (7.4.1), and put $p_\varphi : \bar{M}_\Gamma \to E_\varphi$ for the composite map $\mathrm{pr}_\varphi \circ \kappa_\Gamma$. From (9.5.5), (7.4.2) and (5.4.4) we get the following description:

$$\begin{array}{ccc} p_\varphi : \bar{M}_\Gamma(C) & \longrightarrow & E_\varphi(C) \\ \| & & \| \\ (\Gamma \bar{\backslash} \Omega) & \longrightarrow & C^*/\Lambda\,, \\ & & \\ z & \longmapsto & u_\varphi(z)/u_\varphi(\omega_0)\,. \end{array} \tag{9.5.7}$$

Here, as usual, u_φ is the theta function associated to a representative of φ in Γ. If dw/w is the invariant differential form on E_φ that comes from a coordinate w on C^*,

$$p_\varphi^*\left(\frac{dw}{w}\right) = \frac{u_\varphi'(z)}{u_\varphi(z)}\, dz\,, \tag{9.5.8}$$

which "is" (compare (6.5.1)) the reduction of $\varphi \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ modulo $p = \mathrm{char}\, \mathbb{F}_q$.

(9.6) Now remember that $\Gamma = \Gamma_{\underline{x}}$, $\varphi = \varphi_{\underline{x}}, \ldots$ etc. with $\underline{x} \in S$ as in (8.3.6)–(8.3.8). We can perform the above construction on all the components $J_{\Gamma_{\underline{x}}}$ for all

$$\underline{x} \in S = G(K)\backslash G(\mathfrak{A}_f)/\mathcal{K}_{0,f}(\mathfrak{n})\,.$$

If now $\varphi = \sum_{\underline{x} \in S} \varphi_{\underline{x}}$ is a new eigenform for the *full* Hecke algebra $\mathcal{H}$ as in (8.5), it results from (8.3.8) that the individual $\varphi_{\underline{x}}$ satisfy simultaneously the normalization condition (9.1.3). Let E_φ/K be the global strong Weil curve attached to φ, which is characterized by (8.4.2). Then $E_\varphi \times H$ must be isomorphic with $E_{\varphi_{\underline{x}}}/H$ for $\underline{x} \in S$; in particular, all the $E_{\varphi_{\underline{x}}}$ are H-isomorphic and come from the curve E_φ defined over K.

(9.6.1) Summary. Starting with an eigenform φ as in (9.1), for example the form φ_E attached to an elliptic curve E/K subject to (8.3.1) and (8.3.3), diagrams (9.5.5) and (9.5.7) provide an analytic description of $E_\varphi \times C$, where E_φ/K is the strong Weil curve that corresponds to φ.

(9.7) Examples. We present some examples that illustrate how properties of E_φ may be read off from the newform φ. We assume that $(K, A, \infty) = (\mathbb{F}_q(T), \mathbb{F}_q[T], \infty)$ as in (0.4). Then since A is a principal domain, the Hilbert class field H coincides with K, and S collapses to one point. We are therefore reduced to considering Hecke congruence subgroups $\Gamma = \Gamma_0(\mathfrak{n})$ of $\mathrm{GL}(2, A)$, where $\mathfrak{n}$ is an ideal of $A = \mathbb{F}_q[T]$. In this case, the map

$$j : \bar{\Gamma} \to \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$$

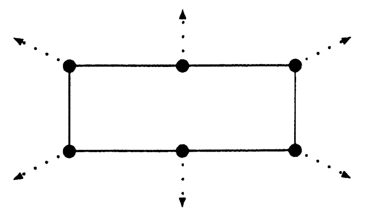
is known to be bijective [34].

Furthermore, we let $q = 2$, such that $w = q_\infty - 1 = q - 1 = 1$, and we need not worry about the divisor d of w that appears in (9.5.2). Another consequence of these requirements is the following:

(9.7.1) Let $E = \mathrm{Tate}(t)$ and $E' = \mathrm{Tate}(t')$ be isogeneous Tate curves over K_∞ with $v_\infty(t) = v_\infty(t')$. Then E and E' are isomorphic.

Now put $\mathfrak{n} = (n)$, $n \in A = \mathbb{F}_2[T]$, $\Gamma = \Gamma_0(\mathfrak{n}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, A) \,|\, n | c\right\}$. Up to coordinate changes in T, there are precisely two different n such that $\bar{M}_\Gamma$ has genus one, given by $n = T^2(T-1)$ and $n = T^3$. For this and the following explicit numerical results, we refer to [9], notably Tabelle 10.2.

(9.7.2) $n = T^2(T-1)$ (Typ (4) of *loc. cit.*, note that Abb. 3 and Abb. 5 unfortunately have been interchanged!) The graph $\Gamma \backslash \mathcal{T}$ is

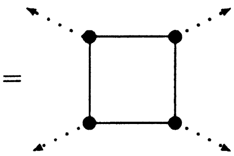


where from each vertex emanates a cusp (= half-line) indicated by $\cdots \blacktriangleright$. The scalar product $(.,.)$ on $\bar{\Gamma} = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is the obvious one (volume of each non-oriented edge = 1), which leads to $(\varphi, \varphi) = 6$ for a basis vector φ. By (9.5.4) we thus have $\bar{M}_\Gamma = E_\varphi \xrightarrow{\cong} \mathrm{Tate}(t)$, where $v_\infty(t) = 6$. Now let E/K be the elliptic curve with equation

$$Y^2 + TXY + TY = X^3,$$

j-invariant $T^8/(T-1)^2$ and conductor $T^2(T-1) \cdot \infty$. It has split multiplicative reduction at ∞ and is therefore isomorphic with $\mathrm{Tate}(t)$, where $v_\infty(t) = -v_\infty(j) = 6$. By (9.7.1) it is K_∞-isomorphic and hence also K-isomorphic with $\bar{M}_\Gamma$.

(9.7.3) $n = T^3$ (Typ (5) of *loc. cit.*).

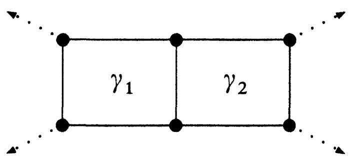
$\Gamma \backslash \mathscr{T} =$  with a cusp emanating from each vertex

Writing $\underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma = \mathbb{Z}\varphi$, we have $(\varphi, \varphi) = 4$, thus $\bar{M}_\Gamma \xrightarrow{\cong} \mathrm{Tate}(t)$ with $v_\infty(t) = 4$. An equation is given by

$$Y^2 + TXY = X^3 + T^2$$

with j-invariant T^4.

(9.7.4) $n = T(T^2 + T + 1)$ (Typ (2) of *loc. cit.*). Here the genus is two, and

$\Gamma \backslash \mathscr{T} =$ 

again with the obvious scalar product on harmonic cochains.

Let γ_1 and γ_2 be the two cycles of length 4, oriented counter-clockwise, and w_1, w_2 the involutions (on $\bar{M}_\Gamma$, thus on $\underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$) attached to the prime divisors

$$\mathfrak{p}_1 = (T) \quad \text{and} \quad \mathfrak{p}_2 = (T^2 + T + 1)$$

of $\mathfrak{n}$. We regard γ_1 and γ_2 as elements of $\underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma$. Then the Hecke eigenforms $\varphi_1 := \gamma_1 + \gamma_2$, $\varphi_2 := -\gamma_1 + \gamma_2$ satisfy

$$w_1(\varphi_1) = \varphi_1 = -w_2(\varphi_1)\,,$$
$$w_2(\varphi_2) = \varphi_2 = -w_1(\varphi_2)\,.$$

Hence $J_\Gamma = \mathrm{Jac}(\bar{M}_\Gamma)$ is isogeneous to $E_1 \times E_2$, where E_i is the elliptic curve (in fact, the strong Weil curve) $\bar{M}_\Gamma / w_i$. We have $(\varphi_1, \varphi_1) = 6$, $(\varphi_2, \varphi_2) = 10$, $(\varphi_1, \varphi_2) = 0$. Furthermore, the minimal products > 0 of φ_1 and φ_2 are given by $(\varphi_1, \gamma_1) = 3$ and $(\varphi_2, \gamma_2) = 5$, thus $E_i \cong \mathrm{Tate}(t_i)$ where $v_\infty(t_1) = 3$ and $v_\infty(t_2) = 5$. Equations are given by

$$E_1: \; Y^2 + (T+1)XY + Y = X^3 + T(T^2 + T + 1)\,,$$
$$j(E_1) = (T+1)^{12}/n^3\,,$$
$$E_2: \; Y^2 + (T+1)XY + Y = X^3 + X^2 + T + 1\,,$$
$$j(E_2) = (T+1)^{12}/T^5(T^2 + T + 1)\,.$$

Note that $\varphi_1 \equiv \varphi_2 \pmod 2$ and $\mathbb{Z}\varphi_1 + \mathbb{Z}\varphi_2$ has index two in $\underline{H}_!(\mathscr{T}, \mathbb{Z})^\Gamma = \mathbb{Z}\varphi_1 + \mathbb{Z}\gamma_1$. Correspondingly, E_1 and E_2 intersect in J_Γ in their common subgroup scheme of two-division points (compare [57], section 5). In view of (6.5.1) this means that the weight two modular forms u_i'/u_i ($u_i := u_{\varphi_i}$, $i = 1, 2$) for Γ agree. As a result, the Hecke action on the two-dimensional space $M_{2,1}^2(\Gamma, C)$ is not semi-simple.

(9.8) Concluding remarks. Quite generally, the Néron type at ∞ of the strong Weil curve E_φ (i.e., v_∞(Tate period)) associated to φ may be read off from (9.5.4). Whereas

$$v_\infty(t) = \min\{(\varphi, \alpha) \mid \alpha \in \bar{\Gamma}, (\varphi, \alpha) > 0\}$$

may be easily calculated from the graph $\Gamma \backslash \mathscr{T}$, the divisor d of $q_\infty - 1$ is more difficult to control. Its determination requires studying the effect of $\mathrm{pr}_\varphi : J_\Gamma \to E_\varphi$ on Néron models. In [12] it is shown that $d = 1$ at least when A is a polynomial ring $\mathbb{F}_q[T]$.

We expect that our explicit description of E_φ will catalyse progress in classical questions on the arithmetic of elliptic curves over function fields. Among these we mention:

– the conjecture of Birch and Swinnerton-Dyer (or rather of Artin and Tate [49]), and the K-analogue of Kolyvagin's results [28];

– formulae of type "Gross-Zagier" [21] that relate heights of Heegner points on E_φ with values of the L-functions $L_E(\chi, s)$;

– Mazur's conjecture on the modular element (as e.g. reported in [48]).

Another important problem is the relationship of congruence numbers of eigenforms φ (i.e., $n \in \mathbb{N}$ for which there exists ψ orthogonal with φ but $\varphi \equiv \psi (\mathrm{mod}\, n)$) with the degree of the corresponding Weil uniformization. In the classical case, and if the level is prime, Zagier showed coincidence ([57], Thm. 3). The same assertion holds (even without the primality assumption on the level) if the base ring A is a polynomial ring ([12], Thm. 3.2), but the general case still resists. Note that some aspects of the game look rather differently in our context. E.g., $p = \mathrm{char}(\mathbb{F}_q)$ itself may appear as a congruence prime, as is shown by the last example (9.7.4), and so the differential form pulled back by a Weil uniformization doesn't suffice to determine a strong Weil curve.

We would also like to give a similarly satisfactory description of Drinfeld modular forms of higher weight through automorphic forms as the one achieved by (6.5.1) for forms of weight two.

Finally, it needs to be determined under which conditions the endomorphism ring of J_Γ is generated by Hecke operators, i.e., the analogue of Mazur's Proposition II, 9.5 in [30] holds.

References

[1] *S. Bosch, U. Güntzer, R. Remmert*, Non-archimedean analysis, Grundl. Math. Wiss. **261**, Springer, Berlin–Heidelberg–New York–Tokyo 1984.

[2] *H. Carayol*, Variétés de Drinfeld compactes, Sém. Bourbaki **756**, Paris 1992.

[3] *P. Deligne*, Formes modulaires et représentations de GL(2), in: Modular Functions of One Variable II, Lect. Notes Math. **349**, Springer (1973), 55–105.

[4] *P. Deligne*, Les constantes des équations fonctionnelles des fonctions L, in: Modular Functions of One Variable II, Lect. Notes Math. **349**, Springer (1973), 501–597.

[5] *P. Deligne, D. Husemöller*, Survey of Drinfeld modules, Contemp. Math. **67** (1987), 25–91.

[6] *V. G. Drinfeld*, Elliptic Modules (Russian), Math. Sbornik **94** (1974), 594–627; English Transl.: Math. USSR-Sbornik **23** (1976), 561–592.

[7] *J. Fresnel, M. van der Put*, Géometrie Analytique Rigide et Applications, Progr. Math. **18**, Birkhäuser, Basel–Boston 1981.

[8] *E.-U. Gekeler*, Drinfeld-Moduln und modulare Formen über rationalen Funktionenkörpern, Bonner Math. Schr. **119** (1980).

[9] *E.-U. Gekeler*, Automorphe Formen über $\mathbb{F}_q(T)$ mit kleinem Führer, Abh. Math. Sem. Univ. Hamburg **55** (1985), 111–146.

[10] *E.-U. Gekeler*, Drinfeld Modular Curves, Lect. Notes Math. **1231**, Springer, Berlin–Heidelberg–New York 1986.

[11] *E.-U. Gekeler*, On the coefficients of Drinfeld modular forms, Invent. Math. **93** (1988), 667–700.

[12] *E.-U. Gekeler*, Analytical construction of Weil curves over function fields, J. Th. Nomb. Bordeaux **7** (1995), 27–49.

[13] *E.-U. Gekeler, M. Reversat*, Some results on the Jacobians of Drinfeld modular curves, in: D. Goss et al. (eds.), The Arithmetic of Function Fields, Columbus 1991, Walter de Gruyter, Berlin–New York 1992.

[14] *S. Gelbart*, Automorphic forms on adele groups, Ann. Math. Stud. **83**, Princeton University Press, Princeton 1975.

[15] *L. Gerritzen*, On non-archimedean representations of abelian varieties, Math. Ann. **196** (1972), 323–346.

[16] *L. Gerritzen*, Unbeschränkte Steinsche Gebiete von $\mathbb{P}^1$ und nichtarchimedische automorphe Formen, J. reine angew. Math. **297** (1978), 21–34.

[17] *L. Gerritzen, M. van der Put*, Schottky Groups and Mumford Curves, Lect. Notes Math. **817**, Springer, Berlin–Heidelberg–New York 1980.

[18] *O. Goldmann, N. Iwahori*, The space of p-adic norms, Acta Math. **109** (1963), 137–177.

[19] *D. Goss*, π-adic Eisenstein Series for Function Fields, Comp. Math. **41** (1980), 3–38.

[20] *D. Goss*, Modular forms for $\mathbb{F}_r[T]$, J. reine angew. Math. **317** (1980), 16–39.

[21] *B.H. Gross, D. Zagier*, Heegner points and derivatives of L-series, Invent. Math. **84** (1986), 225–320.

[22] *G. Harder*, Chevalley groups over function fields and automorphic forms, Ann. Math. **100** (1974), 249–306.

[23] *D. Hayes*, Explicit class field theory in global function fields, in: G.C. Rota (ed.), Studies in Algebra and Number Theory, Academic Press, New York 1979.

[24] *H. Jacquet, R.P. Langlands*, Automorphic forms on GL(2), Lect. Notes Math. **114**, Springer, Berlin–Heidelberg–New York 1970.

[25] *R. Kiehl*, Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie, Invent. Math. **2** (1967), 191–214.

[26] *R. Kiehl*, Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie, Invent. Math. **2** (1967), 256–273.

[27] *U. Köpf*, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen, Schriftenr. Math. Inst. Univ. Münster (2) **7** (1974).

[28] *V.A. Kolyvagin*, Euler Systems, in: P. Cartier et al. (eds.), The Grothendieck Festschrift II, Progr. Math. **87**, Birkhäuser, Boston–Basel–Berlin 1990.

[29] *Y. Manin, V.G. Drinfeld*, Periods of p-adic Schottky groups, J. reine angew. Math. **262/263** (1973), 239–247.

[30] *B. Mazur*, Modular curves and the Eisenstein ideal, Publ. IHES **47** (1977), 33–186.

[31] *J.S. Milne*, Abelian Varieties, in: G.C. Cornell, J.H. Silverman (eds.), Arithmetic Geometry, Springer, New York–Berlin–Heidelberg 1986.

[32] *D. Mumford*, An analytic construction of degenerating curves over local fields, Comp. Math. **24** (1972), 129–174.

[33] *J.F. Myers*, p-adic Schottky groups, Thesis, Harvard Univ. 1973.

[34] *U. Nonnengardt*, Arithmetisch definierte Graphen über rationalen Funktionenkörpern, Diplomarbeit, Saarbrücken 1994.

[35] *A. Ogg*, Elliptic curves and wild ramification, Amer. J. Math. **89** (1967), 1–21.

[36] *M. van der Put*, Les fonctions thêta d'une courbe de Mumford, Groupe d'étude d'analyse ultramétrique 1981/82, Paris 1982.

[37] *M. van der Put*, Discrete groups, Mumford curves and theta functions, Ann. Fac. Sc. Toulouse (6) **I** (1992), 399–438.

[38] *W. Radtke*, Diskontinuierliche Gruppen im Funktionenkörperfall, Dissertation, Bochum 1984.

[39] *W. Radtke*, Diskontinuierliche Gruppen im Funktionenkörperfall, J. reine angew. Math. **363** (1985), 191–200.

[40] *M. Reversat*, Lecture on rigid geometry, in: D. Goss et al. (eds.), The Arithmetic of Function Fields, Columbus 1991, Walter de Gruyter, Berlin–New York 1992.

[41] *M. Reversat*, Sur les revêtements de Schottky des courbes modulaires de Drinfeld, Arch. Math. **66** (1995), 378–387.

[42] *K. Ribet*, On modular representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms, Invent. Math. **100** (1990), 431–476.

[43] *J.-P. Serre*, Corps locaux, Hermann, Paris 1968.
[44] *J.-P. Serre*, Cours d'arithmétique, Presses universitaires de France, Paris 1970.
[45] *J.-P. Serre*, Trees, Springer, Berlin–Heidelberg–New York 1980.
[46] *G. Shimura*, On the factors of the Jacobian variety of a modular function field, J. Math. Soc. Japan **25** (1973), 523–544.
[47] *G. Shimura*, Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. Japan, Tokyo– Princeton 1971.
[48] *K.-S. Tan, D. Rockmore*, Computation of L-series for elliptic curves over function fields, J. reine angew. Math. **424** (1992), 107–135.
[49] *J. Tate*, On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, Sém. Bourbaki **306**, also in: A. Grothendieck, N. H. Kuiper (eds.), Dix exposés sur la cohomologie des schémas, North Holland, Amsterdam–Paris 1968.
[50] *R. Taylor, A. Wiles*, Ring theoretic properties of certain Hecke algebras, Ann. Math. **141** (1995), 553–572.
[51] *J. Teitelbaum*, The Poisson kernel for Drinfeld modular curves, J. Amer. Math. Soc. **4** (1991), 491–511.
[52] *J. Teitelbaum*, Rigid analytic modular forms: An integral transform approach, in: D. Goss et al. (eds.), The Arithmetic of Function Fields, Columbus 1991, Walter de Gruyter, Berlin–New York 1992.
[53] *A. Weil*, Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. **168** (1967), 149–156.
[54] *A. Weil*, Basic Number Theory, Grundl. Math. Wiss. **144**, Springer, Berlin–Heidelberg–New York 1967.
[55] *A. Weil*, Dirichlet series and automorphic forms, Lect. Notes Math. **189**, Springer, Berlin–Heidelberg–New York 1971.
[56] *A. Wiles*, Modular elliptic curves and Fermat's last theorem, Ann. Math. **141** (1995), 443–551.
[57] *D. Zagier*, Modular parametrizations of elliptic curves, Canad. Math. Bull. **28** (1985), 372–384.

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