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MODULAR SYMBOLS FOR $F_q(T)$

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Introduction. Modular symbols were invented by Manin as a tool for studying the arithmetic of modular forms for $SL_2(\mathbf{Z})$. Described in detail in [Man], these modular symbols are elements of the relative homology group $H_1(X, \text{cusps})$ where X is a modular curve obtained as the quotient of the complex upper half-plane by a congruence subgroup of $SL_2(\mathbf{Z})$. This group is generated by the classes of paths joining a to b where a and b belong to $\mathbf{Q} \cup \{\infty\}$. Manin supplied an explicit finite set of generators for $H_1(X, \text{cusps})$ of this type, as well as the complete set of relations among them. This makes it very simple to calculate this H_1 explicitly. His construction also makes the action of the Hecke operators transparent. Using this information, one can exploit the duality between modular forms and $H_1(X, \text{cusps})$ to calculate the structure of the space of modular forms on X.

Modular symbols have proven to be extremely useful for a variety of purposes. For example, Mazur and Swinnerton-Dyer used modular symbols to define p-adic L-functions associated to modular forms; these L-functions play a role in a variety of interesting "p-adic Birch and Swinnerton-Dyer" type conjectures such as those studied initially in [MTT] and [MT]. A generalized, "Λ-adic" modular symbol is a crucial ingredient in the proof by Stevens and Greenberg of the (weight-two) "exceptional zero conjecture" of [MTT]. Modular symbols play a fundamental role in studying congruences between Eisenstein series and cusp forms, as in the work of Mazur ([Maz3]) and Stevens ([St]). In addition, Cremona ([Cr]) has used modular symbols as the basic tool in large scale computations of the Hecke module structure of the space of (weight-two) cusp forms for $\Gamma_0(N)$.

In this paper, we describe a theory of modular symbols for Drinfeld's "upper half-plane" Ω over $\mathbf{F}_q[[1/T]]$. Our modular symbols, which are constructed using the tree $\mathscr{T}$ of $PGL_2(\mathbf{F}_q((1/T)))$, are related to a certain class of cusp forms for GL_2 over (the adeles of) $\mathbf{F}_q(T)$ in the same way that Manin's symbols are related to cusp forms of weight two for GL_2 over the adeles of $\mathbf{Q}$. Many of the applications of Manin's modular symbols transfer immediately to our situation. For example, modular symbols yield a practical method for calculating the Hecke-module structure of certain spaces of cusp forms for GL_2 over $\mathbf{F}_q(T)$, making possible the construction of tables of elliptic curves over $\mathbf{F}_q(T)$ like the famous Antwerp IV tables of elliptic curves over $\mathbf{Q}$ ([MF4]). Gekeler ([Gek2]), using other methods, has constructed tables of this type, but we believe that computations based on modular symbols are more efficient for large-scale, machine-based numerical investigations.

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In support of this claim, we refer to Cremona's success with these techniques in the classical setting ([Cr]).

In addition, our modular symbols yield information on congruences between cusp forms and Eisenstein series and can be used to construct distributions on local rings of $\mathbf{F}_q(T)$.

Perhaps the most important difference between the modular symbols we construct and the classical ones is that we find only "even" symbols. Thus our space of modular symbols is one-half the dimension one would expect from a direct comparison with the classical case. The missing symbols reflect the fact that only half of the cohomology of the upper half-plane Ω can be seen on the tree $\mathcal{T}$; the other half is tied up in étale coverings of Ω which our modular symbols cannot detect.

In the first section of this paper, we establish our notation and conventions, and very briefly recall the relationship between harmonic functions on the tree $\mathcal{T}$ and automorphic forms. In the second section, we define modular symbols and deduce their basic properties. In our main theorems, we prove that the space of $\mathbf{Q}$-valued modular symbols is dual to the space of $\mathbf{Q}$-valued harmonic functions, and we give explicit generators and relations for the space of modular symbols. The first of these results is based on a pairing between the modular symbols for a congruence subgroup Γ of $\mathrm{GL}_2(\mathbf{F}_q[T])$, and the Γ-invariant cuspidal harmonic functions on the tree $\mathcal{T}$. This pairing is nondegenerate over $\mathbf{Q}$, and the determinant of the pairing is intimately related to the arithmetic of Drinfeld's modular curve $X_\Gamma = \Gamma\backslash\Omega$. Indeed, the degree to which this pairing fails to be perfect over $\mathbf{Z}$ is measured by the image of the cuspidal divisor group of X_Γ in the group of connected components of the Neron model of $\mathrm{Jac}(X_\Gamma)$. The generators and relations we describe for modular symbols are formally identical to Manin's original results for the classical symbols; we derive them by a careful study of the action of $\mathrm{GL}_2(\mathbf{F}_q[T])$ on $\mathcal{T}$.

The final section of the paper consists of examples of the various applications of modular symbols. First, we describe how modular symbols are used to compute the special values $L(f, 1, \chi)$ of L-functions associated to twists of a particular automorphic form f. This is a function field version of Birch's formula, and using it one may construct tables like those in Stevens's book ([St]) of special values. Next, we show how modular symbols may be used to construct distributions which, by the function field version of the conjectures in [MTT] and [MT], ought to be related to v-adic periods. We provide a numerical example supporting this conjecture. Evidence for these conjectures has been obtained by a different method by Rockmore and Tan ([TR]). We might say that Rockmore and Tan begin with the equation of an elliptic curve, and then work hard to find the associated modular form; we are able to find the modular form for an elliptic curve rather easily, but then must search hard for the equation of that curve.

At the end of the paper, we observe a congruence on modular symbols analogous to that discovered by Manin and proved by Mazur. Presumably, the proper function field version of Mazur's proof will establish these congruences as well; building the delicate machinary necessary for such a proof is a major open problem in the theory of Drinfeld modular curves.

Notation and conventions. Let $k = \mathbf{F}_q(T)$ be the field of rational functions in T over the finite field $\mathbf{F}_q$. Fix the valuation "at infinity" defined by $|T| = q$, let k_∞ be the completion of k at infinity, and let $\mathcal{O}$ be the valuation ring in k_∞. We also let $A = \mathbf{F}_q[T]$ denote the ring of polynomials in T.

We will be working extensively with the tree $\mathcal{T}$ associated to $\mathrm{PGL}_2(k_\infty)$. Recall that the vertices of $\mathcal{T}$ are the cosets $\mathrm{PGL}_2(k_\infty)/\mathrm{PGL}_2(\mathcal{O})$, and the oriented edges of $\mathcal{T}$ are the cosets $\mathrm{PGL}_2(k_\infty)/\mathcal{N}$ where

$$\mathcal{N} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}) : |c| < 1\right\}.$$

As is well known, $\mathcal{T}$ is a homogeneous tree, with $q + 1$ edges leaving every vertex.

The group $\mathrm{GL}_2(A)$ acts on $\mathcal{T}$ on the left. We will be interested in the "Hecke-type" subgroups of $\mathrm{GL}_2(A)$. If N is an ideal of A, we will use the standard notations:

$$\Gamma_0(N) = \left\{\gamma : \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (\mathrm{mod}\ N)\right\},$$

$$\Gamma_1(N) = \left\{\gamma : \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N)\right\},$$

$$\Gamma(N) = \left\{\gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N)\right\}.$$

By a *congruence subgroup* of $\mathrm{GL}_2(A)$ we will mean a subgroup containing some $\Gamma(N)$.

Fix a vertex x_0 of $\mathcal{T}$. An infinite path, without backtracking, which departs from x_0 is called an *end* of $\mathcal{T}$. The ends of $\mathcal{T}$ are naturally identified with the points of $\mathbf{P}^1(k_\infty)$. The rational ends of $\mathcal{T}$ are those ends which correspond to points of $\mathbf{P}^1(k) \subset \mathbf{P}^1(k_\infty)$. The action of $\mathrm{GL}_2(A)$ on $\mathcal{T}$ extends naturally to the ends of $\mathcal{T}$. All of this data is independent of the choice of the base vertex x_0.

Throughout this paper, we will rely on [Se] for essential facts about the action of $\mathrm{GL}_2(A)$ on $\mathcal{T}$ and its ends.

Much of the arithmetic interest in $\mathcal{T}$ derives from Drinfeld's theory of modular curves for $\mathrm{GL}_2(A)$. Drinfeld has constructed an "upper half-plane" Ω which is closely related to the group $\mathrm{GL}_2(A)$ and to the tree $\mathcal{T}$. In particular, if Γ is a congruence subgroup of $\mathrm{GL}_2(A)$, then Γ acts on Ω and one may construct a quotient $Y_\Gamma = \Gamma\backslash\Omega$. Y_Γ is an affine algebraic curve, and it may be compactified by the addition of finitely many cusps to obtain a complete algebraic curve X_Γ. In addition, there is a map from Ω to $\mathcal{T}$, called the "reduction" map, which is compatible with the action of Γ, and yields a map from Y_Γ to $\Gamma\backslash\mathcal{T}$. For details of this construction, we refer the reader to the works of Drinfeld, Goss, and Gekeler cited in the references. We will call the curves X_Γ "Drinfeld modular curves." If Γ is one of the three special types of subgroups mentioned above, we will refer to the quotient curve $\Gamma\backslash\Omega$ as $X_0(N)$, $X_1(N)$, or $X(N)$, as appropriate.

In the classical case, modular symbols are used to study modular forms. Before introducing our symbols, then, we will describe the modular forms in our situation.

Definition 1. Let R be an abelian group. An R-valued function f on the edges of $\mathcal{T}$ is called an R-valued *harmonic cocycle* if

1. f is alternating; that is, $f(e^*) = -f(e)$ if e^* denotes the opposite of the edge e.
2. f is harmonic; that is, for each vertex v,

$$\sum_{e \mapsto v} f(e) = 0$$

where the sum is over the edges leaving v.

Suppose that Γ is a subgroup of $\mathrm{GL}_2(A)$. If f is Γ-invariant and supported on finitely many edges mod Γ, then f is said to be a cuspidal harmonic cocycle for Γ. We will let $C_{\mathrm{har}}(\Gamma, R)$ denote the group of such functions.

Of particular interest is the group $C_{\mathrm{har}}(\Gamma_0(N), \mathbf{C})$ where $\mathbf{C}$ is the complex numbers. As is explained in [Gek2], $C_{\mathrm{har}}(\Gamma_0(N), \mathbf{C})$ is isomorphic to a certain space of automorphic forms on the adelic group $\mathrm{GL}_2(\mathbf{A}_k)$. By means of this isomorphism, it makes sense to speak of "newforms" in $C_{\mathrm{har}}(\Gamma_0(N), \mathbf{C})$. Before we can state the arithmetic significance of these automorphic forms, we must introduce the Hecke operators.

Let N be an ideal of A and let π be a prime of A which is relatively prime to N. Suppose $f \in C_{\mathrm{har}}(\Gamma_0(N), \mathbf{C})$ is a harmonic cocycle. Let P be the monic irreducible polynomial generating π and define

$$T_\pi f(e) = f\left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} e\right) + \sum_{Q \bmod P} f\left(\begin{pmatrix} 1 & Q \\ 0 & P \end{pmatrix} e\right).$$

This defines the action of the Hecke operator T_π. The operators T_π generate a commutative algebra $\mathbf{T}$ which acts as correspondences on $\Gamma_0(N)\backslash\mathcal{T}$ and on $X_0(N)$, and as linear endomorphisms on $C_{\mathrm{har}}(\Gamma_0(N), \mathbf{C})$.

As is discussed in [Gek2], Drinfeld has proved the Weil-Taniyama-Shimura conjecture for function fields. This means that we may take advantage of the following theorem relating the action of the Hecke operators on harmonic cocycles to elliptic curves.

THEOREM 2. (Drinfeld) *There is a one-to-one correspondence between isogeny classes of elliptic curves E over k having split multiplicative reduction at the place at infinity of k and conductor N away from infinity, and newforms f in $C_{\mathrm{har}}(\Gamma_0(N), \mathbf{C})$ on which $\mathbf{T}$ acts by rational numbers. An elliptic curve E belongs to the isogeny class corresponding to a newform f if and only if, for all primes π of A which are prime to*

N, we have

$$|E(A/\pi)| = q^{\deg(\pi)} + 1 - a_\pi$$

where $T_\pi = a_\pi f$.

Definition of modular symbols. In this section we introduce our modular symbols and establish their basic properties.

Definition 3. The group M of modular symbols is the group of divisors of degree zero supported on the k-rational points of $\mathbf{P}_k^1$. We will let $[r, s]$ denote the divisor $(s) - (r)$.

Most of our arguments in this paper are based on a geometric realization of modular symbols. We think of the modular symbol $[r, s]$ as a path in the tree $\mathcal{T}$ running from the rational end r to the rational end d; there is only one such path.

Suppose x is a subtree of $\mathcal{T}$ with finitely many ends, all of which are rational. Suppose in addition we are given a function α on the edges of x which is nonzero, alternating (meaning that $\alpha(e^*) = -\alpha(e)$) and harmonic (meaning that the sum of the values of α on the edges leaving each vertex is zero). It is not hard to see that the data $\{x, \alpha\}$ determines an element m of M in a unique way. We will frequently view elements m of M as this sort of labeled subtree of $\mathcal{T}$.

The analogy between our modular symbols and Manin's should be clear. In each case, modular symbols are "geodesics" connecting points on the "boundary" of the "upper half-plane."

Definition 4. Suppose that $\Gamma \subset \mathrm{GL}_2(A)$ is a congruence subgroup and that R is a Γ-module. Then the group

$$M_\Gamma(R) = \mathrm{H}_0(\Gamma, M \otimes R)$$

of coinvariants of $M \otimes R$ relative to the Γ action will be called the group of *R-valued modular symbols* for Γ.

The structure of M_Γ. In this section we recall and slightly generalize results of Serre ([Se, Chapter II.2]) on the structure of the group M_Γ. Serre defines the "Steinberg module" of $\mathrm{GL}_2(k)$ to be the kernel of the augmentation map

$$\mathbf{Z}[\mathbf{P}_k^1] \to \mathbf{Z}.$$

Clearly, this kernel is precisely our group of modular symbols. To determine the structure of M and M_Γ, we rely on Serre's alternative resolution of M by Γ-modules, which is based on the decomposition of the tree $\mathcal{T}$ into "stable" and "unstable" regions. Serre limits himself to considering groups which are free from torsion of order prime to p, but we work more generally since we would like to apply our results to $\Gamma_0(N)$. Therefore, we adapt Serre's definition of "stability" to this more general situation.

Definition 5. A vertex v (resp. edge e) is said to be "stable" (relative to Γ) if the stabilizer of v (resp. e) in Γ has order prime to p.

Since any element of Γ fixing an edge e also fixes its bounding vertices, the unstable vertices and edges of $\mathscr{T}$ form a union of connected subtrees of $\mathscr{T}$ which we denote by $\mathscr{T}_\infty$. When Γ is free from torsion of order prime to p, Serre proves that the group $\mathrm{H}_0(\mathscr{T}_\infty, \mathbf{Z})$ of connected components of $\mathscr{T}_\infty$ is isomorphic to $\mathbf{Z}[\mathbf{P}_k^1]$. This is not the case for general Γ; for example, $\mathscr{T}_\infty$ for the full group $\mathrm{GL}_2(A)$ is the entire tree $\mathscr{T}$. However, the argument of [Se, Chapter II.2, Lemma 13] will work provided we make the following assumption on Γ:

$$\Gamma \text{ is contained in some } \Gamma_0(N). \tag{†}$$

We will adopt this hypothesis on Γ from now on.

Following Serre, let L_* be the natural complex with L_1 and L_0 the free abelian groups on the stable edges and vertices of $\mathscr{T}$. Serre proves that there are only finitely many stable vertices and edges modulo the action of Γ. If we let $C_*(\mathscr{T}, \mathbf{Z})$ and $C_*(\mathscr{T}_\infty, \mathbf{Z})$ denote the chain complexes on $\mathscr{T}$ and $\mathscr{T}_\infty$ respectively, we see that the complex L_* fits into the exact sequence

$$0 \to C_*(\mathscr{T}_\infty, \mathbf{Z}) \to C_*(\mathscr{T}, \mathbf{Z}) \to L_* \to 0.$$

The corresponding long exact sequence in homology expresses M as $H_1(L_*)$ and shows that $\mathrm{H}_0(L_*) = 0$. Suppose that we compute the group cohomology $H_1(\Gamma, L_*)$ of Γ with coefficients in the complex L_*. There are two ways to do this, corresponding to the two spectral sequences

$$\left.\begin{aligned} E^2_{pq} &= \mathrm{H}_p(\Gamma, \mathrm{H}_q(L_*)) \\ E^1_{pq} &= \mathrm{H}_q(\Gamma, L_p) \end{aligned}\right\} \Rightarrow \mathrm{H}_{p+q}(\Gamma, L_*).$$

We leave it to the reader to verify that the first of these proves that $\mathrm{H}_1(\Gamma, L_*) = M_\Gamma$.

We will apply the second spectral sequence in a moment. Before doing so, let us observe that, by Shapiro's lemma and Serre's observation that the stable edges and vertices are finite in number mod Γ, the groups $\mathrm{H}_1(\Gamma, L_i)$ are finite for $i = 0$ and $i = 1$. Since both of the $H_0(\Gamma, C_i(\mathscr{T}_\infty, \mathbf{Z}))$ are torsion-free, we obtain from the long exact sequence of group homology two exact sequences

$$0 \to \mathrm{H}_0(\Gamma, C_i(\mathscr{T}_\infty, \mathbf{Z})) \to \mathrm{H}_0(\Gamma, C_i(\mathscr{T}, \mathbf{Z})) \to \mathrm{H}_0(\Gamma, L_i) \to 0 \qquad i = 0, 1.$$

This tells us that the $\mathrm{H}_0(\Gamma, L_i)$ are precisely the relative chains for the pair $(\Gamma\backslash\mathscr{T}, \Gamma\backslash\mathscr{T}_\infty)$, and therefore that

$$\mathrm{H}_1(\Gamma\backslash\mathscr{T}, \Gamma\backslash\mathscr{T}_\infty, \mathbf{Z}) = \mathrm{Ker}(\mathrm{H}_0(\Gamma, L_1) \to \mathrm{H}_0(\Gamma, L_0)).$$

The significance of this relative homology group is explained by the following lemma.

LEMMA 6. *The subgraph* $\Gamma\backslash\mathscr{T}_\infty$ *of* $\Gamma\backslash\mathscr{T}$ *can be retracted to the finitely many ends of* $\Gamma\backslash\mathscr{T}$.

Proof. Serre describes a retraction of $\mathscr{T}_\infty$ onto the rational ends of $\mathscr{T}$. To see how this is done, pick an unstable vertex v in $\mathscr{T}$. Then the p-Sylow subgroup of $\mathrm{Stab}(v)$ fixes a unique rational end of $\mathscr{T}$ (here we are using our assumption (†) on Γ), and the vertex v retracts to this end along the unique path joining it to v.

To establish our lemma, it suffices to prove that $\Gamma\backslash\mathscr{T}_\infty$ is simply connected. If we know this, then there will be a unique path in $\Gamma\backslash\mathscr{T}_\infty$ joining a vertex v to a rational end, and we may define a retraction along this path. Suppose therefore that we have a cycle ε in the quotient. We may lift ε back to a path joining two vertices v_1 and v_2 in $\mathscr{T}_\infty$, such that $\gamma v_1 = v_2$ for some $\gamma \in \Gamma$. Clearly, v_1, and v_2 belong to the same connected component of $\mathscr{T}_\infty$. Therefore, there is a unique rational end x of $\mathscr{T}$ and element $\tau_1 \in \mathrm{Stab}(v_1)$ such that x is the unique fixed point of τ_1 in $\mathbf{P}^1_k$. Since $\gamma\tau_1\gamma^{-1}$ fixes v_2 and v_2 is in the same component of $\mathscr{T}_\infty$ as v_1, the element $\gamma\tau_1\gamma^{-1}$ must also fix x. Therefore $\gamma x = x$. Consider now the paths p_1 and p_2 from x to v_1 and v_2 respectively. Our original path from v_1 to v_2 is equivalent to the path $p_2 - p_1$, but this latter path is trivial in the quotient $\Gamma\backslash\mathscr{T}_\infty$. Thus we see that $\Gamma\backslash\mathscr{T}_\infty$ is simply connected.

Using this lemma, we see that

$$\mathrm{H}_1(\Gamma\backslash\mathscr{T}, \Gamma\backslash\mathscr{T}_\infty, \mathbf{Z}) = \mathrm{H}_1(\Gamma\backslash\mathscr{T}, \text{cusps}, \mathbf{Z})$$

is the cohomology of $\Gamma\backslash\mathscr{T}$ relative to its finitely many ends; these ends correspond to the cusps of Γ.

Let us finally apply the first of our two spectral sequences for computing $\mathrm{H}_*(\Gamma, L_*)$. We obtain from it, and our computations above, the exact sequence

$$\mathrm{H}_1(\Gamma, L_0) \to M_\Gamma \to \mathrm{H}_1(\Gamma\backslash\mathscr{T}, \text{cusps}, \mathbf{Z}) \to 0.$$

Also, as we have previously observed, the group on the left of this sequence is finite, and so M_Γ is, up to torsion, precisely the homology of $\Gamma\backslash\mathscr{T}$ relative to the cusps.

The dimension of M_Γ is given by the formula

$$\dim M_\Gamma \otimes \mathbf{Q} = g(X_\Gamma) + h(X_\Gamma) - 1$$

where $g(X_\Gamma)$ and $h(X_\Gamma)$ are respectively the genus of X_Γ and the number of cusps of Γ. Formulas for these numbers were found by Goss; the reader may look them up in [Go1].

Having identified M_Γ as (more or less) the relative homology of the quotient graph, it is reasonable to ask for the absolute homology. This is supplied by the group of "closed" modular symbols, which we define here.

Definition 7. Let $M_\Gamma^0(R)$ denote the subgroup of $M_\Gamma(R)$ generated by the $[r, s]$ where r and s are Γ-equivalent. For simplicity, let M_Γ^0 denote $M_\Gamma^0(\mathbf{Z})$.

By tracing back through the work we have just done, it is not hard to verify that the quotient of M_Γ^0 by its torsion subgroup is isomorphic to the absolute homology group $H_1(\Gamma\backslash\mathcal{T}, \mathbf{Z})$.

I see some arguments in favor of defining the modular symbols to be the torsion-free quotient of M_Γ, rather than M_Γ. Manin makes such a choice for the classical symbols. I elected not to remove the torsion because, as we hope to explain in a later paper, it contains interesting information about the existence of "unusual" modular forms with coefficients in, say, $\mathbf{Z}/l\mathbf{Z}$ for some prime l different from p. Fortunately, the torsion in M_Γ is usually very small. Indeed, the torsion is $GL_2(\mathbf{F}_q[T])$ is killed by $q^2 - 1$, and so this also kills the torison in M_Γ. In specific cases, one can account for the torsion exactly. For example, using results of Gekeler [Gek2], one can show that, if $\Gamma = \Gamma_0(N)$ for N a prime of odd degree, then M_Γ is torsion-free, while if N is a prime of even degree, then the torsion subgroup is cyclic of order $q + 1$.

Modular symbols and harmonic cocycles. Now that we understand the structure of the group of modular symbols, we may begin to relate them to harmonic cocycles. This relationship is expressed by a pairing between the modular symbols and the harmonic cocycles. In the classical case, this pairing is the deRham pairing given by integration over cycles; we view our pairing as an integration pairing as well, but the actual calculation of the integral is rather trivial.

Definition 8. Suppose $[r, s]$ is a modular symbol and f is an R-valued, cuspidal harmonic cocycle of level N. Then we define

$$\int_r^s f = \text{the sum of the values of } f \text{ along } [r, s],$$

and we extend this "integral" to M_Γ by linearity.

LEMMA 9. *The pairing in Definition* 8 *is well defined. In addition, it enjoys the following properties*:

1. *We have the relation*

$$\int_r^s f + \int_s^t f = \int_r^t f.$$

2. *Let* $\gamma \in GL_2(k)$. *Then*

$$\int_{\gamma(r)}^{\gamma(s)} f = \int_r^s f \circ \gamma.$$

3. *The pairing is symmetric with respect to the action of the Hecke algebra. Suppose for example that* $\Gamma = \Gamma_0(N)$ *for some* N *and that* $\pi = (P)$ *is prime to* N. *Then if*

we define

$$T_\pi[r, s] = [Pr, Ps] + \sum_{Q \bmod P} \left[\frac{r+Q}{P}, \frac{s+Q}{P}\right],$$

we have

$$T_\pi[r, s] \cdot f = [r, s] \cdot T_\pi f.$$

Proof. We know that the quotient $\Gamma_0(N)\backslash\mathcal{T}$ is a finite graph with finitely many ends attached and that these ends correspond to the $\Gamma_0(N)$ equivalence classes of the rational ends of $\mathcal{T}$. Since f is $\Gamma_0(N)$ invariant, the pairing $\int_r^s f$ can be computed on the quotient tree $\Gamma_0(N)\backslash\mathcal{T}$. In addition, since f is cuspidal, it vanishes off all but finitely many edges of this quotient tree. Consequently, the sum involved in computing the "integral" is a finite sum, and the pairing is well defined.

To obtain the first property, observe that traveling from r to s, and then from s to t, takes you from r to t. However, there is a unique path from r to t which does not backtrack. Consequently, adding the first two integrals together yields the sum of the values of f along the unique nonbacktracking path from r to t, with the contribution from any other edges canceled out.

The second property simply reflects the fact that the $\mathrm{GL}_2(k)$ action on the ends is compatible with the action on the edges, and the third property follows immediately from the second since T_π acts through $\mathrm{GL}_2(k)$.

In the next section, we explore some of the arithmetic implications of the pairing between harmonic cocycles and modular symbols.

Modular symbols and the cuspidal divisor group. In this section we employ techniques of Kurihara ([K]) to relate the pairing between modular symbols and harmonic cocycles for Γ to the group of connected components in the Jacobian of X_Γ.

In order to simplify matters, we will restrict the class of Γ which we will consider beyond the restriction imposed in (†) above. Namely, we now require that, for some ideal N of A, we have

$$\text{either } \Gamma_1(N) \subset \Gamma \subset \Gamma_0(N) \qquad \text{or } \Gamma = \Gamma(N). \tag{††}$$

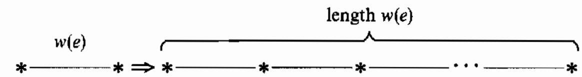
This enables us to consider the most important groups from the point of view of arithmetic, without worrying too much about unpleasant details.

Our calculations in this section are based on finding the intersection graph of a regular model for X_Γ over the ring $R = \mathbf{F}_q[[1/T]]$, and then using this graph to relate the integration pairing to the group of connected components in $\mathrm{Jac}(X_\Gamma)$. To simplify the notation somewhat, let $G^*(\Gamma)$ be the quotient graph $\Gamma\backslash\mathcal{T}$.

Definition 10. Let $G(\Gamma)$ be the smallest subgraph of $G^*(\Gamma) = \Gamma\backslash\mathcal{T}$ containing the stable region of G^*. Assign to every edge of $G(\Gamma)$ the label (or "length")

$$w(e) = |\mathrm{Stab}(e)|.$$

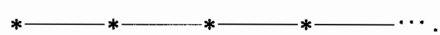
We now describe an adjustment process for the graph $G(\Gamma)$. Suppose e is an edge of $G(\Gamma)$ of length $w(e)$. Subdivide e by adding $w(e) - 1$ vertices in the middle of it; this replaces the single edge e of length $w(e)$ with $w(e)$ edges of length 1:



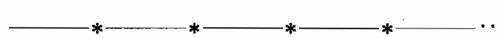
Let $G^{\mathrm{reg}}(\Gamma)$ be the graph which results from making these modifications to $G(\Gamma)$.

PROPOSITION 11. *Assuming that the genus of* X_Γ is greater than 0 *and that hypothesis* (††) *is satisfied, the finite graph* $G^{\mathrm{reg}}(\Gamma)$ *is the intersection graph of a regular model of* X_Γ *over* R.

Proof. We minic here the proof of the characteristic-zero version of this result, due to Kurihara. The only slight complication comes from the cusps. We begin by considering the cases $\Gamma = \Gamma(I)$ or $\Gamma = \Gamma_1(I)$. Since both of these groups are free from torsion of order prime to p, the stabilizer of any stable edge is trivial. This obviously means that $G^{\mathrm{reg}}(\Gamma) = G(\Gamma)$. In addition, each of these groups has the property that the stabilizer of any cusp c consists of the full group of translations $z \mapsto z + b$ for $b \in J$, where J is an ideal of A. This property implies that the stabilizer of an unstable vertex v fixes one edge adjacent to v and transitively permutes the remaining adjacent edges. Consequently, the quotient of the unstable subgraph $\mathcal{T}_\infty$ of $\mathcal{T}$ by Γ consists of finitely many rays of the form



Furthermore, the complement of $G(\Gamma)$ in $G^*(\Gamma)$ consists of finitely many open rays of the form

 (1)

where the initial edge is stable. It follows from Lemma 24 of [Tei] that, in the open curve Y_Γ, the set of points which reduce to an open ray of this form is a punctured open disk, and that in the compactified curve X_Γ the puncture is "filled in." We will now define a covering of X_Γ by affinoids. For each stable edge e in $G^*(\Gamma)$, let $U(e)$ be the set of points which reduce to e or its two bounding vertices. In case both of these vertices are stable, it is clear that $U(e)$ is an affinoid, since $U(e)$ is isomorphic to a similarly defined subset of Ω. If e meets one stable vertex v and one unstable vertex v', it is not hard to see that $U(e)$ is the affinoid $U(v)$ reducing to v with one of the $q + 1$ "holes" in $U(v)$ filled in by a closed disk. Consequently, $U(e)$ is an affinoid in this case as well. Our assumption that the genus of X_Γ is greater than zero rules out the possibility of a stable edge meeting two unstable vertices.

Each $U(e)$ has a canonical reduction. If both bounding vertices of e are stable, then this reduction is two affine lines, punctured at all but one of the rational points and meeting with rational tangents at the remaining one. If one of the vertices is unstable, $U(e)$ is an affine line punctured at all but one of the rational points. Consequently, the formal scheme associated to X_Γ is regular and has intersection graph $G^{\text{reg}}(\Gamma) = G(\Gamma)$, and algebraization ([D1]) gives a regular model for X_Γ with intersection graph G.

To finish our work, we must consider the case where Γ contains $\Gamma_1(N)$, but is not free from torsion of order prime to p. Here we follow Kurihara ([K]). Since $H = \Gamma/\Gamma_1(N)$ is of order prime to p, we obtain a model for X_Γ by taking the quotient of $X_1(N)$ by H, and the resulting model will have intersection graph $H \backslash G(\Gamma)$, which we label with the orders of the stabilizers. Then it follows from Kurihara's work that resolving the quotient singularities introduced by H leads to a model with graph $G^{\text{reg}}(\Gamma)$ obtained by subdividing the edges of $G(\Gamma_1(N))$ in the way that we have described.

We will apply our construction of a regular model for X_Γ in order to obtain information about the group of connected components of the Neron model of its Jacobian, and to relate this group to modular symbols.

LEMMA 12. *Let $C_{\text{har}}^{\text{reg}}(\Gamma)$ be the integer-valued harmonic functions on the edges of the graph $G^{\text{reg}}(\Gamma)$ and let $\Phi(\Gamma)$ be the finite group of connected components of the Neron model of* Jac(X_Γ). *Then there is an exact sequence*

$$0 \to \mathrm{H}_1(G^{\text{reg}}(\Gamma), \mathbf{Z}) \xrightarrow{h} \mathrm{Hom}(C_{\text{har}}^{\text{reg}}(\Gamma), \mathbf{Z}) \to \Phi(\Gamma) \to 0.$$

If x is a cycle in G^{reg}, one evaluates $h(x)$ on a harmonic function f by summing the values of f on the edges in G^{reg} which make up x.

Proof. This is easily seen to be equivalent to Raynaud's formula for Φ([R]).

In order to relate Φ to modular symbols, we proceed in two steps. First, we recover the Γ-invariant, cuspidal harmonic cocycles from certain functions on the subgraph $G(\Gamma)$ of $\Gamma \backslash \mathscr{T}$; then we relate these functions on $G(\Gamma)$ to functions on $G^{\text{reg}}(\Gamma)$.

The first step of this program is quite simple. Let C be the space of alternating functions on the edges of $G(\Gamma)$ which satisfy the relation

$$\sum_{e \mapsto v} \frac{1}{w(e)} f(e) = 0. \tag{2}$$

(Here, as usual, the sum is over the oriented edges e meeting v.) A function in C can be extended by zero to give a function on the edges of $\Gamma \backslash \mathscr{T}$ and then pulled back to the edges of $\mathscr{T}$. It is easy to see that this gives an isomorphism between C and $C_{\text{har}}(\Gamma, \mathbf{Z})$.

The second step in our program is to relate the functions on $G(\Gamma)$ satisfying equation (2) to the harmonic functions on $G^{\text{reg}}(\Gamma)$. The graph $G^{\text{reg}}(\Gamma)$ is obtained from $G(\Gamma)$ by subdividing some edges. Therefore, an integral-valued function f on $G(\Gamma)$ gives rise to a function $\hat{f}$ on $G^{\text{reg}}(\Gamma)$ which takes values in $\mathbf{Q}$. This function $\hat{f}$

is defined by setting $\hat{f} = 1/w(e)$ on the $w(e)$ edges of $G^{\text{reg}}(\Gamma)$ obtained from subdividing an edge e of $G(\Gamma)$. It is clear that if f satisfies equation (2), then $\hat{f}$ will be harmonic in the usual sense on G^{reg}.

LEMMA 13. *The map $f \mapsto \hat{f}$ is an isomorphism.*

Proof. It is clear that $f \mapsto \hat{f}$ is an isomorphism for $\mathbf{Q}$-valued harmonic functions; to prove that the isomorphism is defined over $\mathbf{Z}$ it suffices to show that, if f is a cuspidal harmonic cocycle for Γ, then $f(e)$ is divisible by $w(e) = |\text{Stab}(e)|$ for every stable edge e of $\mathscr{T}$. We know that the stabilizer of e is $\text{GL}_2(\mathbf{F}_q[T])$-conjugate to a subgroup of the group of diagonal matrices. This implies that, if γ stabilizes e, then γ also stabilizes two cusps, and therefore γ stabilizes an entire modular symbol. Indeed, we can say even more. Choose an e such that $\gamma \neq 1$ generates its stabilizer and let e' be an edge adjacent to e, oriented along the path fixed by γ. Then γ fixes e' and e but does not fix any of the other edges, leaving the common vertex of e and e'. If f is harmonic, then one must have $f(e) \equiv f(e') \pmod{|\text{Stab}(e)|}$. Therefore, f is constant mod $|\text{Stab}(e)|$ along the entire modular symbol determined by γ. Since f is cuspidal, however, f is eventually zero along this path, and therefore $f(e)$ is divisible by $w(e) = |\text{Stab}(e)|$. This proves the lemma.

If we combine the various elements of this analysis with what we already know about the relationship between the homology of $\Gamma\backslash\mathscr{T}$ and M_Γ, we obtain the following result.

THEOREM 14. *Suppose that Γ satisfies our hypothesis (††). Then the "integration" pairing between modular symbols and cuspidal harmonic cocycles induces exact sequences*

$$M_\Gamma^0 \to \text{Hom}(C_{\text{har}}(\Gamma, \mathbf{Z}), \mathbf{Z}) \to \Phi \to 0$$

and

$$M_\Gamma \to \text{Hom}(C_{\text{har}}(\Gamma, \mathbf{Z}), \mathbf{Z}) \to \Phi/\Phi^{\text{cusp}} \to 0$$

where Φ^{cusp} is the subgroup of Φ generated by the cuspidal divisors. The kernel of the first map is the torsion subgroup of M_Γ^0.

Proof. The first exact sequence follows from Lemma 12 after substituting the appropriate groups into the exact sequence of that lemma and checking that the integration pairing induces the map h. Since the map from $M^0(\Gamma)$ factors through $H_1(\Gamma\backslash\mathscr{T}, \mathbf{Z})$, the kernel is precisely the torsion group. To obtain the second exact sequence, observe that an element of M_Γ joining two cusps r and s will map to the path in G^{reg} which joins the components where the corresponding cusps reduce. Raynaud's theorem ([R]) tells us that this is precisely the image of $(r) - (s)$ in Φ.

We may apply this result to obtain a version of a theorem of Manin and Drinfeld ([D2]) in the function field situation.

THEOREM 15. *Let F be a monic polynomial in $\mathbf{F}_q[T]$ and let $\Gamma = \Gamma_0(F)$. Suppose that $\pi \in \mathbf{F}_q[T]$ is a prime polynomial of degree d such that $\pi \equiv 1 \pmod F$. Then $1 + q^d - T_\pi$ kills $\Phi^{\text{cusp}}(\Gamma)$.*

Proof. By Theorem 14, it suffices to show that

$$(1 + q^d - T_\pi)[\infty, a] \in M_\Gamma^0.$$

We compute that

$$(1 + q^d - T_\pi)[\infty, a] = [\pi a, a] + \sum_{x \bmod \pi} \left[\frac{a + x}{\pi}, a\right].$$

However, by an elementary calculation, each of the modular symbols appearing on the right-hand side of the above expression does in fact belong to M_Γ^0.

Of course, this result is much weaker than Manin's and Drinfeld's results in the classical case. In the classical case, the calculation we have made in the proof of this theorem shows that the endomorphism $\eta_\pi = 1 + q^d - T_\pi$ actually kills the divisor class $(a) - (\infty)$. This was sufficient to prove that the cuspidal divisors on the classical modular curves are of finite order ([D2]). Gekeler ([Gek1]) has proved that the cuspidal divisor classes are of finite order using methods of Kubert and Lang. Our method suffices to prove only that η_π carries $(a) - (\infty)$ into the connected component of the Neron model of the Jacobian of X_Γ at the infinite place of $\mathbf{F}_q(T)$. This is an inherent limitation of the modular symbols in the function field case and reflects the fact that the upper half-plane Ω is not simply connected. This means that a large part of the cohomology of X_Γ is not "visible" on the tree $\mathscr{T}$.

It would be interesting to understand the group

$$\text{Cusp}^0(\Gamma) = J^0(X_\Gamma) \cap \text{Cusp}(\Gamma)$$

of cuspidal divisor classes lying in the connected component of the fiber of the Jacobian of X_Γ above the place at infinity of k. If N is of degree three, it is not hard to show by the calculations in [Gek2] that $\text{Cusp}^0(\Gamma_0(N))$ is trivial. However, I have no information on more general cases.

This concludes our discussion of modular symbols and the cuspidal group. In the next section, we will describe explicit generators and relations for M_Γ, which can be used for calculations.

Generators and relations for modular symbols. In this section we describe the space of modular symbols in a form which is eminently suitable for calculation and completely analogous to the original description of Manin for the classical symbols.

Suppose that

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a matrix in $\mathrm{GL}_2(A)$. Then we define "basic" modular symbols

$$(g) = [\gamma(\infty), \gamma(0)] = [a/c, b/d].$$

This construction extends by linearity to yield a homomorphism

$$\phi\colon \mathbf{Z}[\mathrm{GL}_2(A)] \to M.$$

LEMMA 16. *The map ϕ is surjective.*

Proof. This is proved exactly in the classical case and is a simple consequence of the Euclidean algorithm. Namely, every symbol can be written as a sum of symbols of the form $[r/s, \infty]$ where r and s are relatively prime elements of $\mathbf{F}_q[T]$. We proceed by induction on the degree of s. If s has degree zero, then $[r/s, \infty] = (g)$ where

$$g = \begin{pmatrix} r & 1 \\ s & 0 \end{pmatrix}$$

Otherwise, we may find a polynomial s_1 of degree less than s which solves the congruence $rs_1 \equiv 1 \pmod{s}$. If we set $r_1 = (rs_1 - 1)/s$, we find a matrix g so that

$$g = \begin{pmatrix} r & r_1 \\ s & s_1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F}_q[T])$$

and $[r/s, \infty] = (g) + [r_1/s_1, \infty]$. Since s_1 has smaller degree than s, the symbol $[r_1/s_1, \infty]$ is in the image of ϕ. By induction, ϕ is surjective.

Let I be the left ideal of $\mathbf{Z}[\mathrm{GL}_2(\mathbf{F}_q[T])]$ which annihilates the modular symbol $[\infty, 0]$. It is clear that I is precisely the kernel of the map ϕ. By finding explicit generators for I, we will be able to find all of the relations among the modular symbols.

Before beginning this calculation, let us recall some facts about the action of $\mathrm{GL}_2(A)$ on $\mathscr{T}$. (See [Se], as usual, for the details.) The quotient of the tree $\mathscr{T}$ by $\mathrm{GL}_2(A)$ is an infinite ray. We may number the vertices in this ray with successive nonnegative integers, the origin being labeled with 0. The "type" of a vertex v of $\mathscr{T}$ is the index assigned to the equivalence class of v in $\mathrm{GL}_2(A)\backslash\mathscr{T}$. This labeling has the property that, if v is of type 0, then all adjacent vertices are of type 1. If v is of type $n > 0$, then v has q adjacent vertices of type $n - 1$, and one adjacent vertex of type $n + 1$. In addition, the stabilizer of a type 0 vertex is conjugate to $\mathrm{GL}_2(\mathbf{F}_q)$, and by choosing one type-0 vertex we may identify the set of all type-0 vertices with the coset space

$$\mathscr{G} = \mathrm{GL}_2(A)/\mathrm{GL}_2(\mathbf{F}_q).$$

Any path of the form (g) for $g \in GL_2(A)$ is $GL_2(A)$ equivalent to $[\infty, 0]$ and therefore passes through a unique vertex of type 0. Conversely, suppose we pick a vertex v of type 0 and ask for the modular symbols $[r, s]$ which pass through v and no other vertex of type 0. It is not hard to see that such a symbol is entirely determined by choosing two vertices v' and v'' adjacent to v. The path from v' to v'' can be extended in only one way to a path joining two rational ends and having v as its only vertex of type 0; this path must then be of the form (g). Since we may identify the vertices adjacent to v with $\mathbf{P}^1_{\mathbf{F}_q}$, there is a well-defined map

$$\iota\colon \bigoplus\nolimits_{\mathscr{G}} \mathrm{Div}^0(\mathbf{P}^1_{\mathbf{F}_q}) \to M.$$

LEMMA 17. *The map ι is an isomorphism.*

Proof. The surjectivity of ι follows from Lemma 16. To see that ι is injective, we will construct an inverse κ to it. Let m be an element of M, viewed as a labeled subtree of $\mathscr{T}$. This labeled subtree determines uniquely a divisor of degree zero on the projective line of edges leaving each vertex of type 0; if we call this divisor $\kappa_v(m)$, then it is clear that $\bigoplus_v \kappa_v$ is an inverse to ι.

A more canonical interpretation of this lemma is given in the following corollary.

COROLLARY 18. *The $GL_2(A)$ module M is induced from $GL_2(\mathbf{F}_q)$. In fact,*

$$M = \mathrm{Ind}_{GL_2(\mathbf{F}_q)}^{GL_2(A)} \mathrm{Div}^0(\mathbf{P}^1_{\mathbf{F}_q}).$$

Let us now apply our results on the structure of M to constructing relations for modular symbols.

COROLLARY 19. *Let I_0 be the ideal in $\mathbf{Z}[GL_2(\mathbf{F}_q)]$ which kills the divisor $(\infty) - (0)$ on $\mathbf{P}^1_{\mathbf{F}_q}$ (in the usual projective action). Then $I = \mathbf{Z}[GL_2(A)]I_0$.*

Proof. Suppose that $x \in I$. Write x in the form

$$\sum_{g \in \mathscr{G}} g x_g$$

where $x_g \in \mathbf{Z}[GL_2(\mathbf{F}_q)]$. Then the modular symbols $g x_g[\infty, 0]$ correspond to elements of $\mathrm{Div}^0(\mathbf{P}^1_{\mathbf{F}_q})$ coming from different type 0 vertices; consequently each $g x_g[\infty, 0] = 0$. This means that $x_g \in I_0$.

The preceding lemma reduces the problem of determining the ideal I to that of determining the ideal I_0. The next lemma carries this out.

LEMMA 20. *The ideal I_0 is generated by the following elements:*
1. *the set of elements of the form $1 - d$, where $d \in GL_2(\mathbf{F}_q)$ is a diagonal matrix;*
2. *the element $1 + w$, where*

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

3. *the element* $1 + t + t^2$, *where*

$$t = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Proof. Let J be the ideal generated by the given elements. Clearly $J \subseteq I_0$. Suppose $a = \sum n_\tau \tau$ belongs to I_0. Define $l(a)$, the length of a, by the formula

$$l(a) = \sum |n_\tau|.$$

We will reduce $l(a)$ to 0 mod J. We begin by using the second relation above to arrange things so that, if $n_\tau \neq 0$, then $n_{\tau wd} = 0$ for all d. Now pick some τ with $n_\tau \neq 0$. Suppose that

$$(\tau(\infty)) - (\tau(0)) = (x) - (y).$$

Then there must be some other τ' with $n_{\tau'} \neq 0$ so that

$$(\tau'(\infty)) - (\tau'(0)) = (y) - (z).$$

The point (z) must be different from (x) since we have already assumed that $\tau' \neq \tau wd$. Pick $\gamma \in \mathrm{GL}_2(\mathbf{F}_q)$ so that $\gamma(x) = \infty$, $\gamma(y) = 0$, and $\gamma(z) = 1$. Then we have the equations

$$\gamma\tau \cdot ((\infty) - (0)) = (\infty) - (0),$$

$$\gamma\tau' \cdot ((\infty) - (0)) = t \cdot ((\infty) - (0)).$$

We conclude that $\gamma\tau = d$ and $\gamma\tau' = td'$ for diagonal matrices d and d'. Consequently,

$$a \equiv a - \tau - \tau' - \gamma t^2 \pmod{J}.$$

But the element on the right-hand side of the above equation has smaller length than a, since we deleted two terms but only added one. Therefore, by repeating this procedure, we can eventually write a directly as an element of J.

If Γ is a congruence subgroup of $\mathrm{GL}_2(A)$, then we know that the action of Γ on the modular symbols (g) is given very simply by the rule $\gamma(g) = (\gamma g)$. Consequently, taking the action of Γ into account, we have the following presentation of M_Γ by generators and relations.

THEOREM 21. *The group* M_Γ *of modular symbols for* Γ *is isomorphic to the quotient of the free abelian group* $\mathbf{Z}[\Gamma\backslash \mathrm{GL}_2(A)]$ *by the relations*

$$(g) - (gd) = 0 \qquad (d \text{ a diagonal matrix}),$$

$$(g) + (gw) = 0,$$

$$(g) + (gt) + (gt^2) = 0,$$

where w and t are the matrices in the statement of the preceding lemma.

Just as in the classical case, these relations can be applied quite simply in practice. To illustrate this, let us work out the case where $\Gamma = \Gamma_0(N)$, and N is a monic irreducible polynomial over $\mathbf{F}_q$. Our notation for this calculation was suggested by Cremona's work in the classical case ([Cr]).

Suppose that a and b are relatively prime polynomials in $\mathbf{F}_q[T]$. Let $M(a, b)$ be the coset

$$M(a, b) = \Gamma_0(N)\begin{pmatrix} u & v \\ a & b \end{pmatrix}$$

in $\Gamma_0(N)\backslash \mathrm{GL}_2(A)$ for any polynomials u and v such that $ub - va \in \mathbf{F}_q^*$. It is easy to see that every coset can be represented in this way, independently of the choice of u and v. Indeed, if u' and v' are another set of polynomials satisfying $u'b - v'a \in \mathbf{F}_q^*$, the fact that a and b are relatively prime means that $zu' - u = ax$ and $zv' - v = bx$ for elements $z \in \mathbf{F}_q^*$ and $x \in \mathbf{F}_q[T]$. From this it follows easily that the matrices with a and b in the bottom row and either u and v or u' and v' in the top row are equivalent. It is also not hard to check that two cosets $M(a, b)$ and $M(a', b')$ are equal if and only if there is a polynomial $\lambda \in A$ which is prime to N such that

$$(\lambda a, \lambda b) \equiv (a', b') \qquad (\mathrm{mod}\ N).$$

From our results above, we conclude that the space $M_\Gamma(\mathbf{Z})$ of modular symbols is the quotient of the free abelian group on the $M(a, b)$ of rank $q^{\deg(N)} + 1$ by the relations

$$M(a, b) - M(da, d'b) = 0 \qquad \text{for } d, d' \in \mathbf{F}_q^*,$$

$$M(a, b) + M(-b, a) = 0,$$

$$M(a, b) + M(-b, a + b) + M(-a - b, a) = 0.$$

These relations can be manipulated quite simply in practice; we work out an example for $q = 7$ and N of degree three in the next section.

Applications.

A numerical example. In this section we will work out a numerical example which will serve to illustrate the applications which we will describe later in the paper. We will set $q = 7$ and $N = T^3 - 2$. The Hecke module structure of the space of modular forms for $\Gamma = \Gamma_0(N)$ has been worked out by Gekeler, in [Gek2], by different techniques. Our discussion of this case will highlight the differences in our methods. In some places we will be able to extend Gekeler's results, while in others we will rely on his more detailed study.

In addition to these calculations, which duplicate Gekeler's in most essentials, we have carried out similar calculations for other primes, and conductors of degree larger than 3. We will describe the details of these calculations in another paper.

We begin the calculation with 344 coset representatives $M(a, b)$ for $\Gamma_0(N)$. Up to equivalence, these are $M(1, 0)$ and $M(x, 1)$ as x runs through representatives for $\mathbf{F}_q[T]/N$. Applying the relations $M(a, b) = M(da, d'b)$ for $d, d' \in \mathbf{F}_q^*$, we reduce the number of representatives to 58, consisting of $M(1, 0)$, $M(0, 1)$, and $M(x, 1)$ for x monic. Using the two term relation $M(a, b) = -M(-b, a)$, we see that $M(1, 1) = 0$, that $M(1, 0) = -M(0, 1)$, and that the remaining 56 representatives are grouped in pairs; thus we have reduced the number of independent symbols to 29. When we apply the three-term relations, the space of symbols is reduced to an 8-dimensional space. Conveniently, this space is represented by elements

$$\{M(0, 1)\} \cup \{M(T + a, 1) \colon a \in \mathbf{F}_q\}.$$

Once we have this basis of "fundamental" symbols, we can compute the action of the Hecke algebra. For example, let T_π be the operator associated to the prime $\pi = (T)$. Then we find the following matrix for T_π in the basis above:

$$\begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -7 & -6 & -6 & -8 & -6 & -8 & -8 \\ 0 & 2 & 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 2 & 1 & 2 \end{bmatrix}$$

The characteristic polynomial $P_\pi(u)$ of this matrix is

$$P_\pi(u) = (u + 4)(u - 8)(u^2 - 3u - 2)^3.$$

By Theorem 14, we know that the space of cusp forms (with values in $\mathbf{Q}$) is the linear dual of the space M_Γ^0 of "closed" modular symbols. In terms of our basis, the group of closed symbols M_Γ^0 is spanned by the $M(T + a, 1)$. Of particular interest is the cusp form f with $f(M(0, 1)) = 2$, and other values defined by the following table. This cusp form is an eigenvector for T_π with eigenvalue -4.

$$\begin{aligned} a &= -3 \;\; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3, \\ f(M(T + a, 1)) &= \;\;\; 2 \;\;\;\;\; 2 \;\;\;\;\;\;\; 3 \;\; 4 \;\; 2 \;\; 2 \;\; 3. \end{aligned} \tag{3}$$

By Drinfeld's theorem, this cusp form corresponds to an elliptic curve E over $\mathbf{F}_q(T)$ with conductor $\infty \cdot N$. Using Gekeler's tabulation of such curves ([Gek2]), we see that the equation of E is

$$y^2 = x^3 - 3T(T^3 + 2)x + 2T^6 - 3T^3 - 1.$$

Besides finding rational eigenvalues, our method is suitable for studying the higher genus components of the Jacobian of $X_0(N)$. For example, if we compute the action of T_μ for the prime $\mu = (T + 1)$, we see that this piece is irreducible of dimension 6 over the Hecke algebra. Indeed, the characteristic polynomial of T_μ on M_Γ^0, factored over $\mathbf{Q}$, is

$$P_\mu(u) = (u - 2)(u^6 + 3u^5 - 22u^4 - 51u^3 + 94u^2 + 40u - 64)$$

showing the elliptic factor and the degree-six factor.

I do not know the exact ring generated by the Hecke operators in this case.

Modular symbols and L-series. In this section we relate modular symbols to values of L-functions. Our goal is the analogue of Birch's formula, which gives a formula for the special value of the L-functions of a twist of an automorphic form in terms of modular symbols [Maz3]. We will base our computations on Mazur's construction of the "modular element."

Let us fix a level N and consider modular symbols and harmonic cocycles for $\Gamma_0(N)$. Suppose that χ is a primitive Dirichlet character for $\mathbf{F}_q[T]$ of conductor F, relatively prime to N, and trivial on $\mathbf{F}_q^*$. In addition, to simplify the notation somewhat, let $[r/s]$ denote the modular symbol $[r/s, \infty]$.

Definition 22. The universal special value $\Lambda(\chi) \in M_{\Gamma^0(N)}(\mathbf{C})$ is defined to be

$$\Lambda(\chi) = \sum_{\substack{a \bmod F \\ a \text{ monic}}} \chi(a) \left[\frac{a}{F}\right].$$

The universal special value plays exactly the same role in the function field case as the analogous expression does for the usual modular symbols. The following theorem illustrates this.

THEOREM 23. *Let f be a cuspidal, complex-valued, harmonic cocycle for $\Gamma_0(N)$. Suppose that f is an eigenvector for the Hecke algebra. Let χ be a nontrivial Dirichlet character of conductor F for $\mathbf{F}_q[T]$ which is trivial on $\mathbf{F}_q^*$. Then*

$$L(f, \chi, 1) = \tau_x \Lambda(\chi) \cdot f$$

where $L(f, \chi, s)$ is the L-function associated to f twisted by χ, and τ_χ is an appropriate Gauss sum. (For the precise definition of this Gauss sum, see [TR, p. 6].*)*

Proof. The proof of this theorem is an exercise in change of notation. We leave it to the reader to verify that Proposition 1 and equation 3.4.8 of [TR] together imply, in our notation, that

$$\tau_\chi^{-1} L(f, \chi, 1) = \sum_{\substack{a \bmod F \\ a \text{ monic}}} \chi(a) \int_{a/F}^{\infty} f,$$

which is precisely the statement of the theorem.

The case where χ is trivial requires somewhat special handling; in this case (as pointed out in [TR]), we have

$$L(f, 1) = \frac{1}{q-1} \int_0^{\infty} f. \tag{4}$$

Let us apply this discussion to the numerical example we have already considered. Once again, we take $q = 7$ and $N = T^3 - 2$. In equation (3), we gave a numerical description of an eigenform f for the Hecke operators on $\Gamma_0(N)$ corresponding to a particular elliptic curve. Since $f(M(0, 1)) = 2$, we see from equation (4) that

$$L(f, 1) = 1/3.$$

Perhaps the best way to view this is as a normalization, since the harmonic cocycle f is determined only up to multiplication by an element of $\mathbf{Q}$.

In Table 1, we tabulate the special values $\Lambda(\chi)\cdot f$ for the quadratic Dirichlet characters which are trivial on $\mathbf{F}_q^*$ and have conductors of degree 2, in those cases where the functional equation does not force $\Lambda(\chi)\cdot f$ to vanish. In this table, the values of 3 reflect the three components of the elliptic curve E at the prime N, and the values of 12 ought (by the Birch and Swinnerton-Dyer conjecture) to imply the existence of a Tate-Shafarevich group of order 4.

Distributions arising from modular symbols. Suppose that f is a harmonic cocycle which represents a new form with rational eigenvalues for $\Gamma_0(N)$. Suppose that π is a prime polynomial which divides N exactly once and that f is an eigenvector for the operator

$$T_\pi f(e) = \sum_{a \bmod \pi} f\left(\begin{pmatrix} 1 & a \\ 0 & \pi \end{pmatrix} e\right) \tag{*}$$

with eigenvalue 1.

Let $\mathcal{O}_\pi$ be the ring of integers in the π-adic completion of $\mathbf{F}_q(T)$. It follows from equation (*) that the function

$$\mu(a + \pi^n \mathcal{O}_\pi) = \left[\frac{a}{\pi^n}\right]\cdot f$$

TABLE 1. Values of $\Lambda(\chi_m)\cdot f$

m	$\Lambda(\chi_m)\cdot f$
1	$\frac{1}{3}$
T^2-2T+3	3
T^2+2T-2	3
T^2-3T-1	3
T^2+T-1	12
T^2+T-3	3
T^2+2T+2	3
T^2+2T+3	12
T^2+T+3	3
T^2+3T-2	3
T^2-3T-2	12
T^2-3T+1	3
T^2-T-1	3

is a **Q**-valued distribution on the compact opens in $\mathcal{O}_\pi$. Furthermore, by Theorem 14, the denominator of this distribution is bounded. Similar distributions have been studied by Galovich and Rosen [Ga]. In the classical case, Mazur and Swinnerton-Dyer ([MSwD]) showed that analogous distributions can be integrated against $\langle x\rangle^s$ to obtain a p-adic L-function. The existence of this measure in the function field setting suggests the existence of an L-function, although a straightforward translation of the classical ideas is blocked by the lack of a logarithm in characteristic p. Nevertheless, recent work of Goss ([Go3]) gives some indication of how such a function might be constructed. We will not pursue this issue here, but we hope to return to it in a later paper.

Even though we do not yet know the correct L-function arising from the distribution μ, we can pursue the formal similarity with the classical case to state a "modular symbols" version of a conjecture derived from the L-function which avoids any use of logarithms.

To study this conjecture, return now to our eigenform f of level N and our selected prime polynomial π which exactly divides N. The function f corresponds to an elliptic curve $E(f)$ over $\mathbf{F}_q(T)$ having totally split reduction at ∞ and at π. The curve $E(f)$ therefore has a Tate multiplicative period $q(E)$ at the place π. In the classical case, the "exceptional zero conjecture" of [MTT] claims a relationship between this q and a certain expression in modular symbols. This conjecture has been proved by Greenberg and Stevens, and a function field version of it has been proposed by Mazur. This function field version has in turn been studied by Tan ([Tan]), who obtains some theoretical results. Tan and Rockmore ([TR]) have recently obtained numerical evidence for the function field conjecture. Our contribution to the function field version of the exceptional zero conjecture is a formal consequence of the existence of the measure μ above, and amounts to giving a characteristic p version of the multiplicative formula for $q(E)$ in [MT, p. 712]. In addition, our method easily lends itself to the collection of computational data, yielding an alternative to the methods of Tan and Rockmore.

Let us define

$$m_\pi = \mathrm{ord}_\pi q(E),$$

$$\tilde{q} = \frac{q(E)}{\pi^{m_\pi}},$$

and

$$q(f) = \lim_{n\to\infty} \prod_{\substack{a \bmod \pi^n \\ (a,\pi)=1}} a^{[a/\pi^n]\cdot f}.$$

Then the function field version of the refined Birch and Swinnerton-Dyer conjecture can be stated

$$\tilde{q}^{[0]\cdot f} = q(f)^{m_\pi}$$

exactly as in [MT, loc. cit].

If we apply our numerical results for $N = \pi = T^3 - 2$ and $q = 7$ in our situation, we find that

$$q(f) = 2 \pmod{\pi},$$

$$\tilde{q} = -1 \pmod{\pi}.$$

We can also check that, for the elliptic curve $E(f)$, we have $m_\pi = 3$. Since

$$q(f)^3 \equiv \tilde{q}^2 \pmod{7},$$

we see that our conjecture holds mod π. This is admittedly not very satisfying, but we hope it is illustrative.

Congruences. In his original work on modular symbols, Manin observed an important congruence. For example, he found that the values of the modular symbols $[a]$ for $X_0(11)$ depended only on a mod 5. Mazur later showed the relationship between the Eisenstein ideal and this congruence, and proved it as part of his study of the Eisenstein ideal. In this section, we will make Manin's observation and raise the hope that Mazur's proof will be carried over to the function field case.

Consider once again our form f associated to the elliptic curve E of conductor $N = T^3 - 2$ with $q = 7$. Gekeler has shown ([Gek4]) that the cuspidal group on the curve X_Γ is of order $57 = (3)(19)$. One expects 3 to function as an "Eisenstein prime" for f, and indeed one can confirm from the equation of E that E has a rational point of order 3.

Following a course suggested by Manin in the classical case, let R be the subring of $\mathbf{F}_q(T)$ of rational functions with denominators prime to N. Define a function $h: R^* \to \mathbf{Z}/3\mathbf{Z}$ by setting

$$h(x) = [x] \cdot f \qquad (\text{mod } 3).$$

Then by direct calculation using the modular symbols for f (which we have supplied above), one can check that h (apparently) depends only on x mod N. Indeed, h appears to be homomorphism from $(R/N)^*/\mathbf{F}_q^*$, which is a cyclic group of order 57, into $\mathbf{Z}/3\mathbf{Z}$.

A somewhat more exotic congruence of this type arises from the case $q = 3$ and $N = T^3 - T + 1$. Here (as Gekeler showed first) the Hecke algebra $\mathbf{T}$ is a simple algebra; in fact, it is the ring of integers in the cyclic cubic field

$$\mathbf{Q}[x]/(x^3 + x^2 - 4x + 1)$$

of discriminant 169, with the Hecke operator associated to the prime $\pi = (T)$ acting as x. There is only one newform f with $T_\pi f = xf$. In this case, Gekeler shows that the cuspidal group is of order 13, and therefore the Eisenstein prime P is the ideal

$$P = (13, 4 - \mathrm{x}) \subset S = \mathbf{Z}[x]/(x^3 + x^2 - 4x + 1).$$

We extend the integration pairing to take values in S. If we let R be the local ring at N in $\mathbf{F}_q(T)$, we may construct a map

$$(R/N)^*/\mathbf{F}_q^* \xrightarrow{h} (S/P) = \mathbf{Z}/(13)\mathbf{Z},$$

$$h(x) = [x] \cdot f.$$

We have checked numerically that this map is an isomorphism between $\mathbf{Z}/(13)\mathbf{Z}$ and the cyclic group $(R/N)^*/\mathbf{F}_q^*$.

In the classical case, the homomorphism h has an interpretation as a sort of "reciprocity law." ([Maz2, Proposition 18.8]). Because of the strong analogy between the classical and the function field situations, developing Mazur's Eisenstein ideal machinery for Drinfeld modular curves should enable one to construct a proof of the preceding congruence in the function field case as well. This is clearly a difficult but important problem in the theory of Drinfeld modular curves.

Conclusions. We hope that our discussion of modular symbols for $\mathbf{F}_q(T)$ illustrates that they may be as useful in the function field setting as in the classical one. We also feel that we have raised many more questions than we have answered. Some questions we have in mind are:

1. What can be done with modular symbols for the general class of Drinfeld modular curves—for example, if the fixed "place at infinity" has degree greater than one, or if the field $\mathbf{F}_q(T)$ is replaced by a more general function field?

2. What is the correct arithmetic interpretation of the distributions we have described?
3. Where are the "odd" modular symbols?
4. Can one prove Mazur's results about the Eisenstein ideal in the setting of Drinfeld modular curves, and thereby prove the congruence we have discussed above?

Besides these theoretical questions, we now have a rather general technique for gathering numerical data about elliptic curves and automorphic forms over function fields. The tables in the Antwerp IV volume ([MF4]) have become a standard reference; computing similar tables for function fields using modular symbols is clearly an important computational project.

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