Note Taker Checklist Form -*MSRI*

Name : Rob Stapleton

E-mail Address/ Phone #: jrstaple@ncsu.edu

Talk Title and Workshop assigned to:
Interactive Parallel Computing in Support of Research in Algebra, Geometry, and Number Theory

Lecturer (Full name): Ilias Kotsireas

Date & Time of Event: 2:00 p.m. Jan 31, 2007

Check List:

(✓) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

(✓) Obtain <u>all</u> presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

(✓) Take down <u>all</u> notes from media provided (blackboard, overhead, etc.)

(✓) Gather <u>all</u> other lecture materials (i.e. Handouts, etc.)

(✓) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: parallel, interactive, combinatorics, design, weighing matrix, Hamilton's Conjecture

2. Please summarize the lecture in 5 or less sentences.
Weighing matricies are important in combinatorics. There has been software developed which uses a parallel implementation to help discover and classify classes of weighing matricies.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

2:00 p.m.
Combinatorial Designs
Ilias Kotsireas

Combinatorial Design Theory:
Is it possibel to arrang eelemnts of a finite set into subsets so that certain properties are satisfied.
Applicatoins to cryptography, optical communications, wireless communciations, coding theory

Weighing Matrices:
A weighing matrix W = W(n,k) of weight k is a square nxn matrix with entries -1,0,1, having k non-zero entries per row and column, with inner product of distinct rows zero.
W . Transpose(W) = kI_n where I_n is the nxn identity matrix.

What can we do?
Plan of attack: Establish pattenrs for the locations of the 5 zeros in solutions for weighing matricies of the type W(2n,2n-5)
This would take us down from 3^(2n) to 2^(2n-5) operations.

Idea: Analyze the wolution sets for W(2n,2n-5) for all odd n up to 15
Since we want the weighing matrix to be constructed from two circulants, when we fix 4 zeroes, the location of the 5th zero is fixed in a very symmetric position. A proof of this would probably imply that there is an infinite class of a certain type of polynomials.

To help, since certain problems still require too many operations, is to calculate certain autocorrelating functions. NPAF and PAF, (non) Periodic Autocorrelating Functions.

NPAF = 0 implies PAF = 0. And these functions are related in other ways as well.
Weighing matricies come from sequences with zero PAF.

Power Spectral Density theorem (PSD): Two sequences can be used to make up ciculant matrices A and B that will give W(2n,k) weighing matrices if and only if a certain relation in the terms of the discrete fourier transform (DFT) hold.

There is a theorem that provides a horizontal relationship between the sequence and the PSD, which allows for faster calculating of the DFT coefficients we need.

A solution to W(2*29,53) can now be found (using the PSD and a string sorting algorithm) within a day with serial programs. However, a solution for W(2*33,61) is still not found. It could be that the algorithm is failing because one of the PSD values is an integer.

It turns out, that there ARE cases where the PSD value can be an integer.

An error bounding algorithm was created to find when the algorithm may fail.

Combinatorial Designs:
constructions, algorithms and new results

Ilias S. Kotsireas
Wilfrid Laurier University
ikotsire@wlu.ca

Combinatorial Design Theory

Is it possible to arrange elements of a finite set into subsets so that certain properties are satisfied?

Existence and non-existence results. Infinite classes.

Tools & concepts from: linear algerbra, algebra, group theory, number theory, combinatorics, symbolic computation, numerical analysis.

Applications to: cryptography, optical communications, wireless communications, coding theory.

- W. D. Wallis, A. P. Street, J. Seberry, *Combinatorics: Room squares, sum-free sets, Hadamard matrices.* Springer-Verlag, 1972.
- A. V. Geramita, J. Seberry, *Orthogonal designs. Quadratic forms and Hadamard matrices.* Marcel Dekker Inc. 1979.
- C. J. Colbourn, J. H. Dinitz, *The CRC handbook of combinatorial designs.* CRC Press, 1996.
- V. D. Tonchev, *Combinatorial configurations: designs, codes, graphs.* Longman Scientific & Technical, John Wiley & Sons, Inc., 1988.
- T. Beth, D. Jungnickel, H. Lenz, *Design theory.* Vols. I, II. Second edition. Cambridge University Press, Cambridge, 1999.
- A. S. Hedayat, N. J. A. Sloane, J. Stufken *Orthogonal arrays. Theory and applications.* Springer-Verlag, 1999.
- D. R. Stinson, *Combinatorial designs, Constructions and analysis.* Springer-Verlag, 2004.
- C. J. Colbourn, J. H. Dinitz, *Handbook of Combinatorial Designs.* Second Edition, Chapman and Hall/CRC Press, 2006.
- K. J. Horadam, *Hadamard Matrices and Their Applications.* Princeton University Press, 2006.

Weighing Matrices

A **weighing matrix** $W = W(n, k)$ of weight k, is a square $n \times n$ matrix with entries $-1, 0, +1$ having k non-zero entries per row and column and inner product of distinct rows zero.

$$W \cdot W^t = k\, I_n$$

Fact:

If there is a $W(2n, k)$, n odd, then $k \leq 2n - 1$ and k is the sum of two squares.

Theorem:

If there exist two circulant matrices A, B of order n each, satisfying $A \cdot A^t + B \cdot B^t = k\, I_n$, then there exists a $W(2n, k)$.

$$W(2n,k) = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix}$$

$W(2n, 2n-1)$ constructed from two circulants: infinite class

$W(2n, 2n-3)$ constructed from two circulants: do not exist

Ten Open Problems: [C. Koukouvinos, J. Seberry, JSPI (81), 1999]

Do there exist

$W(2 \cdot 23, 41)$, $W(2 \cdot 25, 45)$, $W(2 \cdot 27, 49)$, $W(2 \cdot 29, 53)$, $W(2 \cdot 33, 61)$,

$W(2 \cdot 35, 65)$, $W(2 \cdot 39, 73)$, $W(2 \cdot 43, 81)$, $W(2 \cdot 45, 85)$, $W(2 \cdot 47, 89)$

constructed from two circulants?

Common feature: $W(2n, 2n-5)$, for $n = 23, \ldots, 47$.

Odd large weights.

R. Craigen, The structure of weighing matrices having large weights.
Des. Codes Cryptogr. (5) 1995

Plan of attack:

Establish potential patterns for the locations of the 5 zeros in solutions.

$$\text{From } 3^{2n} \sim 2^{3.17n} \text{ ops, down to } 2^{2n-5} \text{ ops.}$$

Idea:

Analyze the solutions sets for $W(2n, 2n-5)$ for all odd n up to $n = 15$. (bash/Maple meta-program, C code generation, supercomputing)

First observation: (4 zeros)

$$\begin{array}{cccccccccc} \star & \ldots & \star & 0 & 0 & 0 & 0 & \star & \ldots & \star \\ a_1 & \ldots & a_{n-2} & a_{n-1} & a_n & b_1 & b_2 & b_3 & \ldots & b_n \end{array}$$

Second observation: (the remaining fifth zero)

$$\underbrace{a_1 \star \ldots \star}_{\frac{n-3}{2}\text{ terms}} \quad 0 \quad \underbrace{\star \ldots \star a_{n-2}}_{\frac{n-3}{2}\text{ terms}} \quad 0 \quad 0 \quad 0 \quad 0 \quad \underbrace{b_3 \star \ldots \star b_n}_{n-2\text{ terms}}$$

$$\underbrace{a_1 \star \ldots \star a_{n-2}}_{n-2\text{ terms}} \quad 0 \quad 0 \quad 0 \quad 0 \quad \underbrace{b_3 \star \ldots \star}_{\frac{n-3}{2}\text{ terms}} \quad 0 \quad \underbrace{\star \ldots \star b_n}_{\frac{n-3}{2}\text{ terms}}$$

CRYSTALIZATION When we fix the 4 zeros as indicated above, then the fifth zero can only appear in exactly two possible places, in a $W(2n, 2n-5)$ solution.

A proof will probably use **Hall polynomials, PAF equations**

Implication: Infinite Class of $W(2n, 2n-5)$

Results:

```
W(2*23,41) solution
-1 -1 -1 -1 -1 1  1 -1 1 -1 0  1  1  1 -1 -1  1 -1 -1 1 -1 0  0
0  0 -1 -1  1 1 -1 -1 1  1 1 -1 -1 -1 -1 -1 -1  1 -1 1 -1 1 -1

W(2*25,45) solution
1 1  1  1 1 -1 -1 1 1 -1 1  0 1 -1  1 -1 -1 -1 -1 1 1  1  1 0 0
0 0 -1 -1 1 -1 -1 1 1  1 1 -1 1  1 -1  1 -1  1 -1 1 1 -1 -1 1 1

W(2*27,49) solution
1 1  1  1  1 1 -1 -1 -1 1 -1 -1  0 -1 -1 1  1 -1 -1  1 -1  1 -1 1 -1 0 0
0 0 -1 -1 -1 1  1 -1  1 1 -1 -1 -1 -1 1 1 -1 -1 -1 -1  1 -1 -1 1 -1 1 -1
```

$W(2 \cdot 29, 53)$ is still out of reach, as it still requires 2^{53} ops.

Periodic & non-periodic autocorrelation function

The 2nd elementary symmetric function in n variables $a_1, \ldots, a_n$

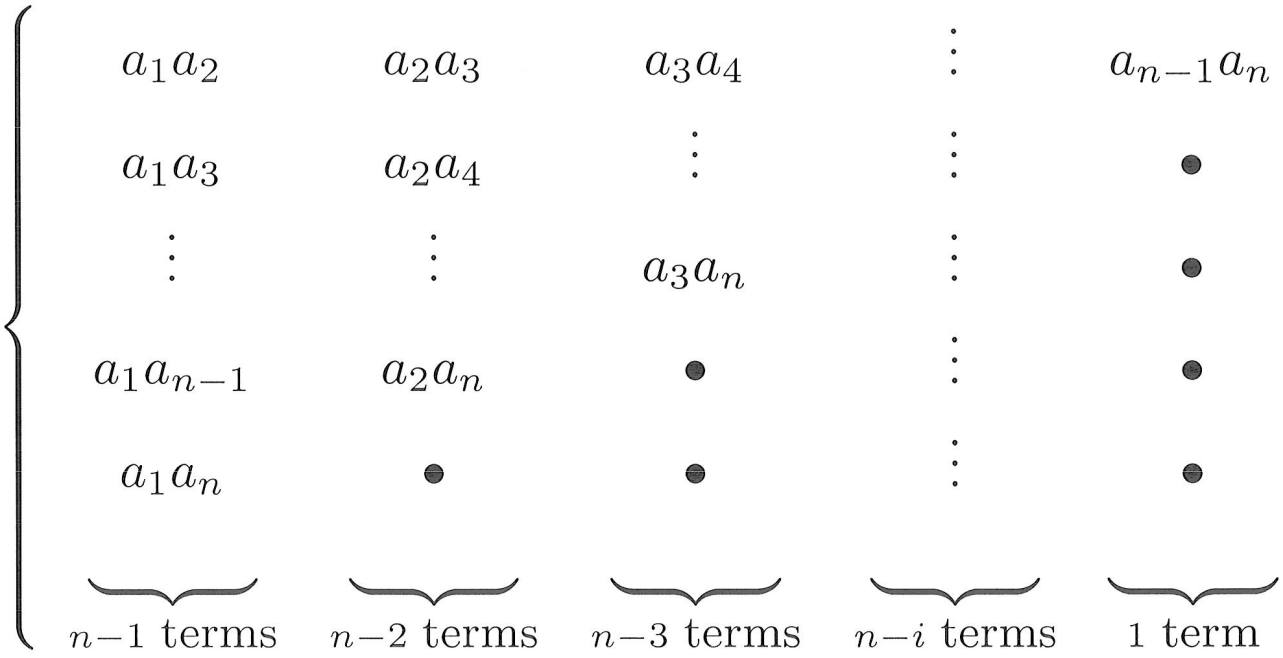
$$\sigma_2 = a_1 a_2 + \cdots a_{n-1} a_n = \sum_{1 \le i < j \le n} a_i a_j$$

plays a pivotal role in building $W(2n, k)$.

PAF and NPAF concepts

σ_2 is made up of $\displaystyle\sum_{i=1}^{n-1} n - i = \frac{n(n-1)}{2} = \binom{n}{2}$

(pairwise different) quadratic monomials:



$$\left\{\begin{array}{ccccccc} a_1a_2 & a_2a_3 & a_3a_4 & \vdots & a_{n-1}a_n & \leftarrow & N_A(1) \\ a_1a_3 & a_2a_4 & \vdots & \vdots & \bullet & \leftarrow & N_A(2) \\ \vdots & \vdots & a_3a_n & \vdots & \bullet & \leftarrow & N_A(3) \\ a_1a_{n-1} & a_2a_n & \bullet & \vdots & \bullet & \vdots & \vdots \\ a_1a_n & \bullet & \bullet & \vdots & \bullet & \leftarrow & N_A(n-1) \end{array}\right.$$

Lemma:

$$N_A(1) + N_A(2) + \ldots + N_A(n-1) = \sigma_2$$

Fact:

$$P_A(s) = N_A(s) + N_A(n-s),\ s = 1, \ldots, n-1$$

Lemma:

$$P_A(1) + P_A(2) + \ldots + P_A(n-1) = 2\sigma_2$$

Fact:

$$NPAF = 0 \Longrightarrow PAF = 0$$

The converse is not always true.

Definition:

Two sequences $A = [a_1, \ldots, a_n]$ and $B = [b_1, \ldots, b_n]$ are said to have zero PAF (resp. NPAF) if

$$P_A(s) + P_B(s) = 0, \;\; i = 1, \ldots, n-1$$

$$\text{resp. } N_A(s) + N_B(s) = 0, \;\; i = 1, \ldots, n-1.$$

Weighing matrices come from sequences with zero PAF.

Fact:

If we can construct two sequences A and B with zero PAF, then we can construct $W(2 \cdot n, k)$ from two circulants.

Power Spectral Density, PSD

PSD Theorem

[Fletcher, Gysin, Seberry, Australas. J. Combin., 23, 2001]

Two sequences $[a_1, \ldots, a_n]$, $[b_1, \ldots, b_n]$ can be used to make up circulant matrices A and B that will give $W(2n, k)$ weighing matrices if and only if

$$PSD([a_1, \ldots, a_n], i) + PSD([b_1, \ldots, b_n], i) = k, \ \forall \ i = 0, \ldots, \frac{n-1}{2}$$

where $PSD([a_1, \ldots, a_n], k)$ denotes the k-th element of the power spectral density sequence, i.e. the square magnitude of the k-th element of the discrete Fourier transform (DFT) sequence associated to $[a_1, \ldots, a_n]$.

The DFT sequence associated to $[a_1, \ldots, a_n]$ is defined as

$$DFT_{[a_1,\ldots,a_n]} = [\,\mu_0, \ldots, \mu_{n-1}\,], \text{ with } \mu_k = \sum_{i=0}^{n-1} a_{i+1}\,\omega^{ik},\ k = 0, \ldots, n-1$$

where $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$ is a primitive n-th root of unity.

The proof is based on the **Wiener-Khinchin Theorem**

- The PSD of a sequence is equal to the DFT of its PAF sequence

$$|\,\mu_k\,|^2 = \sum_{j=0}^{n-1} PAF_A(j)\omega^{jk}$$

- The PAF of a sequence is equal to the inverse DFT of its PSD sequence

$$PAF_A(j) = \frac{1}{n} \sum_{j=0}^{n-1} |\,\mu_k\,|^2\, \omega^{-jk}$$

The **Parseval Theorem** provides a *horizontal* relationship between the elements of a sequence $[a_1, \ldots, a_n]$ and its DFT sequence:

$$\sum_{i=1}^{n} \mid a_i \mid^2 = \frac{1}{n} \sum_{i=1}^{n} PSD([a_1, \ldots, a_n], i)$$

The **PSD theorem** provides a *vertical* relationship between the elements of two sequences $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$.

The **PSD criterion** for $W(2n, k)$ states that:

if for a certain sequence $[a_1, \ldots, a_n]$ there exists $i \in \{1, \ldots, \frac{n-1}{2}\}$ with the property that $PSD([a_1, \ldots, a_n], i) > k$, then this sequence cannot be used to construct $W(2n, k)$.

Important Consequence: we can now **decouple** the PAF equations, roughly corresponding to cutting down the complexity by half.

Algorithm: String Sorting

Begin with

$$PSD([b_1, \ldots, b_n], i) = k - PSD([a_1, \ldots, a_n], i), \ \forall \ i = 0, \ldots, \frac{n-1}{2}$$

and take integer parts

$$[PSD([b_1, \ldots, b_n], i)] = \begin{cases} k - 1 - [PSD([a_1, \ldots, a_n], i)], & \text{is not an integer} \\ k - [PSD([a_1, \ldots, a_n], i)], & \text{is an integer} \end{cases}$$

A pair of vectors $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ can be **encoded** as the concatenation of the integer parts of the first $\frac{n-1}{2}$ components of their PSD vectors:

$$[b_1, \ldots, b_n] \longrightarrow [PSD([b_1, \ldots, b_n], 1)] \ldots$$

$$[a_1, \ldots, a_n] \longrightarrow k - 1 - [PSD([a_1, \ldots, a_n], 1)] \ldots$$

Using the above encoding, the condition that a pair of sequences $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$ can be used as the first rows of circulants to construct $W(2n, k)$ weighing matrices, can be simply phrased by saying that their corresponding string encodings are equal.

Therefore we see that the search for weighing matrices is essentially a string sorting problem.

A solution for $W(2 \cdot 29, 53)$ can now be found within a day, with serial programs.

However: A solution for $W(2 \cdot 33, 61)$ was still not found.

Is it possible that $[\mathbf{PSD}([\mathbf{a_1}, \ldots, \mathbf{a_n}], \mathbf{i})]$ can be an integer?

Rounding Error Treatment

LEMMA Let n be an odd integer such that $n \equiv 0 \, (mod \; 3)$ and let $m = \frac{n}{3}$. Let $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$ the principal n-th root of unity. Let $[a_1, \ldots, a_n]$ be a sequence with elements from $\{-1, 0, +1\}$. Then we have that $DFT([a_1, \ldots, a_n], m)$ can be evaluated explicitly in closed form and $PSD([a_1, \ldots, a_n], m)$ is a non-negative integer. The explicit evaluations are given by

$$DFT([a_1, \ldots, a_n], m) = \left(A_1 - \frac{1}{2}A_2 - \frac{1}{2}A_3\right) + \left(\frac{\sqrt{3}}{2}A_2 - \frac{\sqrt{3}}{2}A_3\right) i$$

$$PSD([a_1, \ldots, a_n], m) = A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_1A_3 - A_2A_3$$

where

$$A_1 = \sum_{i=0}^{m-1} a_{3i+1}, \quad A_2 = \sum_{i=0}^{m-1} a_{3i+2}, \quad A_3 = \sum_{i=0}^{m-1} a_{3i+3}.$$

Sketch of proof: Acknowledgement: Doron Zeilberger

$DFT([a_1, \ldots, a_n], m)$ is a linear combination of $\omega^0, \omega^m, \omega^{2m}$

$$DFT([a_1, \ldots, a_n], m) = \sum_{i=0}^{n-1} a_{i+1}\, \omega^{im} =$$

$$= \left(\sum_{i=0}^{m-1} a_{3i+1}\right) \omega^0 + \left(\sum_{i=0}^{m-1} a_{3i+2}\right) \omega^m + \left(\sum_{i=0}^{m-1} a_{3i+3}\right) \omega^{2m}$$

$$A_1\omega^0 + A_2\omega^m + A_3\omega^{2m}.$$

$\omega^m = e^{\frac{2\pi i}{3}}$ and $\omega^{2m} = e^{\frac{4\pi i}{3}}$ are the roots of the cyclotomic polynomial $\Phi_3(x) = x^2 + x + 1$ and can be evaluated explicitly as:

$$\omega^m = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \;\; \omega^{2m} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Solutions for $W(2 \cdot 33, 61)$ were found. `http://www.cargo.wlu.ca/weighing/`

Lemma 11. *Let* $k = a^2 + b^2$, $k \leqslant 2n - 1$. *Then there exist* $W(2n,k)$ *constructed from two circulants except for*

1. $n = 23,\ k \in \{41, 45\}$;
2. $n = 25,\ k \in \{41, 45\}$;
3. $n = 27,\ k \in \{9, 18, 29, 36, 37, 41, 45, 49, 50\}$;
4. $n = 29,\ k \in \{9, 18, 29, 36, 37, 41, 45, 49, 50, 53\}$;
5. $n = 31,\ k \in \{9, 18, 29, 45, 49, 53, 58\}$;
6. $n = 33,\ k \in \{29, 37, 41, 49, 53, 58, 61, 65\}$;
7. $n = 35,\ k \in \{9, 18, 29, 37, 41, 45, 49, 53, 58, 61, 65, 68\}$;
8. $n = 39,\ k \in \{29, 41, 49, 53, 58, 61, 68, 71, 72\}$.

In addtion there exist $W(2n,k)$ *not constructed from two circulants for*

1. $n = 23,\ k \in \{45\}$;
2. $n = 27, 29, 31,\ k \in \{9, 18\}$;
3. $n = 31, 33, 35, 39,\ k \in \{29\}$.