Definite quaternion algebras and triple-product L-functions

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Objective

The formulas of Gross-Kudla, Böcherer–Schulze-Pillot, Watson and Ichino express central critical values of triple-product L-functions $L(\pi_1 \times \pi_2 \times \pi_3, \frac{1}{2})$ in terms of values of trilinear forms

$$\ell : V_{\pi_1} \otimes V_{\pi_2} \otimes V_{\pi_3} \longrightarrow \mathbb{C}$$

on specific test vectors.

I will discuss joint work with Marco Seveso in which we show that, in the definite case, **the trilinear forms themselves can be constructed in p-adic families**, implying the existence of corresponding 3-variable p-adic L-functions.

Outline

1. Introduction: Special values and arithmetic
   - why? examples
   - families of L-functions and families of special values
   - p-adic variation
2. Automorphic forms and L-functions
   - elliptic modular forms
   - L-functions
   - p-adic families of modular forms
   - automorphic forms on quaternion algebras
3. Special value formulas
   - formulas of Gross, Gross-Kudla
   - higher weight analogues
   - p-adic variation
   - a theorem

1. Special values and arithmetic

Special values of L-functions encode arithmetic invariants.

Prototypical example: K number field, $\mathcal{O} \subset K$ ring of integers

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}} N(I)^{-s} = \sum_{n=1}^{\infty} r_n n^{-s}, \quad \Re(s) > 1,$$

where $\quad N(I) = |\mathcal{O}/I|, \quad r_n = \#$ of ideals of $\mathcal{O}$ of norm n

Theorem: $\zeta_K(s)$ admits analytic continuation to $\mathbb{C} - \{1\}$. Moreover,

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|d_K|}}.$$

Quadratic fields and their discriminants

$$m \text{ squarefree, } K = \mathbb{Q}(\sqrt{m}): \quad d_K = \begin{cases} m & \text{if } m \equiv 1 \pmod 4 \\ 4m & \text{otherwise.} \end{cases}$$

- d_K characterizes K:

$$d \in \mathcal{D} := \{d_K : K \text{ quadratic}\} : \quad K_d = \mathbb{Q}(\sqrt{d}) \text{ has disc. } d$$

- $d \in \mathcal{D} \rightsquigarrow$ Kronecker character:

$$\chi_d : (\mathbb{Z}/d\mathbb{Z})^\times \to \{\pm 1\}, \quad \chi_d(x) = \left(\frac{d}{x}\right)$$

- Dirichlet L-function:

$$L(\chi_d, s) = \sum_{(n,d)=1} \chi_d(n) n^{-s}, \quad \Re(s) > 1$$

Quadratic class number formula

$$K = K_d : \quad \zeta_K(s) = \zeta(s) L(\chi_d, s)$$

Since $\operatorname{res}_{s=1} \zeta(s) = 1$,

$$L(\chi_d, 1) = \begin{cases} \dfrac{h_d \log |u_d|}{\sqrt{d}} & \text{if } d > 0, \\[2ex] \dfrac{\pi h_d}{\sqrt{-d}} & \text{if } d < -4, \end{cases}$$

$$\text{where} \qquad \mathcal{O}_d^\times / \{\pm 1\} = \langle u_d \rangle,$$

$$\mathrm{Cl}_d = \{0 \neq I \subset \mathcal{O}_d\}/\sim, \quad I \sim J \Leftrightarrow aI = bJ, \ a, b \in \mathcal{O}_d, \ ab \neq 0$$

$$\text{Class number:} \quad h_d := |\,\mathrm{Cl}_d\,| < \infty.$$

Using the formula

$$h_d = L(\chi_d, 1)\frac{\sqrt{-d}}{2\pi},$$

the special values at $s = 1$ of the family of L-functions $\{L(\chi_d, 1) : d \in \mathcal{D}^-\}$ can be used to study the behaviour of h_d as $d \to \infty$.

Theorem: (Siegel, 1935) For every $\epsilon > 0$, there is a $C_\epsilon > 0$ such that

$$h_d > C_\epsilon d^{1/2-\epsilon} \quad \forall\, d \in \mathcal{D}^-.$$

Theorem: (Goldfeld, 1976; Gross-Zagier, 1983) For every $\epsilon > 0$, there is an effectively computable constant $C_\epsilon > 0$ such that

$$h_d > C_\epsilon(\log|d|)^{1-\epsilon} \quad \forall\, d \in \mathcal{D}^-.$$

Another interesting family

Consider the (nonunitary) character

$$\rho_{2k} : I_{\mathbb{Q}} \longrightarrow \mathbb{C}^{\times}, \quad \rho_{2k}(n) = n^{2k}.$$

$$L(\rho_{2k}, s) = \sum_{n=1}^{\infty} n^{2k} n^{-s} = \sum_{n=1}^{\infty} n^{-(s-2k)} = \zeta(s-2k)$$

Consider the special values at $s = 1$ of L-functions in the family $\{L(\rho_{2k}, s) : k > 0\}$:

Theorem: (Euler) $L(\rho_{2k}, 1) = -\frac{B_{2k}}{2k} \in \mathbb{Q}$.

The values at $s = 1$ of this family of L-functions display p-adic continuity in the "variable" ρ_{2k}...

Kummer's congruences:

$$2k \equiv 2\ell \pmod{(p-1)p^{n-1}} \implies$$

$$\left(1 - p^{-(1-2k)}\right)\frac{B_{2k}}{2k} \equiv \left(1 - p^{-(1-2\ell)}\right)\frac{B_{2\ell}}{2\ell} \pmod{p^n}$$

or, written differently,

$$\left|L^*(\rho_{2k},1) - L^*(\rho_{2\ell},1)\right|_p \leq p^{-n},$$

where

$$L^*(\rho_{2k},s) = (1-\rho_{2k}(p)p^{-s})L(\rho_{2k},s).$$

Theorem: (Kubota–Leopoldt, 1964) There is a continuous function

$$\zeta_p : \mathbb{Z}_p \times 2\mathbb{Z}/(p-1)\mathbb{Z} \longrightarrow \mathbb{Z}_p$$

extending the mapping

$$2k \mapsto L^*(\rho_{2k},1) \quad \forall\, k \in \mathbb{Z}.$$

Choose an embeddings $\bar{\mathbb{Q}} \subset \mathbb{C}$ and $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$.

Let Φ be a collection of objects to which we can attach L-functions.

The philosophy of p-adic variation

Suppose that

- Φ has a p-adic analytic structure – it's a "p-adic family"
- we can make sense of $L(\varphi, s_0)$ as an algebraic number for all $\varphi \in \Phi$.

Then we should investigate the p-adic properties of the function

$$L_p(\,\cdot\,) : \varphi \mapsto e^{(p)}(\varphi, s_0)L(\varphi, s_0),$$

where $e^{(p)}(\varphi, s)$ is the factor at p in the Euler product for $L(\varphi, s)$.

2. Automorphic forms and L-functions

Elliptic modular forms

$$f(z) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i n z}, \quad z \in \mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$$

holomorphic, *weight* k, *level* N:

$$(f|\gamma)(z) = f(z) \qquad \text{for}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \left\{ \gamma \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod{N} \right\} \subset \mathrm{SL}_2(\mathbb{Z})$$

(and holomorphic at the cusps $\neq \infty$)

- a *cusp form* if $a_0(f) = 0$ (and vanishes at the other cusps $\neq \infty$)
- Notation: $M_k(N)$ for modular forms, $S_k(N)$ for cusp forms

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holomorphic, *weight* k, *level* N:

$$(f|\gamma)(z) := (\det\gamma)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z) \qquad \text{for}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \left\{\gamma \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod{N}\right\} \subset \mathrm{SL}_2(\mathbb{Z})$$

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Arithmeticity of modular forms

- subgroup structure of $\mathrm{SL}_2(\mathbb{Z}) \rightsquigarrow$ Hecke operators T_n
- the T_n are self-adjoint and commute pairwise $\Rightarrow$ they're simultaneously diagonalizable
- systems of Hecke eigenvalues are algebraic integers
- If f is a normalized, cuspidal eigenform $(a_1(f) = 1)$, then $f|T_n = a_n(f)f$.

p-adic families of modular forms

Suppose

- f is a normalized eigenform of weight k_0 and level N,
- $\operatorname{ord}_p a_p(f) < k_0 - 1 \quad$ ("small slope").

Theorem: (Hida, Coleman) There is a p-adic domain

$$\Omega \subset \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$$

with $k_0 \in \Omega$ and analytic functions $\mathbf{a}_n$ on Ω such that:

- for each $k \in \Omega \cap \mathbb{Z}$, $k > k_0$, $\mathbf{a}_n(k)$ is algebraic for all n, and

  $$\mathbf{f}_k := \sum_{n=1}^{\infty} \mathbf{a}_n(k) q^n \in \bar{\mathbb{Q}}[[q]] \subset \mathbb{C}[[q]], \quad q := e^{2\pi i z},$$

  is the Fourier expansion of an eigenform of weight k,
- $\mathbf{f}_{k_0} = f$

p-adic family of Eisenstein series

$$E_{2k}^{(p)}(q) = \frac{L^*(\rho_{2k}, 1)}{2} + \sum_{n=1}^{\infty} \sigma_{2k-1}^{(p)}(n) q^n \in M_{2k}(p)$$

$$\text{where} \quad \sigma_{2k-1}^{(p)}(n) = \sum_{\substack{d \mid n \\ p \nmid d}} d^{2k-1}$$

$$k \equiv \ell \pmod{(p-1)p^{N-1}} \Longrightarrow E_{2k}^{(p)} - E_{2\ell}^{(p)} \in p^N \mathbb{Z}_{(p)}[[q]]$$

- Cuspidal examples of p-adic families typically don't have such simple q-expansions, i.e., nice formulas for the $\mathbf{a}_n$.

L-functions of modular forms

Let $f \in S_k(N)$ be a normalized, primitive eigenform.

$$\Re(s) > k : \quad L(f,s) = \prod_{\ell \nmid N} \left(1 - a_\ell(f)\ell^{-s} + \ell(\ell^{-s})^2\right)^{-1} \prod_{\ell | N} (\cdots)$$

Mellin transform:

$$\Lambda(f,s) = N^{s/2} \int_0^\infty f(iy) y^s \frac{dy}{y} = L(f,s) N^{s/2} (2\pi)^{-s} \Gamma(s)$$

Theorem: (Hecke) $\Lambda(f,s)$ has analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(f,s) = w_N(f)\Lambda(f,k-s), \quad w_N(f) \in \{\pm 1\} \quad (f|W_N = w_N(f)f).$$

- We need more than classical modular forms to study the L-functions of modular forms!

Quaternion algebras

- A quaternion $\mathbb{Q}$-algebra B is a 4-dimensional central, simple $\mathbb{Q}$-algebra.
- There is a finite set F of places of $\mathbb{Q}$ such that

$$B \otimes_{\mathbb{Q}} \mathbb{Q}_v \begin{cases} \cong M_2(\mathbb{Q}_v) & \text{if } v \notin F, \\ \text{is a division algebra} & \text{if } v \in F. \end{cases}$$

- If $v \in F$, v is said to *ramify in* B. F characterizes B, up to isomorphism. The *discriminant of* B is the quantity

$$\prod_{\ell \in F, \ell \neq \infty} \ell.$$

- B is called *definite* $\iff B_\infty \cong \mathbb{H} \iff \infty \in F$

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij, \qquad i^2 = j^2 = k^2 = -1,\ ji = -ij$$

Quaternionic orders and ideals

- An *order R in B* is a subring of B free of rank 4 over $\mathbb{Z}$.
- An *Eichler order R of level N^+*, $(N^+, N^-) = 1$, is an order such that for all $\ell \nmid N^-$,

$$R \otimes \mathbb{Z}_\ell \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_\ell) : c \in N\mathbb{Z}_\ell \right\}$$

- $R_{N^-,N^+} :=$ Eichler order of level N^+ in the quaternion algebra of discriminant N^-
- A rank 4 $\mathbb{Z}$-submodule I of B is called a *left R-ideal* if

$$R = \{x \in B : xI \subset I\}.$$

- $\mathcal{I}_R :=$ set of left R-ideals.

Automorphic forms on definite quaternion algebras

$$R = R_{N^-,N^+}: \quad M_2(N^-, N^+) := \{f : \mathcal{I}_R/B^\times \longrightarrow \mathbb{Q}\}$$

ideal class representatives: $\mathcal{I}_R/B^\times = \{[I_i] : i = 1, \ldots, h\}$

standard basis: $e_i(I_j) := \delta_{i,j}$

inner product: $\langle F, G \rangle := \sum_{i=1}^{h} \frac{1}{w_i} f(I_i) g(I_i)$

cusp forms: $\mathcal{S}_2(N^-, N^+) := \ker\langle \cdot\,, \mathbf{1} \rangle \subset M_2(N^-, N^+)$

Hecke operators

ideal structure of $R_{N^-,N^+} \rightsquigarrow$ operators T_n on $M_2(N^-, N^+)$ & S_2

self-adjoint: $\langle F | T_n, G \rangle = \langle F, G | T_n \rangle$

theta-function: $\Theta(F, G)(q) := \sum_{n=1}^{\infty} \langle F | T_n, G \rangle q^n$

Proposition: (Brandt matrices $B(n)$) We have:

$$\Theta(e_i, e_j) = \frac{1}{2w_j} \sum_{x \in I_j^{-1} I_i} q^{N(x)/N(I_j^{-1} I_i)} = \sum_{n=0}^{\infty} B_{i,j}(n) q^n,$$

where $B_{i,j}(n) = \#$ of ideals of norm n right equivalent to $I_j^{-1} I_i$.

Eichler's correspondence (aka. Jacquet-Langlands)

$$\mathbb{T} := \mathbb{Q}[T_n : (n, N^-) = 1] \subset \mathrm{End}_{\mathbb{Q}} S_2(N^-, N^+),$$

$$N := N^- N^+.$$

Theorem: (Eichler) The map

$$\Theta : S_2(N^-, N^+) \otimes_{\mathbb{T}} S_2(N^-, N^+) \longrightarrow S_2(N)^{N^-\text{-new}}$$

is an isomorphism.

Corollary: Fix $0 \neq \star \in S_2(N)$. Then

$$\Theta_\star : S_2(N^-, N^+) \longrightarrow S_2(N)^{N^-\text{-new}}, \quad \Theta_\star(F) = \Theta(F, \star)$$

is a $\mathbb{T}$-equivariant isomorphism.

3. Special value formulas

Suppose:

- $N = N^- N^+$ is squarefree,
- d is a fundamental discriminant, $d < -4$, $(d, N) = 1$.
- $\chi_d(\ell) = -1$ for all $\ell | N^-$,
- $\chi_d(\ell) = +1$ for all $\ell | N^+$.

Let $\psi : G_K^{ab} \to \mathbb{C}^\times$ be a finite order, anticyclotomic character.

Theorem:

(Gross, 1987; Hatcher; Dagigh; Xue; Yuan–Zhang–Zhang, 2011)
There is a linear functional

$$\ell_\psi : S_2(N^-, N^+) \to \mathbb{Q}(\psi)$$

such that for all Hecke eigenforms $F \in S_2(N^-, N^+)$,

$$\frac{L(\Theta_\star(F), \psi, 1)}{\|\Theta_\star(F)\|^2} = d^{-1/2} \frac{|\ell_\psi(F)|^2}{\|F\|^2}.$$

Gross-Kudla formula

$$f, g, h \in S_2(N), \quad \Sigma = \{\ell | N : -a_\ell(f) a_\ell(g) a_\ell(h) = -1\}.$$

Suppose $|\Sigma|$ is odd.

$$B := \text{quaternion } \mathbb{Q}\text{-algebra ramified at } \Sigma \cup \{\infty\} \qquad \text{(definite)}$$

$$N^- := \prod_{\ell \in \Sigma} \ell, \quad N^+ = N/N^-.$$

$$\exists\, F, G, H \in S_2(N^-, N^+) \text{ s.t. } \Theta_\star(F) = f,\ \Theta_\star(G) = g,\ \Theta_\star(H) = h$$

Theorem: (Gross-Kudla, 1992; Böcherer–Schulze-Pillot; Ichino)

$$\frac{L(f \times g \times h, 2)}{\|f\|^2 \|g\|^2 \|h\|^2} \doteq \frac{\left| \sum_i w_i^{-2} F(I_i) G(I_i) H(I_i) \right|^2}{\|F\|^2 \|G\|^2 \|H\|^2}$$

Higher weight

Let F be a field that splits B:

$$B \otimes F \cong M_2(F). \qquad (\text{e.g. } F = \mathbb{Q}_\ell,\ \ell \nmid N^-)$$

$$P_k = \{P(x,y) \in F[x,y] : P(tx,ty) = t^k P(x,y)\}$$
$$V_k = \mathrm{Hom}(P_k, F)$$

$$\gamma \in \mathrm{SL}_2(F) : \quad (\gamma P)(x,y) = P\big((x,y)\gamma\big), \quad (\ell\gamma)(P) = \ell(\gamma P)$$

$$M_{k+2}(N^-, N^+) = \{f : \mathcal{I}_R \longrightarrow V_k : f(I\gamma) = f(I)\gamma\ \forall\ \gamma \in B^\times\}$$

Gross-Kudla in higher weight

weight 2: $$\frac{L(f \times g \times h, 2)}{\|f\|^2\|g\|^2\|h\|^2} \doteq \frac{\left|\sum_i w_i^{-2} F(I_i) G(I_i) H(I_i)\right|^2}{\|F\|^2\|G\|^2\|H\|^2}$$

What's the analogue of the <u>trilinear form</u>

$$(F, G, H) \mapsto \sum_i w_i^{-2} F(I_i) G(I_i) H(I_i)$$

for $F \in M_{k+2}(N^-, N^+)$, $G \in M_{\ell+2}(N^-, N^+)$, $H \in M_{m+2}(N^-, N^+)$?

To answer this, we need some representation theory of SL_2:

- self-duality of highest weight representations
- Clebsch-Gordan decomposition

$$P_k^\iota = P_k \text{ with "reversed" action } P\gamma := \gamma^\iota P,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Highest weight vectors:

$$E_k \in P_k^\iota,\ E_k(x,y) = x^k; \qquad \delta_{(1,0)} \in V_k,\ \delta_{(1,0)}(P) = P(1,0)$$

$$E_k \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = t^k E_k, \qquad \delta_{(1,0)} \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = t^k \delta_{(1,0)}$$

$$E_k \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} = E_k, \qquad \delta_{(1,0)} \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} = \delta_{(1,0)}$$

Proposition: There is a unique $\mathrm{SL}_2(F)$-equivariant map $\varphi : V_k \to P_k^\iota$ such that $\varphi(\delta_{(1,0)}) = E_k$. It is an isomorphism.

Proposition: (Clebsch-Gordan)

$$\ell, m \text{ even}, \ \ell > m: \qquad V_\ell \otimes V_m = \bigoplus_{\substack{\ell-m<i<\ell+m \\ i \text{ even}}} V_i$$

Assume: $k > \ell > m$ even, $k < \ell + m$, $s := \dfrac{-k+\ell+m}{2}$

$$\text{then} \qquad \exists\,!\, \varphi_{k,\ell,m} : V_k \longrightarrow V_\ell \otimes V_m = \left(P_\ell^{x_1,y_1} \otimes P_m^{x_2,y_2}\right)^\iota$$

$$\text{such that} \qquad \varphi_{k,\ell,m}(\delta_{(1,0)}) = x_1^{\ell-s} x_2^{m-s} (x_1 y_2 - x_2 y_1)^s =: E_{k,\ell,m}.$$

Trilinear form: Since $(V_\ell \otimes V_m)^\vee = V_\ell \otimes V_m$,

$$\varphi_{k,\ell,m} \in \mathrm{Hom}(V_k, V_\ell \otimes V_m) = \mathrm{Hom}(V_k \otimes V_\ell \otimes V_m, F).$$

$\varphi_{k,\ell,m}$, induces

$$\Phi_{k,\ell,m} : M_{k+2}(N^-, N^+) \otimes M_{\ell+2}(N^-, N^+) \otimes M_{m+2}(N^-, N^+) \longrightarrow F.$$

Theorem: (Böcherer–Schulze-Pillot, 1995) Suppose f_k, g_ℓ, and h_m have weights $k+2$, $\ell+2$, and $m+2$, respectively, and that $\Theta_\star(F_k) = f_k$, $\Theta_\star(G_\ell) = g_\ell$, and $\Theta_\star(H_m) = h_m$. Then

$$\frac{L(f_k \times g_\ell \times h_m, c_{k,\ell,m})}{\|f_k\|^2 \|g_\ell\|^2 \|h_m\|^2} \doteq \frac{\left|\Phi_{k,\ell,m}(F_k \otimes G_\ell \otimes H_m)\right|^2}{\|F_k\|^2 \|G_\ell\|^2 \|H_m\|^2},$$

where

$$c_{k,\ell,m} = \frac{k+\ell+m+4}{2}.$$

Question: What happens if we vary f_k, g_ℓ, and h_m in p-adic families?

The universal highest weight representation

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p) : a \in \mathbb{Z}_p^\times,\ c \in p\mathbb{Z}_p \right\}$$
$$= \{\gamma \in \mathrm{GL}_2(\mathbb{Q}_p) : \mathbb{Z}_p\gamma \subset \mathbb{Z}_p\}$$

Theorem: (Stevens) There exist:

- a $\mathbb{Q}_p$-Fréchet algebra R_{univ} and a *universal weight*
$$\varphi_{\mathrm{univ}} : \mathbb{Z}_p^\times \longrightarrow R_{\mathrm{univ}},$$
- a Fréchet R_{univ}-module $\mathcal{D}_{\mathrm{univ}}$ and a vector $\delta_{\mathrm{univ}} \in \mathcal{D}_{\mathrm{univ}}$ such that $(\mathcal{D}_{\mathrm{univ}}, \delta_{\mathrm{univ}})$ is a highest weight Σ-module for φ_{univ},

such that for any highest weight representation (V, v) of $\mathrm{SL}_2(\mathbb{Q}_p)$ with weight φ, there are unique maps

$$\rho : R_{\mathrm{univ}} \longrightarrow \mathbb{Q}_p, \quad \rho : \mathcal{D}_{\mathrm{univ}} \otimes_\rho \mathbb{Q}_p \longrightarrow V$$

such that $\rho \circ \varphi_{\mathrm{univ}} = \varphi$ and $\rho(\delta_{\mathrm{univ}}) = v$.

$$\begin{aligned} \mathcal{A} = \mathcal{A}(\mathbb{Z}_p^\times \times \mathbb{Z}_p) &:= \text{locally analytic functions on } \mathbb{Z}_p^\times \times \mathbb{Z}_p \\ \mathcal{D}_{\mathrm{univ}} = \mathcal{D}(\mathbb{Z}_p^\times \times \mathbb{Z}_p) &:= \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}, \mathbb{Q}_p) \\ &= \text{locally analytic distributions on } \mathbb{Z}_p^\times \times \mathbb{Z}_p \\ \delta_{\mathrm{univ}} &= \delta_{(1,0)} \in \mathcal{D} \text{ univ. highest weight vector} \end{aligned}$$

Universality: Given (V, v) highest weight representation of $\mathrm{SL}_2(\mathbb{Q}_p)$, define

$$J : \mathbb{Z}_p^\times \times \mathbb{Z}_p \to V, \quad J(x, y) = v \begin{pmatrix} x & y \\ & 1 \end{pmatrix}$$

and

$$\rho : (\mathcal{D}_{\mathrm{univ}}, \delta_{\mathrm{univ}}) \to (V, v), \quad \rho(\mu) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} J(x, y) d\mu(x, y).$$

p-adic families of automorphic forms

$$\mathbf{M}(N^-, N^+) := \mathcal{D}_{\text{univ}}\text{-valued automorphic forms for } R_{N^-,N^+}$$

$$\rho_k : \mathcal{D}_{\text{univ}} \to V_k, \quad \rho_k(\delta_{\text{univ}}) = \delta_{(1,0)}$$

Theorem: (Chenevier, 2003) Given an eigenform f_k in $M_{k+2}(N^-, N^+)$, with

$$\text{ord}_p\, a_p(f_k) < k + 1,$$

there is a unique $\mathbf{f}$ in $\mathbf{M}(N^-, N^+)$ such that $\rho_k(\mathbf{f}) = f_k$.

p-adic families of trilinear forms

Setting $X = \mathbb{Z}_p^\times \times \mathbb{Z}_p$, we might look for a diagram like:

$$\begin{array}{ccc} \mathcal{D}_{\text{univ}} \otimes \mathcal{D}_{\text{univ}} \otimes \mathcal{D}_{\text{univ}} & \xrightarrow{\varphi_{\text{univ}}} & R_{\text{univ}} \otimes R_{\text{univ}} \otimes R_{\text{univ}} \\ \downarrow{\scriptstyle \rho_k \otimes \rho_\ell \otimes \rho_m} & & \downarrow{\scriptstyle \rho_k \otimes \rho_\ell \otimes \rho_m} \\ V_k \otimes V_\ell \otimes V_m & \xrightarrow[\varphi_{k,\ell,m}]{} & \mathbb{Q}_p \otimes \mathbb{Q}_p \otimes \mathbb{Q}_p = \mathbb{Q}_p \end{array}$$

We don't get this. We get a similar diagram, but with

$$(\rho_k \otimes \rho_\ell \otimes \rho_m) \circ \varphi_{\text{univ}} = (e_{k,\ell,m}^{(p)} \cdot \varphi_{k,\ell,m}) \circ (\rho_k \otimes \rho_\ell \otimes \rho_m)$$

$$\text{where} \quad e_{k,\ell,m}^{(p)} = \text{Euler-like factor at } p.$$

Constructing φ_{univ}

Recall that we constructed

$$\varphi_{k,\ell,m} \in \text{Hom}(V_k, V_\ell \otimes V_m) = \text{Hom}(V_k \otimes V_\ell \otimes V_m, F).$$

by identifying a highest weight k vector

$$E_{k,\ell,m} \in V_\ell \otimes V_m = (P_\ell^{x_1,y_1} \otimes P_m^{x_2,y_2})^\iota.$$

To construct φ_{univ} we do the same thing, but with universal objects instead of the Vs and Ps.

- "Clebsch-Gordan in families"

A theorem

Let f_k, g_ℓ, and h_m be in $S_k(N_f)$, $S_\ell(N_g)$, and $S_m(N_h)$, respectively such that

$$k,\ \ell,\ m \text{ are even}, \quad m - \ell < k < m + \ell.$$

Set $N = \gcd(N_f, N_g, N_h)$, suppose that

$$\Sigma := \{q | N : -w_q(f_k) w_q(g_\ell) w_q(h_m) = -1\}$$

has <u>odd</u> size, and let

$$N^- = \prod_{q \in \Sigma} q, \quad N^+ = N/N^-.$$

Suppose $p \nmid N$ and that

$$\operatorname{ord}_p a_p(f_k) < k-1, \quad a_p(g_\ell) < \ell - 1, \quad a_p(h_m) < m-1.$$

Let $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ be the p-adic families through f_k, g_ℓ, and h_m.

Theorem: (G–Seveso, 2012) There is a p-adic analytic function $\mathcal{L}_p$ such that

$$\frac{L(\mathbf{f}_\kappa \times \mathbf{g}_\lambda \times \mathbf{h}_\mu, c_{\kappa,\lambda,\mu})}{\|\mathbf{f}_\kappa\|^2 \|\mathbf{g}_\lambda\|^2 \|\mathbf{h}_\mu\|^2} \doteq C_p \cdot C_{N_f, N_g, N_h} \cdot \mathcal{L}_p(\kappa, \lambda, \mu)$$

for all positive, even integers κ, λ, and μ p-adically close to k, ℓ, and m, respectively, in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}$.

Thanks!

Questions?