Function fields and number fields

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Topics:

- FF/NF analogies
- Curves over FFs, their L-functions, Jacobians, conjectures
- Automorphic forms on GL_2 over a function field
- Drinfeld modular curves and forms

FF/NF analogies:

$\mathcal{C}$ a curve over $\mathbb{F}_q$. $F = \mathbb{F}_q(\mathcal{C})$. E.g., $\mathcal{C} = \mathbb{P}^1$, $F = \mathbb{F}_q(t)$.

F has a countable set of places (non-trivial absolute values) v, the valuation rings $\mathcal{O}_v$ are DVRs, the residue fields $\mathcal{O}_v/\mathfrak{m}_v = \mathbb{F}_{q_v}$ are finite. E.g., places of $\mathbb{F}_q(t)$ correspond to monic irreducibles in $\mathbb{F}_q[t]$ and $t = \infty$. In general, they correspond to closed points of $\mathcal{C}$ $=$ Galois orbits of points with $\overline{\mathbb{F}}_q$ coordinates.

If $\infty_1, \ldots, \infty_r$ are places of F, then the ring of elements of F which are integral/regular outside $\{\infty_1, \ldots, \infty_r\}$ is a Dedekind domain. ("S-integers").

F has a zeta-function

$$\zeta(F,s) = \zeta(\mathcal{C},s) = \prod_{v \in \mathcal{C}} (1 - q_v^{-s})^{-1} = \exp\left(\sum_{n \geq 1} \frac{\#\mathcal{C}(\mathbb{F}_{q^n})}{n} q^{-ns} \right)$$

Convergence, analytic continuation, functional equation, RH.

SPECTRAL INTERPRETATION: there are cohomology groups with Frob action so that

$$\zeta(\mathcal{C},s) = \prod_{i=0}^{2} \det\left(1 - q^{-s} Fr_q | H^i\right)^{(-1)^{i+1}} = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

L-functions of curves over F

Varieties over F give rise to L-functions, (cohomology), conjectures on points/cycles (BSD, Tate). E.g., take a curve X of genus g over F. E.g., g=1, X an elliptic curve.

$L(X, s)$ is an Euler product over places of F. The local factor is the numerator of the zeta function of the reduction of X at v, which has degree $\leq 2g$, exactly $2g$ at most places. (This is somewhat naive at the bad places ...better to think of L as an invariant of the representation of G_F on some $H^1(X)$. Serre gave the good recipe for local factors in this optic.)

To analyze $L(X, s)$, it's useful to make a global model of X:

$$\mathcal{X} \to \mathcal{C}$$

(smooth, proper, minimal, ...)

Then up to adjustments at bad places,

$$\zeta(\mathcal{X}, s) = \prod_{x \in \mathcal{X}} (1 - q_x^{-s})^{-1} = \prod_{v \in \mathcal{C}} \prod_{x \to v} (1 - q_x^{-s})^{-1}$$

$$= \prod_{v \in \mathcal{C}} \zeta(\mathcal{X}_v, -s(\deg v)) = \prod_{v \in \mathcal{C}} \frac{P_v(q_v^{-s})}{(1 - q_v^{-s})(1 - q_v^{1-s})}$$

$$= \frac{\zeta(\mathcal{C}, s)\zeta(\mathcal{C}, s-1)}{L(X, s)(**)}$$

The spectral interpretation of $\zeta(\mathcal{X}, s)$ leads to a complete analysis of the analytic properties of $L(X, s)$: analytic continuation, functional equation, RH.

$L(X, s)$ is a rational function in q^{-s} (usually a polynomial) ... so calculating it requires only a finite amount of data.

Jacobians and BSD

X has a Jacobian J_X, an abelian variety of dimension g over F.
(When $g = 1$ and X has an F-rational point, $J_X = X$.)

MWLN theorem: $J_X(F)$ is a finitely generated abelian group.

BSD conjecture: $\mathrm{ord}_{s=1} L(X, s) = \mathrm{Rank}\, J_X(F)$.

This is open (even in $g = 1$), but a lot is known: ord $\geq$ Rank.
ord $=$ Rank iff a Tate-Shafarevich group is finite. If so, refined conjecture holds.

BSD and Tate

Everything known about BSD in FF case comes from the connection between X/F and $\mathcal{X} \to \mathcal{C}$.

Let $NS(\mathcal{X})$ be the Neron-Severi group of $\mathcal{X}$—divisors modulo algebraic equivalence. It's a finitely generated abelian group.

BSD is equivalent to Tate: $-\operatorname{ord}_{s=1} \zeta(\mathcal{X}, s) = \operatorname{Rank} NS(\mathcal{X})$.

The order of pole of $\zeta(\mathcal{X}, s)$ is the multiplicity of a certain eigenvalue of Fr on some $H^2(\mathcal{X})$ (spectral interpretation) and this multiplicity is at least the rank of $NS(\mathcal{X})$ because of a cycle class map $NS(\mathcal{X}) \to H^2$. The full story uses a lot of ℓ-adic and p-adic cohomological technology.

BSD/Tate examples

There are a few general classes of surfaces for which one knows the Tate conjecture a priori (and so BSD for a related Jacobian):

rational surfaces, abelian surfaces, K3 surfaces, products of curves

any surface dominated by a surface for which one knows Tate

Using this, one can give examples of (simple) Jacobians of every dimension for which BSD holds and for which the rank is large.

Things it would be nice to be able to compute efficiently

- $L(X, s)$. There is a naive point counting algorithm and a fancier algorithm by A. Lauder. Having a catalog of $L(X, s)$'s (say for a bunch of Xs of genus 1) would be interesting.
- $J_X(F)$: provably full calculations when $g = 1$, anything at all in higher genus.
- other invariants that enter into BSD: heights, regulators, T-S groups, ...

Automorphic forms

Let $\mathbb{A}_F$ be the adeles of F and $\mathcal{O}_F$ the everywhere integral adeles. Let $G = \mathrm{GL}_2$. Fix a compact open subgroup $K \subset G(\mathcal{O}_F)$. (E.g., $K = \Gamma_0(\mathfrak{n})$).

We consider functions (with values in a field of characteristic zero) on the double coset space

$$Y_K = G(F)\backslash G(\mathbb{A}_F)/K.$$

When $F = \mathbb{Q}$ we use strong approximation to push everything to the component at infinity, and find that

$$Y_K = \Gamma_0(N)\backslash G(\mathbb{R})/O_2(\mathbb{R}) = \Gamma_0(N)\backslash\mathcal{H}.$$

Something similar happens over a function field. When $F = \mathbb{F}_q(t)$, we have

$$Y_K = \Gamma\backslash G(F_\infty)/G(\mathcal{O}_\infty)$$

where Γ is a congruence subgroup of $G(\mathbb{F}_q[t])$. Over a general F, we get a union of components $\Gamma_i\backslash G(F_\infty)/G(\mathcal{O}_\infty)$.

$G(F_\infty)/G(\mathcal{O}_\infty)$ is an analogue of the upper half plane. It turns out to be (the edges of) a regular tree of degree $q+1$. (picture).

The quotient $\Gamma\backslash G(F_\infty)/G(\mathcal{O}_\infty)$ turns out to be (the edges of) a finite graph with finitely many infinite rays (cusps) attached. (picture)

There is a notion of a cusp form and these turn out to vanish on the cusps. In particular, the set of cusp forms is finite dimensional.

There are Hecke operators (indexed by the places of F) and the space of automorphic forms can be decomposed into spaces of eigenforms.

Eigenforms give rise to L-functions with good analytic properties.

Modularity 1

Fix a place ∞ of F. It turns out that to every elliptic curve E over F with split multiplicative reduction at ∞, there corresponds an automorphic form f of the type described above, characterized by $L(E, \chi, s) = L(f, \chi, s)$ for all (idele class) characters χ. The "level" K of f is determined by the bad reduction places of E. Moreover, f determines E up to isogeny.

Since f is a function on a combinatorially defined space, this gives a way to find (the L-functions of) all elliptic curves with given bad reduction. Gekeler worked out the first case of this, which is already quite non-trivial. Others continue this work.

More things it would be nice to be able to compute efficiently

- The graph Y_K, say for $F = \mathbb{F}_q(t)$, K an analogue of $\Gamma_0(N)$ (cf. Gekeler et al.)
- Hecke operators on Y_K
- Eigenfunctions
- $L(f, \chi, s)$ given f (Rockmore-Tan?). This might provide a line of attack on the Goldfeld conjecture.

Drinfeld modular curves

Recall that over $\mathbb{Q}$, $\Gamma_0(N)\backslash\mathcal{H}$ is the set of complex points $Y(\mathbb{C})$ of an algebraic curve defined over $\mathbb{Q}$ (or over a number field in a more general setup). This can't be the case for a function field F, since Y_K is combinatorial in nature.

Let's replace

$$G(F_\infty)/G(\mathcal{O}_\infty) = \text{ tree}$$

with a different analogue of the upper half plane:

$$\mathcal{H} = \mathbb{P}^1(\hat{\overline{F}}_\infty) \setminus \mathbb{P}^1(F_\infty)$$

(analogue of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) = \mathcal{H}_\pm$).

Let $\mathbb{A}_F^f$ be the finite adeles of F (∞-component removed) and set

$$\mathcal{Y}_K = G(F)\backslash \left(G(\mathbb{A}_F^f) \times \mathcal{H} \right) /K^f$$

which turns out to be a union of components

$$\Gamma_i\backslash\mathcal{H}$$

Unwinding everything exactly as in the classical case shows that $\mathcal{Y}_K$ is a moduli space of lattices (f.g. $\mathbb{F}_q[t]$-submodules of $\overline{F}_\infty$) of rank 2 up to scaling. (Compare with rank 2 $\mathbb{Z}$-submodules of $\mathbb{C}$.) Ultimately, these are analytic versions of Drinfeld modules and $\mathcal{Y}_K$ turns out to be the set of $\hat{\overline{F}}_\infty$ points of a curve defined over an extension of F. Its compactification is X_K, a Drinfeld modular curve.

Modularity 2

Another version of modularity is that for an elliptic curve over F with split multiplicative redution at ∞, there is a surjective morphism of curves over F:

$$X_K \to E.$$

(So E is a factor of the Jacobian of X_K.)

As in the classical picture, X_K carries a system of Heegner points associated to orders in certain quadratic extensions of F and defined over ring class fields of these extensions.

A non-analogy

There are some things that play out differently in the function field setting. For example, note that differential forms on X_K are characteristic p objects since X_K is a curve over F. Thus the classical relationship

"cusp forms of weight 2 $\leftrightarrow$ differential on the modular curve"

can't hold here.

There is a connection mediated by the reduction map

$$\mathcal{H} \to \text{geometric realization of the tree}$$

(pictures) but lots of pathologies occur (notably, failure of multiplicity one) and the situation is not well-understood.

Yet more things it would be nice to be able to compute efficiently

- Equations for the modular curve X_K
- The analytic parameterization $X_K(\hat{\overline{F}}_\infty) \to E(\overline{F}_\infty)$ (Gekeler-Reversat)
- Images of Heegner points, analytically or algebraically
- ...