Progress Report: Statistics over function fields

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Sage Days 21: Function Fields
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Summary

Goal: Use ellff library to investigate statistics of zeros of elliptic curve L-functions in function field.

Successfully installed library on Miller's laptop.

Gathered data on ranks and first zero above the central point.

Plan: add additional functions to ellff library.

Sage Code using ellff (which is not yet finalized)

```
import sage.libs.ellff as ellff
R = ZZ['T']
R.inject_variables ()

def test (p = 5) : F = GF (p)
R.< t > =  F['t']
K = Frac (R)
return ellff.ellff_EllipticCurve (K, [0, -1 - t, 0, t, 0])

def twist (E, f, tables = False, force = False, verbose = False) : E_twist =
 E.quadratic_twist (f, tables = tables, force = force, verbose = verbose)

if verbose : print "finite  bad  reduction:"
print "     M_sp  : ", E_twist.__finite _M _sp
print "     M_ns  : ", E_twist.__finite _M _ns
print "      A    : ", E_twist.__finite _A

print
print "refined  finite  additive  reduction:"
print "     I^*      : ", E_twist.__finite _I _star
print "    II,II^*   : ", E_twist.__finite _II, E_twist.__finite _II _star
print "   III,III^* : ", E_twist.__finite _III, E_twist.__finite _III _star
print "    IV,IV^*   : ", E_twist.__finite _IV, E_twist.__finite _IV _star

return E_twist

def pullback (E, f, tables = False, force = False, verbose = False) : E_pullback =
 E.pullback (f, tables = tables, force = force, verbose = verbose)

if verbose : print "finite  bad  reduction:"
print "     M_sp  : ", E_pullback.__finite _M _sp
print "     M_ns  : ", E_pullback.__finite _M _ns
print "      A    : ", E_pullback.__finite _A

print
print "refined  finite  additive  reduction:"
print "     I^*      : ", E_pullback.__finite _I _star
print "    II,II^*   : ", E_pullback.__finite _II, E_pullback.__finite _II _star
print "   III,III^* : ", E_pullback.__finite _III, E_pullback.__finite _III _star
print "    IV,IV^*   : ", E_pullback.__finite _IV, E_pullback.__finite _IV _star

return E_pullback
```

Sage Code using ellff (which is not yet finalized)

```
p = 5
R.<t> = GF(p)['t']
R.inject_variables()
print

E = test(p)

print "finite bad reduction:"
print "    M_sp  : ", E.__finite_M_sp
print "    M_ns  : ", E.__finite_M_ns
print "     A    : ", E.__finite_A
print
print "L-fcn = ", E.L_function()
```

Sage Code using ellff (which is not yet finalized)

```
# CREATE DATA

data_list = [[p, E.a4, E.a6]]
data_vec = [[p, E.a4, E.a6]]
print "Printing information on our initial elliptic curve"
print E
print "Prime is ", p
Edisc = 4 * (E.a4) ^3 + 27 * (E.a6) ^2
print "a4 = ", E.a4, " a6 = ", E.a6, " and disc = ", Edisc
for a in range (p) : for b in range (p) : for c in range (p) : for d in range (p) : for e in range (p) : f =
 a + b * t + c * t^2 + d * t^3 + e * t^4 + t^5
rk = 0
args = []
args_list = []
# print (f, Edisc, gcd (f, Edisc))
if gcd (f, diff (f)).degree () == 0 and gcd (f, Edisc).degree () == 0 : E_twist =
 twist (E, f, tables = True, force = True)
L = E_twist.L_function ()
# print "f = ", f, " : ", L, ", ", factor (L), ", ", "HERE"
for pi, ex in list (L.factor ()) : v = []
# print "ex = ", ex, ", pi = ", pi, ", pi.roots() = ", pi.roots (CDF),
for r, m in pi.roots (CDF) : # print "r = ", r, ", m = ", m
```

Sage Code using ellff (which is not yet finalized)

```
assert m == 1
v.append (r.arg ())
for i in range (ex) : if r.arg () == 0 : rk = rk + 1
args_list.append (r.arg ())
args.append ([ex, v])
# print "v = ", v
# build the data file consisting of the twist,
         L - function, # sign of the f.e., the rank, and the zeroes
# data_vec stores a zero with its multiplicities as a vector
# data_list stores a zero as many times as its multiplicity
data_vec.append ([f, L, E_twist.sign, rk, args])
data_list.append ([f, L, E_twist.sign, rk, args_list])

         for i in range (len (data_list) - 1) :
          if data_list [i + 1] [1].degree () ≠ data_list [2] [1].degree () :
           raise ValueError ("Degree of L-function at %s is %s" % (i + 1, data_list [i + 1] [1]))
if data_list [i + 1] [1].degree () ≠ len (data_list [i + 1] [4]) :
          raise ValueError ("Not enough zeros found at %s" % (i + 1))

         data_str = str (data_list).replace ('[', '{').replace (']', '}')

fname = str (E.a4) + "-" + str (E.a6) + ".dat"
file = open (fname, "w")
file.write (data_str)
file.close ()
```

Rank of degree 4 twists

$$E : y^2 = x^3 + (3 + 2t + 3t^2)x + (4 + 4t + 4t^2 + 4t^3).$$

Twisting by square-free $a + bt + ct^2 + dt^3 + t^4$ relatively prime to discriminant.

Data incomplete for $p = 11$

	$p = 5$ (345)	$p = 7$ (1573)	$p = 11$ (5000)
Rank 0	39.13	41.96	42.53
Rank 1	51.01	50.16	49.86
Rank 2	9.86	7.69	7.33
Rank 3	0.00	0.19	0.27
Rank 4	0.00	0.00	0.02

First normalized zero above central point

$$E : y^2 = x^3 + (3 + 2t + 3t^2)x + (4 + 4t + 4t^2 + 4t^3).$$

Twisting by square-free $a + bt + ct^2 + dt^3 + t^4$ relatively prime to discriminant.

Data incomplete for $p = 11$

	$p = 5$ (345)	$p = 7$ (1573)	$p = 11$ (5000)
mean Rank 0	.458	.437	.432
mean all even	.367	.369	.368

First normalized eigenangle above 0: 23,040 SO(4) matrices: Mean = .357; 23,040 SO(6) matrices: Mean = .325, $N \to \infty$ scaling limit: Mean = .321.

First normalized zero above central point

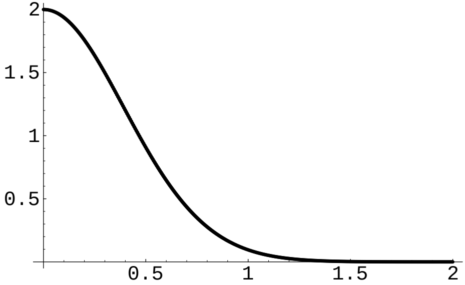

Figure: First zero for $N \to \infty$ limit of SO($2N$).

Histograms for first zero (same curve and degree twists as above)

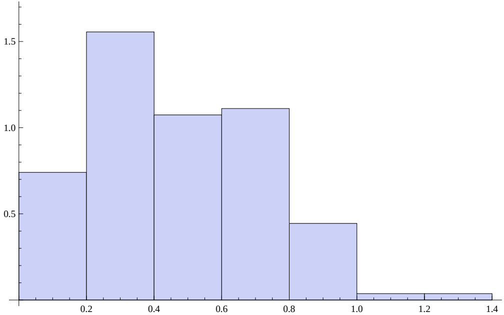

Figure: First zero for $p = 5$ rank 0 curves: Mean = .458

Histograms for first zero (same curve and degree twists as above)

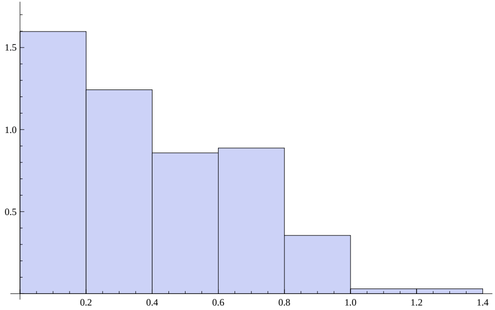

Figure: First zero for $p = 5$ rank even curves: Mean = .367

Histograms for first zero (same curve and degree twists as above)

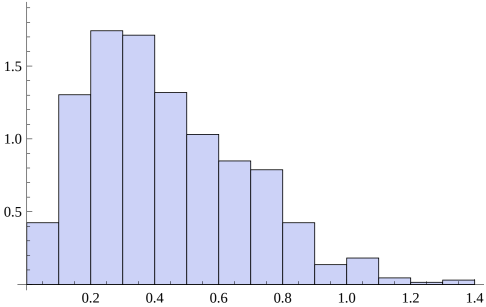

Figure: First zero for $p = 7$ rank 0 curves: Mean = .437

Histograms for first zero (same curve and degree twists as above)

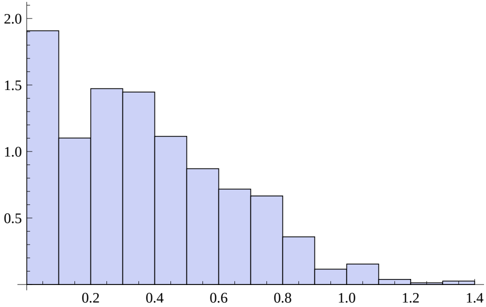

Figure: First zero for $p = 7$ rank even curves: Mean = .369

Histograms for first zero (same curve and degree twists as above)

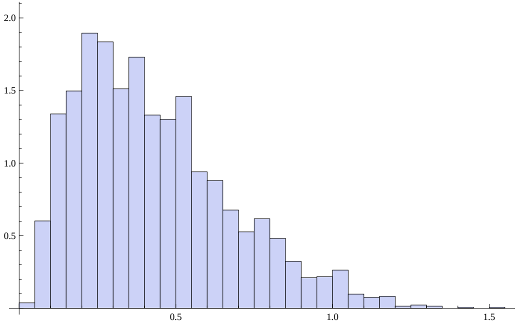

Figure: First zero for $p = 11$ rank 0 curves: Mean = .432

Histograms for first zero (same curve and degree twists as above)

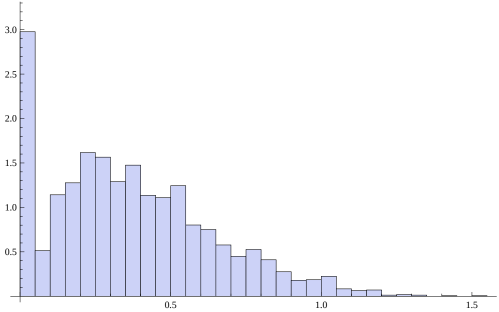

Figure: First zero for $p = 11$ rank even curves: Mean = .368