Fundamental Domains of Some Drinfeld Modular Curves

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Abstract. We construct fundamental domains for arithmetic subgroups of $\Gamma = \mathrm{GL}_2(\mathbb{F}_q[t])$. Given $\Delta \supseteq \Gamma(\mathfrak{a})$ we construct a contracted form $\overline{\mathcal{T}}$ of the Bruhat-Tits tree $\mathcal{T}$ and a fundamental domain $\overline{\mathcal{F}}$ of Δ acting on $\overline{\mathcal{T}}$. We define a lift of $\overline{\mathcal{F}}$ to $\mathcal{F} \subset \mathcal{T}$ called the "bipartite" lift. We show that $\mathcal{F}$ is a fundamental domain of Δ acting on $\mathcal{T}$ precisely when $\overline{\mathcal{F}}$ is "Δ-compressed."

1. Introduction

Fix a finite field $k = \mathbb{F}_q$, let $A = k[t]$ and $K = k(t)$. We denote the ∞-adic completion of K by K_∞.

The arithmetic group $\Gamma = \mathrm{GL}_2(A)$ acts on the Bruhat-Tits tree $\mathcal{T}$ and the quotient $X = \Gamma\backslash\mathcal{T}$ is a half-line. Given a proper ideal $\mathfrak{a} \subset A$ one may consider the arithmetic subgroup $\Gamma(\mathfrak{a}) \subset \Gamma$ with the induced action on $\mathcal{T}$. Similarly one may consider an arbitrary arithmetic subgroup $\Delta \supseteq \Gamma(\mathfrak{a})$. The quotient $X_\Delta = \Delta\backslash\mathcal{T}$ is a connected graph which is the union of a finite graph X_Δ° and finitely many cusps. The genus of the finite graph and the number of cusps have been computed by Gekeler.

We say that a subtree $\mathcal{F} = \mathcal{F}(\Delta) \subset \mathcal{T}$ is a fundamental domain for Δ if the image of $\mathcal{F}$ surjects onto X_Δ and is a bijection of edges. A fundamental domain always exists and we may assume without loss of generality that it is connected. Given Δ we show how to compute a fundamental domain.

2. Properties of Connected Fundamental Domains

In this section we study properties satisfied by any (connected) fundamental domain $\mathcal{F}$ for $\Gamma(\mathfrak{a})$. We assume throughout that $\mathfrak{a} \subset A$ is a proper ideal. We start by recalling the construction of a fundamental domain of Γ (cf. section II.1.6 of [**S**]). Let $\mathcal{O} \subset K_\infty$ be the valuation ring; the valuation of $f \in K^\times$ is $-\deg(f)$.

Fix $V = K_\infty^2$ and let $\{e_1, e_2\}$ denote the canonical basis. For $n \geq 0$, let v_n denote the vertex in $\mathcal{T}$ corresponding to the lattice $\mathcal{O}t^n e_1 \oplus \mathcal{O}e_2$. The subtree $\mathcal{F}(\Gamma) \subset \mathcal{T}$

spanned by $\{v_n\}_{n\geq 0}$ is a half-line starting at v_0. Let

$$\Gamma_0 = \mathrm{GL}_2(k) \subset \Gamma, \qquad \Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma : a, d \in A^\times, \deg(b) \leq n\} \right\}, \quad n \geq 1.$$

We recall without proof the following lemma from [*loc. cit.*].

Lemma 2.0.1.

(1) *The v_n are pairwise Γ-inequivalent.*
(2) *Γ_n is the stabilizer of v_n in Γ.*
(3) *Γ_0 acts transitively on the set of edges starting at v_0.*
(4) *For $n \geq 1$, Γ_n fixes the edge from v_n to v_{n+1} and operates transitively on the remaining edges from v_n.*

Proof. This is proposition 3 on page 119 of [*loc. cit.*]. $\square$

The lemma implies that $\mathcal{F}(\Gamma)$ is a fundamental domain of Γ (cf. corollary to proposition II.3 of [*loc. cit.*]).

Definition: Let $v \in \mathcal{T}$ be a vertex and Γ_v denote the stabilizer of v in $\mathcal{T}$. We say that v is of *type* n if $v = \gamma v_n$ for some $\gamma \in \Gamma$, in which case $\Gamma_v = \gamma \Gamma_n \gamma^{-1}$.

More generally, for an arithmetic subgroup $\Delta \supseteq \Gamma(\mathfrak{a})$, we denote the stabilizer of v by Δ_v. Let $\mathcal{T}_n \subset \mathcal{T}$ be the vertices of type n.

Corollary 2.0.2. *Suppose $n \geq 1$ and $v \in \mathcal{T}_n$. Then Γ_v acts transitively on the edges from v to $\mathcal{T}_{n-1}$. It fixes the unique edge from v to $\mathcal{T}_{n+1}$.*

Proof. The follows from part 4 of lemma 2.0.1 and the identification of Γ_v above. $\square$

For every $n \geq 0$, $\mathcal{T}_n$ is closed under the action of Γ, a fortiori under the action of Δ. We denote the subtree of all vertices of type at most n by $\mathcal{T}_{\leq n}$, and similarly for $\mathcal{T}_{\geq n}$. They are also closed under the action of Γ.

Lemma 2.0.3. *For every $n \geq 0$, the quotient $\Gamma(\mathfrak{a})\backslash\mathcal{T}_{\leq n}$ is a finite graph.*

Proof. By definition $\Gamma\backslash\mathcal{T}_{\leq n}$ is the finite subgraph of $\Gamma\backslash\mathcal{T}$ spanned by the first $n+1$ vertices. It is a quotient of the graph $\Gamma(\mathfrak{a})\backslash\mathcal{T}_{\leq n}$ by the finite group $\Gamma(\mathfrak{a})\backslash\Gamma$, hence must be finite. $\square$

We define the cusps of $X_{\mathfrak{a}} = \Gamma(\mathfrak{a})\backslash\mathcal{T}$ to be the infinite half-lines such that the origin of each is connected to at least three vertices in $X_{\mathfrak{a}}$ and every other vertex in the cusp is connected to exactly two. We denote the complement $X_{\mathfrak{a}}^\circ$ and call it the finite part.

Fix a fundamental domain $\mathcal{F} = \mathcal{F}(\Gamma(\mathfrak{a})) \subset \mathcal{T}$ of $\Gamma(\mathfrak{a})$ and let $\mathcal{F}_n$ denote $\mathcal{F} \cap \mathcal{T}_n$. We assume without loss of generality that $\mathcal{F}$ is connected. We define the cuspidal part of $\mathcal{F}$ and the finite part $\mathcal{F}^\circ$ to be the inverse image of the cusps of $X_{\mathfrak{a}}$ and $X_{\mathfrak{a}}^\circ$ respectively.

COROLLARY 2.0.4. *There exists $N \geq 0$, such that for every $n \geq N$, $\mathcal{F}^\circ \subset \mathcal{T}_{\leq n}$ and every cusp intersects $\mathcal{T}_n$ in at least one point.*

PROOF. The first part is trivial because $\mathcal{F}^0$ is a finite graph. Let $\mathcal{L} \subset \mathcal{F}$ be a cusp. By definition, $\mathcal{L}$ is an infinite graph which maps injectively to the quotient $\Gamma(\mathfrak{a})$, and hence by the previous lemma, it has finite intersection with $\mathcal{F}_{\leq N}$ for every $N \geq 0$. It is also connected, so intersects $\mathcal{F}_n$ for every $n \gg 0$. □

We will see below that one may take $N = d - 1 = \deg(\mathfrak{a}) - 1$ and that for every $n \geq N$ each cusp will intersect $\mathcal{F}_n$ in exactly one point.

For any v in $\mathcal{T}$ of positive type, part 4 of lemma 2.0.1 implies that there is a unique edge from v to $\mathcal{T}_{n+1}$. We denote the edge $\varepsilon(v)$ and the terminating point $\tau(v)$.

LEMMA 2.0.5. *Suppose $n \geq 0$ and $v \in \mathcal{T}_n$.*

(1) *If $n < d$, then edges from v are $\Gamma(\mathfrak{a})$-inequivalent.*
(2) *If $n \geq d$, the stabilizer of v in $\Gamma(\mathfrak{a})$ fixes $\varepsilon(v)$ and acts transitively on the remaining edges from v.*

PROOF. Let $\gamma \in \Gamma$ be an element such that $v = \gamma v_n$. Then $\Gamma_v = \gamma \Gamma_n \gamma^{-1}$, so

$$\Gamma(\mathfrak{a})_v = \Gamma_v \cap \Gamma(\mathfrak{a}) = \gamma \Gamma_n \gamma^{-1} \cap \Gamma(\mathfrak{a}) = \gamma(\Gamma_n \cap \Gamma(\mathfrak{a}))\gamma^{-1}.$$

The last equality holds because $\Gamma(\mathfrak{a})$ is a normal subgroup of Γ. Therefore, we may conjugate by γ^{-1} and reduce to the case $v = v_n$. In this case, the lemma follows from lemma 2.0.1. □

The last property imposes a strong uniformity on $\mathcal{F}_n$ for $n \geq d$.

LEMMA 2.0.6. *Suppose $n \geq d$ and $v \in \mathcal{F}_n$. There are exactly two edges in $\mathcal{F}$ from v, one of which is $\varepsilon(v)$.*

PROOF. Part 4 of lemma 2.0.5 implies that for every $n \geq d$ and $v \in \mathcal{F}_n$, there is at most one edge from v to $\mathcal{F}_{n-1}$. For a fixed v, this implies every geodesic of length $m \geq 1$ in $\mathcal{F}$, starting at v and going through $\tau(v)$, must terminate at $\tau^{(m)}(v)$. Hence there must be at least one edge in $\mathcal{F}$ from v to $\mathcal{F}_{n-1}$, otherwise $\mathcal{F}$ would not be connected. It remains to show that $\varepsilon(v)$ is in $\mathcal{F}$.

Because $\mathcal{F}$ is a fundamental domain, there are a unique edge e in $\mathcal{F}$ and some $\gamma \in \Gamma(\mathfrak{a})$ such that $\gamma e = \varepsilon(v)$. From the above, there is a unique edge e' in $\mathcal{F}$ from $v' = \gamma^{-1} v$ to $\mathcal{F}_{n-1}$. Also, $\gamma e'$ is the unique edge in $\mathcal{F}$ from v to $\mathcal{F}_{n-1}$. Because $\mathcal{F}$ is a fundamental domain, we must have $\gamma e' = e'$, which implies $v' = \gamma v' = v$. Hence $\varepsilon(v) = \varepsilon(v') = e$ is in $\mathcal{F}$. □

COROLLARY 2.0.7. *For every $n \geq d$ and $v \in \mathcal{F}_n$, the half-line in $\mathcal{T}$ spanned by $\{\tau^{(m)}(v)\}_{m \geq 0}$ is contained in a unique cusp $\mathcal{L} \subset \mathcal{F}$. In particular, there is a one-to-one correspondance between elements of $\mathcal{F}_n$ and cusps.*

The proof is straightforward.

Theorem 2.0.8. *Every cusp $\mathcal{L} \subset \mathcal{F}$ attaches to $\mathcal{F}^\circ$ in $\mathcal{F}_{d-1}$. Moreover, if $d > 1$, there is a one-to-one correspondance between elements of $\Gamma(\mathfrak{a})\backslash\mathcal{F}_{d-1}$ and cusps. For $d = 1$, there is a unique point $v \in \mathcal{F}_0$ and a one-to-one correspondance between elements of $\mathcal{F}_1$ and cusps.*

Proof. By corollary 2.0.7, every $v \in \mathcal{F}_d$ is contained in a unique cusp $\mathcal{L} \subset \mathcal{F}$. Further, there is a unique edge starting at v and ending in $\mathcal{F}_{n-1}$, hence $\mathcal{L}$ connects to $\mathcal{F}^\circ$ in $\mathcal{F}_{\leq d-1}$. On the other hand, part 3 of lemma 2.0.6 implies that for every $v' \in \mathcal{F}_{d-1}$ there are at least three $\Gamma(\mathfrak{a})$-inequivalent edges e_i of $\mathcal{F}$ and elements $\gamma_i \in \Gamma(\mathfrak{a})$, such that $\{\gamma_i e_i\}$ are $\Gamma(\mathfrak{a})$-inequivalent edges from v'. Hence $\mathcal{L}$ must attach to $\mathcal{F}^\circ$ in $\mathcal{F}_{d-1}$.

For $d = 1$ and $v \in \mathcal{F}_0$, every edge starting at v terminates in $\mathcal{F}_1$ and is contained in a unique cusp $\mathcal{L} \subset \mathcal{F}$. Hence for every $v' \in \mathcal{F}$ distinct from v, the segment in $\mathcal{F}$ connecting v to v' is contained in a unique cusp $\mathcal{L} \subset \mathcal{F}$. In particular, v' is of positive type, hence $\mathcal{F}_0 = \{v\}$. This implies there is a one-to-one correspondance between cusps $\mathcal{L} \subset \mathcal{F}$ and edges starting at v, which in turn correspond bijectively to elements of $\mathcal{F}_1$.

Finally, suppose $d \geq 1$. From the above, distinct cusps attach at $\Gamma(\mathfrak{a})$-inequivalent points of $\mathcal{F}_{d-1}$. Conversely, let $v \in \mathcal{F}_{d-1}$. Because $\mathcal{F}$ is a fundamental domain there is a $\gamma \in \Gamma(\mathfrak{a})$ such that $e = \gamma^{-1}\varepsilon(v)$ is contained in $\mathcal{F}$. Further, e is contained in a unique cusp $\mathcal{L} \subset \mathcal{F}$ which attaches to $\mathcal{F}^\circ$ at γv. Therefore cusps $\mathcal{L} \subset \mathcal{F}$ correspond bijectively to elements of $\Gamma(\mathfrak{a})\backslash\mathcal{F}_{d-1}$. $\square$

Remark: From the proof we see, for $d = 1$, that $\mathcal{F}$ is the union of $q + 1$ cusps all starting at the same $v \in \mathcal{T}_0$.

Let $\Delta \supset \Gamma(\mathfrak{a})$ denote an arithmetic subgroup and $\mathcal{F}(\Delta) \subset \mathcal{T}$ a connected fundamental domain of Δ acting on $\mathcal{T}$. We may assume without loss of generality that $\mathcal{F}(\Delta) \subset \mathcal{F} = \mathcal{F}(\Gamma(\mathfrak{a}))$ and view it as a fundamental domain of $\Gamma(\mathfrak{a})\backslash\Gamma$ acting on $\mathcal{F}$.

Corollary 2.0.9. *Every cusp $\mathcal{L} \subset \mathcal{F}(\Delta)$ attaches to the finite part $\mathcal{F}(\Delta)^\circ$ at a vertex in $\mathcal{F}(\Delta)_{\leq d-1}$. In particular, for every $n \geq d$ and $v \in \mathcal{F}(\Delta)_n$, there are precisely two edges from v in $\mathcal{F}$, one of which is $\varepsilon(v)$.*

Proof. The finite group $\Gamma(\mathfrak{a})\backslash\Delta$ permutes the cusps of $\mathcal{F} = \mathcal{F}(\Gamma(\mathfrak{a}))$. Hence every cusp $\mathcal{L} \subset \mathcal{F}(\Delta)$ contains a unique cusp of $\mathcal{F}$ and is disjoint from the remaining cusps. Theorem 2.0.8 implies $\mathcal{L}$ attaches to $\mathcal{F}(\Delta)^\circ$ in $\mathcal{F}(\Delta)_{\leq d-1}$. $\square$

3. Bipartite Trees

Let $\mathcal{T}$ be the Bruhat-Tits tree. Throughout we let $\mathcal{F} \subset \mathcal{T}$ denote a connected subtree such that $\mathcal{F}_0$ is non-empty. We fix an ideal $\mathfrak{a} \subset A$ and write $d = \deg(\mathfrak{a})$.

Let $\mathcal{T}_+ \subset \mathcal{T}$ denote the vertices of positive type.

Definition: Suppose $v, v' \in \mathcal{T}_+$. We say v' *orbits* v if there is an $m \geq 0$ such that $v = \tau^{(m)}(v')$. Let $\mathcal{T}(v)$ denote the subtree of $\mathcal{T}$ spanned by the vertices orbiting v and $\mathcal{F}(v) = \mathcal{T}(v) \cap \mathcal{F}$. We call it the *constellation* centered at v.

If one imagines bodies orbiting a common point, the following lemma motivates the terminology.

Lemma 3.0.10. *Suppose $n \geq 1$ and $v \in \mathcal{T}_n$. Then the stabilizer in Γ of $\mathcal{T}(v)$ is Γ_v. It acts transitively on $\mathcal{T}(v)_m$ for every $1 \leq m \leq n$.*

Proof. The first part follows by observing that $\gamma\mathcal{T}(v) = \mathcal{T}(\gamma v)$ for every $\gamma \in \Gamma$. The second part is corollary 2.0.2. $\square$

The first property in the following definition is motivated by corollary 2.0.9.

Definition: We say $\mathcal{F}$ is a *bipartite* tree if it satisfies the following properties:

- `(B1)`: For every $n \geq d$ and $v \in \mathcal{F}_n$, there are precisely two edges from v in $\mathcal{F}$, one of which is $\varepsilon(v)$.
- `(B2)`: For every $1 < n < d$ and $v \in \mathcal{F}_n$, $\mathcal{F}(v)$ is the union of the segments from $\mathcal{F}(v)_1$ to v.
- `(B3)`: For every $v \in \mathcal{F}_1$, there is an edge from $\mathcal{F}_0$ to v.

For $v \in \mathcal{T}_+$, let $\mathcal{L}(v)$ denote the half-line spanned by $\varepsilon(\tau^{(m)}(v))$ for $m \geq 0$. We call it the cusp starting at v.

Lemma 3.0.11. *Suppose $\mathcal{F}, \mathcal{F}'$ satisfy `(B1)` and `(B2)`. Then $\mathcal{F}_{\leq 1} = \mathcal{F}'_{\leq 1}$ if and only if $\mathcal{F} = \mathcal{F}'$.*

Proof. One direction is trivial: if $\mathcal{F} = \mathcal{F}'$, then $\mathcal{F}_{\leq 1} = \mathcal{F}'_{\leq 1}$. Property `(B1)` is equivalent to assuming that $\mathcal{F}$ is the union of $\mathcal{F}_{\leq d-1}$ and the cusps $\mathcal{L}(v)$ starting at each $v \in \mathcal{F}_{d-1}$. Hence $\mathcal{F} = \mathcal{F}'$ if $\mathcal{F}_{\leq d-1} = \mathcal{F}'_{\leq d-1}$. Both properties imply $\mathcal{F}$ is the union of $\mathcal{F}_{\leq 1}$ and the cusps starting at each $v \in \mathcal{F}_1$. In particular, $\mathcal{F}$ may be recovered from $\mathcal{F}_{\leq 1}$, so $\mathcal{F} = \mathcal{F}'$ if $\mathcal{F}_{\leq 1} = \mathcal{F}'_{\leq 1}$. $\square$

Suppose that $v, v' \in \mathcal{T}_1$. If they do not belong to a common constellation, then we say that v, v' do not meet and define the distance $d(v, v') = \infty$. This happens if and only if the segment between them contains a vertex in $\mathcal{T}_0$. Otherwise, we define $d(v, v')$ to be the radius of the smallest constellation $\mathcal{T}(v'')$ containing both vertices, and call $\mu(v, v') = v''$ the meeting point; if $v'' \in \mathcal{T}_m$, then $d(v, v') = m - 1$.

Lemma 3.0.12. *If $\mathcal{F}$ satisfies property `(B1)` and $v, v' \in \mathcal{F}_1$, then either $d(v, v') \leq d - 2$ or $d(v, v') = \infty$.*

Proof. The segment from v to v' has trivial intersection with each cusp. Corollary 2.0.9 implies it is confined to $\mathcal{F}_{\leq d-1}$. If $m = d(v, v') < \infty$, then their meeting point lies on this segment, hence is in $\mathcal{F}_{\leq d-1}$. Therefore $m \leq d - 2$. $\square$

Let $\sim$ denote the equivalence relation on $\mathcal{T}_1$ such that $v \sim v'$ if $d(v,v') \le d-2$ and $\overline{\mathcal{T}}$ the quotient tree $\mathcal{T}_{\le 1}/\sim$. For any $\gamma \in \Gamma$ we have $\mu(\gamma v, \gamma v') = \gamma v'' = \gamma\mu(v,v')$, hence d is Γ-equivariant. This allows us to "restrict" the action of Γ on $\mathcal{T}_{\le 1}$ to the quotient $\overline{\mathcal{T}}$. Restricting $\sim$ to any $\mathcal{F}$, we let $\overline{\mathcal{F}}$ denote the quotient tree $\mathcal{F}_{\le 1}/\sim$.

Corollary 3.0.13. *If $\mathcal{F}$ satisfies* (B1)*, then $\overline{\mathcal{F}}$ connected.*

Proof. If $\overline{v}, \overline{v}'$ are vertices in $\overline{\mathcal{F}}$, it suffices to construct the segment connecting $\overline{v}$ to $\overline{v}'$. Let v, v' be vertices in $\mathcal{F}_{\le 1}$ lying over $\overline{v}, \overline{v}'$, and consider the segment s connecting them in $\mathcal{F}$. Lemma 3.0.12 implies it is confined to $\mathcal{F}_{\le d-1}$. We construct the segment connecting $\overline{v}$ to $\overline{v}'$ by contracting each subgment of s, contained within a constellation $\mathcal{T}(v'')$ centered in $\mathcal{F}_{d-1}$, to the point $\overline{v}''$ in $\overline{\mathcal{F}}_1$ associated to v''. □

Let $\overline{\mathcal{F}} \subset \overline{\mathcal{T}}$ denote a connected subtree such that $\overline{\mathcal{F}}_0$ is non-empty. One may lift it to $\mathcal{F}_{\le 1} \subset \mathcal{T}_{\le 1}$ by lifting the edges, then to $\mathcal{F} \subset \mathcal{T}$ by adjoining $\mathcal{L}(v)$ for every $v \in \mathcal{F}_1$. We call $\mathcal{F}$ the *bipartite lift* of $\overline{\mathcal{F}}$. As the following lemma shows, given a bipartite $\mathcal{F}$, the bipartite lift of $\overline{\mathcal{F}}$ is $\mathcal{F}$ again.

Lemma 3.0.14. *Suppose $\mathcal{F}, \mathcal{F}'$ are bipartite. Then $\overline{\mathcal{F}} = \overline{\mathcal{F}}'$ if and only if $\mathcal{F} = \mathcal{F}'$.*

Proof. One direction is trivial: if $\mathcal{F} = \mathcal{F}'$, then $\overline{\mathcal{F}} = \overline{\mathcal{F}}'$. Suppose that $\overline{\mathcal{F}} = \overline{\mathcal{F}}'$. The edges of $\overline{\mathcal{F}}$ correspond bijectively to those of $\mathcal{F}_{\le 1}$. Property (B3) assert that for every $v \in \mathcal{F}_1$ there is an edge from v to $\mathcal{F}_0$, hence $\mathcal{F}_{\le 1}$ may be recovered from $\overline{\mathcal{F}}$. In particular, $\mathcal{F}_{\le 1} = \mathcal{F}'_{\le 1}$, and lemma 3.0.12 implies $\mathcal{F} = \mathcal{F}'$. □

Fix an arithmetic subgroup $\Delta \supseteq \Gamma(\mathfrak{a})$. If $\mathcal{F}$ is a fundamental domain of Δ, then $\overline{\mathcal{F}}$ is a fundamental domain of Δ acting on $\overline{\mathcal{T}}$. However, if $\overline{\mathcal{F}}$ is a fundamental domain of Δ on $\overline{\mathcal{T}}$, then the bipartite lift $\mathcal{F}$ is *not* necessarily a fundamental domain of Δ on $\mathcal{T}$.

Suppose $\overline{e}, \overline{e}'$ are edges $\overline{\mathcal{T}}_0$ to $\overline{\mathcal{T}}_1$. We lift each to $\mathcal{T}_{\le 1}$ and let v, v' be the endpoints in $\mathcal{T}_1$. We define

$$d(\overline{e}, \overline{e}') := \begin{cases} 0 & \text{if } \overline{e} = \overline{e}' \\ d(v,v') + 1 & \text{otherwise} \end{cases}. \tag{3.0.1}$$

If $\overline{v} \in \overline{\mathcal{T}}_1$ and $v \in \mathcal{T}_{d-1}$ is the center of the associated constellation, then one easily verifies that $\Delta_{\overline{v}} = \Delta_v$.

Definition: We say that $\overline{\mathcal{F}} \subset \overline{\mathcal{T}}$ is Δ-*compressed* if it satisfies the following properties:

(D1): For every $\overline{v} \in \overline{\mathcal{F}}_1$ and every pair of edges $\overline{e}_1, \overline{e}_2$ from $\overline{\mathcal{F}}_0$ to $\overline{v}$,

$$d(\overline{e}_1, \overline{e}_2) = \min\{d(\overline{e}_1, \overline{e}_3) : \overline{e}_3 \in \Delta_{\overline{v}}\,\overline{e}_2\}.$$

(D2): For every $\overline{v} \in \overline{\mathcal{F}}_1$ and every $\gamma \in \Delta$, $\gamma\overline{v} \in \overline{\mathcal{F}}_1$ if only if $\gamma \in \Delta_{\overline{v}}$.

Suppose $\overline{e}_1, \overline{e}_2$ are edges from $\overline{\mathcal{F}}_0$ to $\overline{v}$. If they are $\Delta_{\overline{v}}$-equivalent, property (D1) implies $d(\overline{e}_1, \overline{e}_2) = 0$, which happens if and only if $\overline{e}_1 = \overline{e}_2$. Thus for every $\overline{v} \in \overline{\mathcal{F}}_1$, the edges from $\overline{\mathcal{F}}_0$ to $\overline{v}$ are pairwise Δ-inequivalent. Property (D2) implies the vertices in $\overline{\mathcal{F}}_1$ are pairwise Δ-inequivalent, hence so are the cusps $\mathcal{L}(v) \subset \mathcal{F}$ for every $v \in \mathcal{F}_{d-1}$. Together the properties imply that the edges of $\overline{\mathcal{F}}$, hence the edges of $\mathcal{F}_{\leq 1}$, are pairwise Δ-inequivalent.

LEMMA 3.0.15. *Suppose $\overline{\mathcal{F}} \subset \overline{\mathcal{T}}$ is connected and $\overline{\mathcal{F}}_0$ is non-empty. The edges of the bipartite lift $\mathcal{F}$ are pairwise Δ-inequivalent if and only if $\overline{\mathcal{F}}$ is Δ-compressed.*

PROOF. From the remarks preceding the lemma we see that properties (D1) and (D2) are necessary for the edges of $\mathcal{F}$ to be pairwise Δ-inequivalent, hence it suffices to show that they are sufficient.

Suppose $\overline{\mathcal{F}}$ is Δ-compressed and e_1, e_2 are distinct Δ-equivalent edges of $\mathcal{F}$. They must lie in $\mathcal{F}(v)$ for some $v \in \mathcal{F}_{d-1}$ because the edges of $\mathcal{F}_{\leq 1}$ are pairwise Δ-inequivalent. By property (B2) there are (distinct) vertices $v', v'' \in \mathcal{F}(v)_1$ such that e_1, e_2 lie on the segments in $\mathcal{F}(v)$ connecting v', v'' to their meeting point $\mu(v', v'')$. If e_1', e_2' are the edges of the segments ending at $\mu(v', v'')$ and $e_1 = \gamma e_2$ with $\gamma \in \Delta_v$, then $e_1' = \gamma e_2'$, which implies $d(v', \gamma v'') < d(v', v'')$.

By property (B3) there exist distinct edges e_1'', e_2'' from $\mathcal{F}_0$ to v', v''. They contract to distinct edges $\overline{e}_1{}'', \overline{e}_2{}''$ from $\overline{\mathcal{F}}_0$ to the point $\overline{v} \in \overline{\mathcal{F}}_1$ corresponding to v. In particular,

$$d(\overline{e}_1,'', \gamma\overline{e}_2{}'') - 1 = d(v', \gamma v'') < d(v', v'') = d(\overline{e}_1{}'', \overline{e}_2{}'') - 1,$$

which contradicts (D1). The first equality results from (3.0.1), for the edges of $\overline{\mathcal{F}}_{\leq 1}$ are Δ-inequivalent, hence $\overline{e}_1{}'' \neq \gamma\overline{e}_2{}''$. The second follows similarly. □

We now have most of the pieces for the following theorem.

THEOREM 3.0.16. *Let $\overline{\mathcal{F}}$ be a connected fundamental domain of Δ acting on $\overline{\mathcal{T}}$. The bipartite lift $\mathcal{F}$ is a connected fundamental domain of Δ acting on $\mathcal{T}$ if and only if $\overline{\mathcal{F}}$ is Δ-compressed.*

PROOF. As we observed before, one direction is trivial. It remains to show that the bipartite lift $\mathcal{F}$ of a fundamental domain $\overline{\mathcal{F}}$ of Δ on $\overline{\mathcal{T}}$ is one of Δ on $\mathcal{T}$. By lemma 3.0.15 the edges of $\mathcal{F}$ are pairwise Δ-inequivalent, hence $\mathcal{F}$ is a fundamental domain of Δ on $\mathcal{T}$, unless there is an edge of $\mathcal{T}$ which is Γ-inequivalent to every edge of $\mathcal{F}$.

Suppose e is an edge from $\mathcal{T}_n$ to $\mathcal{T}_{n+1}$. If $n = 0$, the contraction $\overline{e}$ is Δ-equivalent to an edge of $\overline{\mathcal{F}}$, hence e is Δ-equivalent to an edge of $\mathcal{F}$. If $n \geq 1$, we may choose a vertex $v \in \mathcal{T}_1$ such that $e = \varepsilon(\tau^{(n-1)}(v))$ and an edge e' from $\mathcal{T}_0$ to v. By the above, $\gamma e'$ is an edge of $\mathcal{F}$ for some $\gamma \in \Delta$, hence γv is in $\mathcal{F}_1$. Because $\mathcal{F}$ is connected and satisfies (B1) and (B2), the edge $\gamma\varepsilon(\tau^{(m)}(v))$ is in $\mathcal{F}$ for every $m \geq 0$. In particular, γe is in $\mathcal{F}$. □

4. Constructing Fundamental Domains in $\overline{\mathcal{T}}$

Let $\overline{\mathcal{T}}$ be the contracted Bruhat-Tits tree associated to $\Gamma(\mathfrak{a})$ as in section 3 and $\Delta \supseteq \Gamma(\mathfrak{a})$ an arithmetic subgroup. In this section we construct a fundamental domain $\overline{\mathcal{F}}$ of Δ acting on $\overline{\mathcal{T}}$ and show that it is Δ-compressed. If one applies theorem 3.0.16, then the bipartite lift $\mathcal{F}$ will be a connected fundamental domain of Δ acting on $\mathcal{T}$.

Let e_0 the "canonical" edge of $\mathcal{T}$ from v_0 to v_1 (cf. section 2). We denote the contracted edge by $\overline{e}_0$ and the contracted vertices by $\overline{v}_i$. Let $B \subset \Gamma_0$ denote the Borel subgroup of upper triangular matrices. We recall the following facts without proof (cf. lemmas 2.0.5 and 3.0.10).

Lemma 4.0.17. *$\Gamma_{\overline{v}_0} = \Gamma_0$, $\Gamma_{\overline{v}_1} = \Gamma_{d-1}$, and $\Gamma_{\overline{e}_0} = \Gamma_{\overline{v}_0} \cap \Gamma_{\overline{v}_1} = B$.*

We refer to vertices in $\overline{\mathcal{T}}_0$ as *nodes* and those in $\overline{\mathcal{T}}_1$ as *cusps*. We call an edge $\overline{e}$ from a node to a cusp a *positive* edge, and we associate to it the coset γB where $\gamma \in \Gamma$ satisfies $\gamma \overline{e}_0 = \overline{e}$. We obtain a bijection with the coset space Γ/B.

Corollary 4.0.18. $\{$ *positive edges of* $\overline{\mathcal{T}}$ *connected to* $\overline{e}_0$ $\} \xrightarrow{1-1} \{$ *cosets of* Γ/B $\}$.

The cosets $\Gamma_{\overline{v}_0}/B$ correspond bijectively to the edges of $\overline{\mathcal{T}}$ from $\overline{v}_0$ and those of $\Gamma_{\overline{v}_1}/B$ to the edges of $\overline{\mathcal{T}}$ to $\overline{v}_1$. We choose representatives of the cosets as follows.

Viewing the coset space $\Gamma_{\overline{v}_0}/B$ as $\mathbb{P}^1(k)$, we index the cosets by z, and let $\{\omega_z\}$ be the coset representatives

$$\left\{ \omega_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in k \right\} \cup \left\{ \omega_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

B is a normal subgroup of $\Gamma_{\overline{v}_1}$, and the coset space $\Gamma_{\overline{v}_1}/B$ is a group isomorphic to $R := k[t]/t^{d-1}$. We index the cosets by $b \in R$, so $\deg(b) \le d-2$, and let $\{\gamma_b\}$ be the coset representatives

$$\left\{ \gamma_b = \begin{pmatrix} 1 & t \cdot b \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $S_0 = \{\omega_z\} - \{\omega_0\}$, $S_1 = \{\gamma_b\} - \{\gamma_0\}$ and S be the disjoint union $S_0 \cup S_1$. We refer to elements of S as *letters*, elements of S_0 as *nodal* letters and elements of S_1 as *cuspidal* letters.

Definition: An *S-word* is a finite product $w = \lambda_1 \cdots \lambda_n$ of letters. When viewed as an element of Γ, the empty word corresponds to the identity element. We say that an S-word is *reduced* if every pair of successive letters λ_i, λ_{i+1} has both a nodal and a cuspidal letter. Moreover, we say it is *nodal* (resp. *cuspidal*) if the last letter is nodal (resp. cuspidal); the empty word is both nodal and cuspidal.

To every S-word w we associate the edge $\overline{e}_w = w\,\overline{e}_0$ in $\overline{\mathcal{T}}$, associating $\overline{e}_0$ to the empty word.

Lemma 4.0.19. *Suppose w is a reduced S-word.*

(1) *If w is cuspidal and $\overline{v}$ is the node of $\overline{e}_w$, then the nodal letters $\lambda \in S_0$ correspond bijectively to the other edges of $\overline{\mathfrak{T}}$ from $\overline{v}$ via $\lambda \mapsto \overline{e}_{w\cdot\lambda}$.*
(2) *If w is nodal and $\overline{v}$ is the cusp of $\overline{e}_w$, then the cuspidal letters $\lambda \in S_1$ correspond bijectively to the other edges of $\overline{\mathfrak{T}}$ to $\overline{v}$ via $\lambda \mapsto \overline{e}_{w\cdot\lambda}$.*

PROOF. We will prove the first statement, leaving the analogous proof of the second statement to the reader. Let w be a cuspidal (reduced) S-word and $\overline{v}$ the node of $\overline{e}_w$. Suppose $\lambda \in S_0$ and $w' = w \cdot \lambda$. Then $w\lambda w^{-1}$ is an element of $\Gamma_{\overline{v}}$, hence

$$\overline{e}_{w'} = (w\lambda)\,\overline{e}_0 = (w\lambda w^{-1}w)\,\overline{e}_0 = (w\lambda w^{-1})\,\overline{e}_w$$

is an edge whose node is also $\overline{v}$. The coset $w'B$ is distinct from wB because $w^{-1} \cdot w' = \lambda \notin B$, hence $\overline{e}_w$ and $\overline{e}_{w'}$ are distinct edges by corollary 4.0.18. If $\lambda, \lambda' \in S_0$ are distinct, then $(w \cdot \lambda)^{-1}(w \cdot \lambda') = \lambda^{-1}\lambda' \notin B$, hence $\overline{e}_{w\cdot\lambda}$ and $\overline{e}_{w\cdot\lambda'}$ are distinct edges of $\overline{\mathfrak{T}}$ to $\overline{v}$. □

We say that two edges of $\overline{\mathfrak{T}}$ are adjacent if they are distinct and have a vertex in common.

COROLLARY 4.0.20. $\{$ *reduced S-words* $\} \xrightarrow{1-1} \{$ *positive edges of $\overline{\mathfrak{T}}$ connected to $\overline{e}_0$* $\}$.

PROOF. Suppose $\overline{e}$ is a positive edge of $\overline{\mathfrak{T}}$. Let $\overline{e}_0, \dots, \overline{e}_n$ be the unique sequence of adjacent edges in $\overline{\mathfrak{T}}$ from $\overline{e}_0$ to $\overline{e}_n = \overline{e}$. By lemma 4.0.19 and induction on i, there is a unique reduced S-word $w = \lambda_1 \cdots \lambda_n$ such that $\overline{e}_i = \overline{e}_{\lambda_1\cdots\lambda_i}$ for $0 \le i \le n$. □

We denote the set of reduced S-words by Σ. Combining corollaries 4.0.18 and 4.0.20 we see that Σ is a set of coset representatives of Γ/B.

LEMMA 4.0.21. *Suppose $w, w' \in \Sigma$ are distinct. If $\overline{e}_w, \overline{e}_{w'}$ are not adjacent edges or their common vertex is a node, then $d(\overline{e}_w, \overline{e}_{w'}) = \infty$. Otherwise, $w^{-1}w'B = \gamma_b B$ for a unique $\gamma_b \in S_1$ and $d(\overline{e}_w, \overline{e}_{w'}) = \deg(b) + 1$.*

PROOF. The first part follows from the definition of $d(\overline{e}_w, \overline{e}_{w'})$ (cf. section 3). Suppose have the cusp $\overline{v}$ in common. We may assume either w is nodal or both w, w' are cuspidal.

In the first case, $w' = w\gamma_b$ for a unique $\gamma_b \in S_1$, by part 2 of lemma 4.0.19. Because $d(\cdot,\cdot)$ is Γ-equivariant we have

$$d(\overline{e}_w, \overline{e}_{w'}) = d(w\overline{e}_0, w'\overline{e}_0) = d(w\overline{e}_0, w\gamma_b\overline{e}_0) = d(\overline{e}_0, \gamma_b\overline{e}_0).$$

Therefore it suffices to prove $d(\overline{e}_0, \gamma_b\overline{e}_0) = \deg(b) + 1$ for every $\gamma_b \in S_1$.

In the second case $w = w''\gamma_{b'}$ and $w' = w''\gamma_{b''}$ for a unique nodal $w'' \in \Sigma$ and distinct $\gamma_{b'}, \gamma_{b''} \in S_1$, again by part 2 of lemma 4.0.19. Writing $b = b'' - b'$ we have

$$d(\overline{e}_w, \overline{e}_{w'}) = d(w''\gamma_{b'}\overline{e}_0, w''\gamma_{b''}\overline{e}_0) = d(\overline{e}_0, \gamma_b\overline{e}_0).$$

Again it suffices to prove that $d(\overline{e}_0, \gamma_b\overline{e}_0) = \deg(b) + 1$.

Let $\overline{\mathfrak{F}}$ be the union of the edges $\overline{e}_0, \gamma_b\overline{e}_0$ and $\mathfrak{F}$ the bipartite lift. We denote the lifts of the cusps of $\overline{e}_0, \gamma_b\overline{e}_0$ by $v, \gamma_b v$. From the definition of γ_b we see that it is an element of $\Gamma_{\deg(b)+1} - \Gamma_{\deg(b)}$. Moreover, their meeting point $m(v, \gamma_b v)$ is the only vertex fixed by γ_b on the segments from $v, \gamma_b v$ to v'. By lemma 2.0.1 we must have $m(v, \gamma_b v) \in \mathcal{T}_{\deg(b)+1}$, so $d(v, \gamma_b v) = \deg(b)$. Finally, the identity $d(\overline{e}_0, \gamma_b \overline{e}_0) = d(v, \gamma_b v) + 1$ implies the lemma. □

Recall that $R = k[t]/t^{d-1}$. For any element $f \in R$ and $0 \leq n \leq \deg(f)$, let $f_n \in k$ denote the nth coefficient of f, so that $f(t) = \sum_n f_n \cdot t^n$ in R. We choose a bijection $\sigma : k^\times \to \{1, \ldots, q-1\}$ and impose the lexical ordering on R as follows.

Definition: If $f, g \in R$ are distinct elements, then we define $f < g$ if either of the following conditions hold:

(O1): $\deg(f) < \deg(g)$;
(O2): $\deg(f) = \deg(g)$ and $\sigma(f_{\deg(f-g)}) < \sigma(g_{\deg(f-g)})$.

We consider the induced order on the cuspidal letters: $\gamma_b < \gamma_{b'}$ if $b < b'$.

We call a positive edge *nodal* (resp. *cuspidal*) if the corresponding reduced S-word w is nodal (resp. cuspidal); $\overline{e}_0$ is the unique edge which is both cuspidal and nodal. Every cusp $\overline{v}$ of $\overline{\mathcal{T}}$ belongs to a unique nodal edge $\overline{e}_w$, for lemma 4.0.19 implies that the remaining edges to $\overline{v}$ correspond bijectively to cuspidal letters via $\gamma_b \mapsto \overline{e}_{w \cdot \gamma_b}$. We consider the induced order on the positive edges of $\overline{\mathcal{T}}$ to $\overline{v}$: $\overline{e}_w < \overline{e}_{w\cdot\gamma_b} < \overline{e}_{w\cdot\gamma_{b'}}$ if $\gamma_0 < \gamma_b < \gamma_{b'}$.

Let $\overline{\mathcal{F}} \subset \overline{\mathcal{T}}$ be connected and assume $\overline{\mathcal{F}}_0$ is non-empty. For any cusp $\overline{v}$ let $\overline{\mathcal{F}}(\overline{v})$ denote the union of the edges containing $\overline{v}$.

Lemma 4.0.22. *If $\overline{v}$ is a cusp of $\overline{\mathcal{T}}$ and $\overline{e}_1 \leq \overline{e}_2 \leq \overline{e}_3$ are positive edges of $\overline{\mathcal{T}}(\overline{v})$, then*

$$d(\overline{e}_1, \overline{e}_2) \leq d(\overline{e}_1, \overline{e}_3).$$

Proof. The lemma follows immediately when $\overline{e}_1 = \overline{e}_2$ or $\overline{e}_2 = \overline{e}_3$, hence we assume $\overline{e}_1 < \overline{e}_2 < \overline{e}_3$. It suffices to prove the following: if $\overline{e}_1 < \overline{e}_2, \overline{e}_3$ and $d(\overline{e}_1, \overline{e}_2) > d(\overline{e}_1, \overline{e}_3)$, then $\overline{e}_2 > \overline{e}_3$.

Let $\overline{e}_w$ be the unique nodal edge adjacent to every $\overline{e}_i$ and $b_1 < b_2, b_3$ the unique elements such that $\overline{e}_i = \overline{e}_{w\cdot\gamma_{b_i}}$; $b_1 = 0$ if and only if $\overline{e}_1 = \overline{e}_{w\cdot\gamma_0} = \overline{e}_w$. Let $d_i = \deg(b_i)$ and $d_{i,j} = \deg(b_i - b_j)$. We observe that

$$d_{3,1} = d(\overline{e}_1, \overline{e}_3) - 1 < d(\overline{e}_1, \overline{e}_2) - 1 = d_{2,1},$$

by assumption. Also, by assumption, $\overline{e}_1 < \overline{e}_2, \overline{e}_3$, so lemma 4.0.21 implies $d_1 \leq d_2, d_3$. If $d_1 < d_3$, then

$$d_3 = d_{3,1} < d_{2,1} \leq d_2,$$

so $\overline{e}_2 > \overline{e}_3$. If $d_1 = d_3 < d_2$, then $\overline{e}_2 > \overline{e}_3$. Hence we may assume $d_1 = d_2 = d_3$.

Because $d_{3,1} < d_{2,1}$, b_3 and b_1 have more leading coefficients in common than b_2 and b_1 do. That is, b_3 is closer than b_2 to b_1, in the lexical order. Finally, b_1 precedes both b_2 and b_3, hence b_3 must precede b_2. That is, $\overline{e}_2 > \overline{e}_3$. □

The following is the main theorem of the section.

THEOREM 4.0.23. *Assume* $\overline{\mathcal{F}}$ *satisfies* (D2). *Suppose for every cusp* $\overline{v}$ *that* $\overline{\mathcal{F}}(\overline{v})$ *satisfies the following properties:*

(1) *the positive edges of* $\overline{\mathcal{F}}(\overline{v})$ *are* Δ*-inequivalent;*
(2) *for every positive edge* $\overline{e}$ *of* $\overline{\mathcal{F}}(\overline{v})$ *we have* $\overline{e} = \min\{\gamma\overline{e} : \gamma \in \Delta_{\overline{v}}\}$.

Then $\overline{\mathcal{F}}$ *is* Δ*-compressed.*

PROOF. We must show that $\overline{\mathcal{F}}$ satisfies (D1): for every cusp $\overline{v}$ of $\overline{\mathcal{F}}$ and every pair of positive edges $\overline{e}_1, \overline{e}_2$ in $\overline{\mathcal{F}}(\overline{v})$,

$$d(\overline{e}_1, \overline{e}_2) = \min\{d(\overline{e}_1, \overline{e}_3) : \overline{e}_3 \in \Delta_{\overline{v}}\,\overline{e}_2\}. \tag{4.0.2}$$

Suppose $\overline{e}_1 < \overline{e}_2$ are positive edges of $\overline{\mathcal{F}}(\overline{v})$. By assumption $\overline{e}_2 \leq \gamma\overline{e}_2$ for every $\gamma \in \Delta_{\overline{v}}$, hence applying lemma 4.0.22 with $\overline{e}_3 = \gamma\overline{e}_2$ we obtain (4.0.2) as desired.

For $\overline{e}_2 < \overline{e}_1$ we observe that (4.0.2) is symmetric because $d(\cdot,\cdot)$ is Γ-equivariant and symmetric:

$$\min\{\, d(\overline{e}_1, \gamma\overline{e}_2) : \gamma \in \Delta_{\overline{v}}\,\} = \min\{\, d(\overline{e}_2, \gamma^{-1}\overline{e}_1) : \gamma^{-1} \in \Delta_{\overline{v}}\,\}.$$

Hence we may apply the above argument swapping $\overline{e}_1$ and $\overline{e}_2$. $\square$

It is now quite simple to describe an algorithm for constructing $\overline{\mathcal{F}} = \overline{\mathcal{F}}(\Delta)$ inductively. We choose a node $\overline{v}$ of $\overline{\mathcal{T}}$, say $\overline{v} = \overline{v}_0$, and let $\overline{\mathcal{F}}_{(0)}$ denote the tree whose single vertex is $\overline{v}$. Without loss of generality, we may assume $\overline{v}$ is in $\overline{\mathcal{F}}$. Because $\overline{\mathcal{F}}$ has a finite number of edges, the theorem implies it suffices to construct a strictly increasing sequence of Δ-compressed trees $\overline{\mathcal{F}}_{(0)} \subset \cdots \subset \overline{\mathcal{F}}_{(m)} = \overline{\mathcal{F}}$.

DEFINITION: Suppose $\overline{v}\,'$ is a cusp in $\overline{\mathcal{T}}$. Let $\overline{\mathcal{M}}(\overline{v}\,')$ denote the maximal subtree of $\overline{\mathcal{T}}(\overline{v}\,')$ satisfying property 2 of the theorem.

The idea is to choose a sequence of Δ-inequivalent cusps $\overline{v}\,'_1, \ldots, \overline{v}\,'_m$ in $\overline{\mathcal{T}}$ such that $\overline{\mathcal{M}}(\overline{v}\,'_{n+1})$ is is adjacent to $\overline{\mathcal{F}}_{(n)}$ and to define

$$\overline{\mathcal{F}}_{(n+1)} := \overline{\mathcal{F}}_{(n)} \cup \overline{\mathcal{M}}(\overline{v}\,'_{n+1}).$$

Such a sequence always exists: if $\overline{e}$ is an edge from $\overline{\mathcal{F}}_{(n)}$ to a cusp $\overline{v}\,'_{n+1}$ of $\overline{\mathcal{T}} - \Delta\overline{\mathcal{F}}_{(n)}$, then it is the unique nodal edge in $\overline{\mathcal{T}}(\overline{v}\,'_{n+1})$, so lies in $\overline{\mathcal{M}}(\overline{v}\,'_{n+1})$. Further, the sequence is finite, because the number of cusps m is known to be finite. One uses the following lemma to show inductively that $\overline{\mathcal{F}}_{(n)}$ is Δ-compressed for all n.

LEMMA 4.0.24. *Suppose* $\overline{\mathcal{F}}, \overline{\mathcal{M}}$ *satisfy the hypothesis of theorem 4.0.23. If their cusps are pairwise* Δ*-inequivalent, then* $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$ *satisfies the hypothesis of the theorem.*

PROOF. The edges of $\overline{\mathcal{F}}, \overline{\mathcal{M}}$ are pairwise Δ-inequivalent, because their cusps are pairwise Δ-inequivalent. Let $\overline{\mathcal{G}}$ denote $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$. By assumption, it satisfies (D2). Any cusp $\overline{v}$ of $\overline{\mathcal{G}}$ is either a cusp of $\overline{\mathcal{F}}$ or a cusp of $\overline{\mathcal{M}}$. In the first case $\overline{\mathcal{G}}(\overline{v}) = \overline{\mathcal{F}}(\overline{v})$ and in the second $\overline{\mathcal{G}}(\overline{v}) = \overline{\mathcal{M}}(\overline{v})$. In either case $\overline{\mathcal{G}}(\overline{v})$ satisfies the properties in theorem 4.0.23, hence $\overline{\mathcal{G}}$ is Δ-compressed. $\square$

4.1. Minimal Spanning Tree (Sketch/Draft). In order to ensure that we can find a minimal spanning tree, we must choose the sequence of cusps $\overline{v}'_1, \ldots, \overline{v}'_m$ carefully. Starting with $\overline{v}_0$ we choose cusps adjacent to $\overline{v}_0$. Once we have exhausted the Δ-equivalence classes of all such cusps, so that the resulting tree is $\overline{\mathcal{F}}_{(i)}$, we choose a new node $\overline{v}$ in $\overline{\mathcal{F}}_{(i)}$ and precede as before.

LEMMA 4.1.1. *For each Δ-equivalence class of nodes in $\overline{\mathcal{F}}$, there is at most one node adjacent to at least two edges of $\overline{\mathcal{F}}$.*

PROOF. Indeed, the only time a node is adjacent to two or more edges is when it is one of the nodes fixed in the construction for which we add all adjacent cusps. Hence if we later chose a Δ-equivalent node, then we will have already added cusps Δ-equivalent to every adjacent cusp, hence will add no extra edges. □

For each node $\overline{v}$ adjacent to two or more edges (in $\overline{\mathcal{F}}$) we must discard the edges adjacent to all Δ-equivalent nodes distinct from $\overline{v}$.

CLAIM 4.1.2. *Every node $\overline{v} \in \overline{\mathcal{F}}$ adjacent to exactly one edge $\overline{e} \in \overline{\mathcal{F}}$ lies on the 'boundary'.*

If all nodes in a given Δ-equivalence class are adjacent to exactly one edge, then we may choose any one node, its adjacent edge, and discard the edges adjacent to the other Δ-equivalent nodes.

- If I discard an edge, then it means that its node is Δ-equivalent to another node of $\overline{\mathcal{F}}$.
- I recognize that a node is Δ-equivalent to another node by recognizing that adjacent edges are Δ-equivalent. Hence I need to keep track of the elements of Δ taking the respective edges to each other.

4.2. Hecke Operators. The important data with respect to a minimal spanning tree $\overline{\mathcal{F}}_t$ is the collection of edges $\mathcal{E}$ in the complement $\overline{\mathcal{F}} - \overline{\mathcal{F}}_t$. The genus g of $Y(\Delta)$ is equal to the number of edges in $\mathcal{E}$.

Given two Δ-equivalent nodes $\overline{v}_1, \overline{v}_2 \in \overline{\mathcal{F}}$ we need an algorithm for determining whether an edge $\overline{e}$ lies on the geodesic from $\overline{v}_1$ to $\overline{v}_2$. If it doesn't, we define $\chi(\overline{v}_1, \overline{v}_2; \overline{e}) = 0$. Otherwise, we also need to know its orientation with respect to the geodesic, i.e. whether its node or cusp is closer to $\overline{v}_1$. In the first case we define $\chi(\overline{v}_1, \overline{v}_2; \overline{e}) = 1$ and in the second we define $\chi(\overline{v}_1, \overline{v}_2; \overline{e}) = -1$.

Given an element $\gamma \in \mathrm{GL}_2(K) \cap M_2(A)$ we need an algorithm for finding $\gamma' \in \mathrm{GL}_2(A)$ and $m \in \mathbb{Z}$ such that

$$\gamma \cdot \mathrm{PGL}_2(\mathcal{O}_\infty) = \gamma' \begin{pmatrix} T^m & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{PGL}_2(\mathcal{O}_\infty).$$

For $\mathrm{PGL}_2(\mathcal{O}_\infty)$ is the stabilizer of the 'canonical' edge e_0 and such cosets correspond bijectively to (oriented) edges of $\mathcal{T}$.

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