# MODULO ONE UNIFORM DISTRIBUTION OF CERTAIN FIBONACCI-RELATED SEQUENCES

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Let $\{x_j\}_1^\infty$ be a sequence of real numbers with corresponding fractional parts $\{\beta_j\}_1^\infty$, where $\beta_j = x_j - [x_j]$ and the bracket denotes the greatest integer function. For each $n \geq 1$, we define the function $F_n$ on $[0,1]$ so that $F_n(x)$ is the number of those terms among $\beta_1, \cdots, \beta_n$ which lie in the interval $[0,x)$, divided by n. Then $\{x_j\}_1^\infty$ is said to be uniformly distributed modulo one iff $\lim_{n\to\infty} F_n(x) = x$ for all $x \in [0,1]$.

In other words, each interval of the form $[0,x)$ with $x \in [0,1]$, contains asymptotically a proportion of the $\beta_n$'s equal to the length of the interval, and clearly the same will be true for any sub-interval $(\alpha,\beta)$ of $[0,1]$. The classical Weyl criterion [1, p. 76] states that $\{x_j\}_1^\infty$ is uniformly distributed mod 1 iff

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} e^{2\pi i\nu x_j} = 0 \quad \text{for all } \nu \geq 1 .$$

An example of a sequence which is uniformly distributed mod 1 is $\{n\xi\}_{n=0}^\infty$, where $\xi$ is an arbitrary irrational number. (See [1, p. 81] for a proof using Weyl's criterion.)

The purpose of this paper is to show that the sequence $\{\ln F_n\}_1^\infty$ and $\{\ln L_n\}_1^\infty$ are uniformly distributed mod 1. More generally, we show that if $\{V_n\}_1^\infty$ satisfies the Fibonacci recurrence

$$V_{n+2} = V_{n+1} + V_n$$

for $n \geq 1$ with $V_1 = K_1 > 0$ and $V_2 = K_2 > 0$ as initial values, then $\{\ln V_n\}_1^\infty$ is uniformly distributed mod 1. Toward this end, the following two lemmas are helpful.

Lemma 1. If $\{x_j\}_1^\infty$ is uniformly distributed mod 1 and $\{y_j\}_1^\infty$ is a sequence such that

$$\lim_{j\to\infty} (x_j - y_j) = 0 ,$$

then $\{y_j\}_1^\infty$ is uniformly distributed mod 1.

Proof. From the hypothesis and the continuity of the exponential function, it follows that

$$\lim_{j\to\infty}\left(e^{2\pi i\nu x_j} - e^{2\pi i\nu y_j}\right) = 0 .$$

But it is well known [2, Theorem B, p. 202], that if $\{\gamma_n\}$ is a sequence of real numbers converging to a finite limit L, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_1^n \gamma_j = L .$$

Taking

$$\gamma_j = e^{2\pi i\nu x_j} - e^{2\pi i\nu y_j} ,$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_1^n\left(e^{2\pi i\nu x_j} - e^{2\pi i\nu y_j}\right) = 0 .$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_1^n e^{2\pi i\nu x_j} = 0$$

by Weyl's criterion, we also have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} e^{2\pi i \nu y_j} = 0$$

and the sufficiency of Weyl's criterion proves the sequence $\{y_j\}_1^\infty$ to be uniformly distributed mod 1.

Lemma 2. If $\alpha$ is an algebraic number, then $\ln \alpha$ is irrational.

Proof. Assume, to the contrary, that $\ln \alpha = p/q$, where p and q are non-zero integers. Then $e^{p/q} = \alpha$, so that $e^p = \alpha^q$. But $\alpha^q$ is algebraic, since the algebraic numbers are closed under multiplication [1, p. 84]. Thus $e^p$ is algebraic, in turn implying e is algebraic. But e is known to be transcendental [1, p. 25] so that a contradiction is obtained.

Theorem. Let $\{V_n\}_1^\infty$ be a sequence generated by the recursion formula

$$V_{n+2} = V_{n+1} + V_n$$

for $n \geq 1$ given that $V_1 = K_1 > 0$ and $V_2 = K_2 > 0$. Then the sequence $\{\ln V_n\}_1^\infty$ is uniformly distributed modulo one.

Proof. The recursion (difference equation) has general solution

$$V_n = C_1\alpha^n + C_2\beta^n ,$$

where $\alpha,\beta$ are the roots of the equation $x^2 - x - 1 = 0$ and $C_1$, $C_2$ are constants determined by the initial conditions. Thus

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} ,$$

while $C_1\alpha + C_2\beta = K_1$ and $C_1\alpha^2 + C_2\beta^2 = K_2$. Now,

$$|V_n - C_1\alpha^n| = |C_2\beta^n|$$

for $n \geq 1$, so that, noting $|\beta| < 1$, we have

$$\lim_{n \to \infty} |V_n - C_1\alpha^n| = 0 .$$

Moreover, from the fact that $\{V_n\}_1^\infty$ is an increasing positive sequence,

$$\left|1 - \frac{C_1\alpha^n}{V_n}\right| = \left|\frac{V_n - C_1\alpha^n}{V_n}\right| \leq \frac{1}{K_1}\left|V_n - C_1\alpha^n\right| ,$$

so that

$$\lim_{n \to \infty} \frac{C_1\alpha^n}{V_n} = 1 .$$

Thus,

$$\lim_{n \to \infty} \ln\left(\frac{C_1\alpha^n}{V_n}\right) = 0 ,$$

or equivalently,

(2) $$\lim_{n \to \infty} [\ln (C_1\alpha^n) - \ln V_n] = 0 .$$

Since $\alpha$ is algebraic $(\alpha^2 = \alpha + 1)$, it follows from Lemma 2 that $\ln \alpha$ is irrational and consequently [1, p. 84] that

$$\{n \ln \alpha\}_1^\infty = \{\ln (\alpha^n)\}_1^\infty$$

is uniformly distributed mod 1. Then

$$\{\ln (C_1\alpha^n)\}_1^\infty$$

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