Neumann to Dirichlet Map

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Abstract

The purpose of this paper is to derive the Neumann to Dirichlet Map of a connected electrical network and to explicate some of its properties, namely, linearity, symmetry, positive semi-definiteness, the fact that row and column sums are equal to zero, and that the map is not necessarily a Kirchhoff matrix. Also, relationships among entries in the map will be discussed, with a brief discourse concerning the duals of graphs.

Contents

1 Derivation of Neumann to Dirichlet Map

1.1 Explication

Consider the electrical network $\Gamma = (G, \gamma)$ with V being the set of vertices, ∂V the set of boundary vertices, and $\text{int}V = V - \partial V$ the set of interior vertices. We will further require that G be a graph with boundary such that every connected component of G contains a boundary vertex. Let K be the Kirchhoff matrix of Γ, partitioned in the conventional manner with boundary vertices followed by interior vertices such that

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Let ψ be a column vector corresponding to the Neumann data on the boundary and u be a column vector corresponding to the electric potential at the vertices of G (also partitioned in the conventional manner such that $u = \begin{bmatrix} x \\ y \end{bmatrix}$). Thus,

$$Ku = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} u = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \psi \\ 0 \end{bmatrix} \tag{1}$$

Also note $e^T\psi = 0$ where $e^T = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$. Notice that u is not uniquely determined by ψ, since the addition of a constant vector to a solution will also yeild a solution.

Lemma 1.1. Let $\Gamma = (G, \gamma)$ be a connected electrical network, where V is the vertices. Let K be the Kirchhoff matrix of Γ, partitioned in the conventional manner. Then, the submatrix C of K is nonsingular if and only if $\partial V \neq \emptyset$.

Proof. By definition, the row and column sums of K are zero. That is, the constant vector e is in the nullspace of K, which implies that $\det K = 0$. Furthermore, the determinant of any principal proper submatrix of K is nonzero. If $\partial V = \emptyset$, then $C = K$, which implies that C is singular. On the other hand, if $\partial V \neq \emptyset$, then C is a principal proper submatrix of K, which implies that C is nonsingular. □

Lemma 1.2. Let $\Gamma = (G, \gamma)$ be an electrical network and K be the Kirchhoff matrix of Γ, partitioned in the conventional manner. Then, the submatrix C of K is nonsingular if and only if G is *bounded*, i.e. every connected component of G contains a boundary vertex.

Proof. Say that G consists of n connected components. Call these components $G_1, \dots, G_n = (V_1, E_1), \dots, (V_n, E_n)$. Order the interior nodes of G such that

$$C = \begin{bmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C_n \end{bmatrix},$$

where C_k corresponds to interior-interior connections within the kth connected component of G. It follows that

$$\det C = \prod_{i=1}^{n} \det C_i.$$

That is, C is singular if and only if there exists C_i such that C_i is singular.

By the previous lemma, C is nonsingular if and only if for all $i \in [1, n]$, $\partial V_i \neq \emptyset$. The claim follows. $\square$

1.2 Derivation

In order to ensure the uniqueness of u given ψ, we will require that

$$e^T x = 0 \tag{2}$$

That is to say that the sum of the electric potential around the boundary is equal to zero. While distinguishing an electric ground might seem more natural in a physical situation and would also ensure uniqueness, the motivation for the specified requirement is to retain symmetry, and this symmetry will be seen to be rather convenient and useful.

With (1) and (2), we have

$$\begin{aligned} Ax + By &= \psi \\ B^T x + Cy &= 0 \end{aligned}$$

As previously mentioned the preceding equations do not uniquely define x. Since C is a principal proper submatrix of K and G is connected, C is invertible, so

$$\begin{aligned} y &= -C^{-1} B^T x \\ \psi &= (A - BC^{-1} B^T) x =: \Lambda x \end{aligned}$$

Expressing these equations in matrix form,

$$\begin{bmatrix} \Lambda \\ e^T \end{bmatrix} x = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$$

and by multiplying on the left with $\begin{bmatrix} \Lambda & \epsilon \end{bmatrix}$, we get

$$\begin{aligned} \begin{bmatrix} \Lambda & \epsilon \end{bmatrix} \begin{bmatrix} \Lambda \\ e^T \end{bmatrix} x &= \Lambda \psi \\ (\Lambda^2 + E) x &= \Lambda \psi \end{aligned} \tag{3}$$

where $E = ee^T$. Note that $\Lambda^2 + E$ is invertible.

Proof. Assume

$$\begin{aligned}
(\Lambda^2 + E)x = \begin{bmatrix} \Lambda & \epsilon \end{bmatrix} \begin{bmatrix} \Lambda \\ e^T \end{bmatrix} x = 0 \\
\Rightarrow x^T \begin{bmatrix} \Lambda & \epsilon \end{bmatrix} \begin{bmatrix} \Lambda \\ e^T \end{bmatrix} x = 0 \\
\Rightarrow \|\Lambda x\|^2 + \|e^T x\|^2 = 0
\end{aligned}$$

$\Lambda x = 0$ if and only if x is a constant vector, and $e^T x = 0$ if and only if the entries of x sum to zero. The only vector that satisfies both of these conditions is the zero vector, thus $\Lambda^2 + E$ is invertible. $\square$

Finally, we may conclude $x = (\Lambda^2 + E)^{-1}\Lambda\psi =: H\psi$ where H (capital eta) is *a* Neumann to Dirichlet Map (and subsequently *the* Neumann to Dirichlet Map).

Notice that in (3) any non-zero scalar multiple of e or e^T will have no effect on our imposed stipulation, (2), due to its homogeneity, and the previous argument will still follow. Let us consider the case where an arbitrary scalar, α, is introduced. Then let $H_\alpha = (\Lambda^2 + \alpha^2 E)^{-1}\Lambda$ which is an equally valid Neumann to Dirichlet Map.

Remark 1.3. $H_\alpha = H$ for all $\alpha \neq 0$.

Proof. Let us take the derivative of H_α with respect to α.

$$\begin{aligned}
\frac{d}{d\alpha} H_\alpha &= \frac{d}{d\alpha}(\Lambda^2 + \alpha^2 E)^{-1}\Lambda \\
&= -(\Lambda^2 + \alpha^2 E)^{-1}(2\alpha E)(\Lambda^2 + \alpha^2 E)^{-1}\Lambda \\
&= -(\Lambda^2 + \alpha^2 E)^{-1}(2\alpha E)\Lambda(\Lambda^2 + \alpha^2 E)^{-1} \\
&= 0
\end{aligned}$$

since Λ commutes with $(\Lambda^2 + \alpha^2 E)^{-1}$, which will be proved further on in (4), and because the row and column sums of Λ are equal to zero. Thus we may conclude that $H_\alpha = (\Lambda^2 + \alpha^2 E)^{-1}\Lambda = (\Lambda^2 + E)^{-1}\Lambda = H$ for all $\alpha \neq 0$. $\square$

1.3 Abstract Justification

Let $D = \{\xi : e^T\xi = 0\}$ and n be the number of boundary vertices for a given electrical network. Since the set of currents produced by Λ always sums to zero, we may consider Λ a map from $\mathbb{R}^n \to D$, i.e. $\Lambda : \mathbb{R}^n \to D$. If we restrict the domain of Λ to D, then $\Lambda|_D : D \to D$ is an invertible map, and we will denote its inverse by H. Notice that H is not defined on all of $\mathbb{R}^n$, but we may extend its domain to include vectors that correspond to illegal currents (ones disobeying Kirchhoff's Law) where H orthogonally projects such a vector to a legal current vector. This is done by requiring that $H(e) = 0$ and extending H linearly.

Remark 1.4. $\Lambda = (H^2 + E)^{-1}H$

Proof. For the sake of understanding, we will assume $\Lambda = (H^2+E)^{-1}H$ and by biconditional statements deduce an indisputably true statement, thus verifying our assumption, $\Lambda = (H^2+E)^{-1}H$. Note that the argument is valid in reverse order but is not at all intuitive.

$$
\begin{aligned}
& \Lambda = (H^2 + E)^{-1}H \\
\iff & H = (\Lambda^2 + E)^{-1}\Lambda \\
\iff & H = (((H^2 + E)^{-2}H^2 + E)^{-1}(H^2 + E)^{-1}H) \\
\iff & H = ((H^2 + E)^{-2}H^2 + nE)^{-1}H \\
\iff & ((H^2 + E)^{-1}H^2 + nE)H = H \\
\iff & (H^2 + E)^{-1}H^3 = H \\
\iff & H^3 = H(H^2 + E)
\end{aligned}
$$

which is obviously true, thus $\Lambda = (H^2 + E)^{-1}H$. □

2 Properties of Neumann to Dirichlet Map

2.1 Symmetric

Remark 2.1. The Neumann to Dirichlet Map is symmetric.

Proof. Let $H = (\Lambda^2 + E)^{-1}\Lambda$ be the Neumann to Dirichlet Map for some electrical network, $\Gamma = (G, \gamma)$.

Since row and column sums of Λ are equal to zero,

$$
\begin{aligned}
(\Lambda^2 + E)\Lambda &= \Lambda^3 = \Lambda(\Lambda^2 + E) \\
\Lambda(\Lambda^2 + E)^{-1} &= (\Lambda^2 + E)^{-1}\Lambda \qquad (4)
\end{aligned}
$$

Since the inverse of a transpose is the transpose of an inverse, the square of a transpose is the transpose of a square, and E is symmetric,

$$H = (\Lambda^2 + E)^{-1}\Lambda = \Lambda(\Lambda^2 + E)^{-1} = H^T$$

□

2.2 Positive Semi-Definite

Remark 2.2. The Neumann to Dirichlet Map is positive semi-definite.

Proof. We know that Λ, the response matrix for any given electrical network, $\Gamma = (G, \gamma)$, is positive semi-definite. Therefore,

$$\begin{aligned}\psi^T H \psi &= x^T \psi \\ &= x^T \Lambda x \geq 0\end{aligned}$$

Also notice that $\alpha\epsilon \in \ker(H)$ where α is an arbitrary constant. □

2.3 Row and Column Sums Zero

Remark 2.3. The Neumann to Dirichlet Map has row and column sums equal to zero.

Proof. Since $\Lambda\epsilon = 0$, then $(\Lambda^2 + E)^{-1}\Lambda\epsilon = H\epsilon = 0$, and by symmetry column sums are also equal to zero. □

2.4 Diagnal Entries Positive

Remark 2.4. The diagnal entries of the Neumann to Dirichlet Map are positive.

Proof. Let H_i be the ith column of H.

$$H_i = He_i$$

where e_i is the ith unit basis vector (for our purposes we will need it to be a column vector). Then,

$$He_i = H(e_i - \alpha e)$$

In order to have a legal current (one obeying Kirchhoff's Law), $1 - \alpha n = 0$, where n is equal to the number of boundary vertices which implies that $\alpha = 1/n$. Furthermore,

$$(e_i - e/n)_j = \begin{cases} (n-1)/n & \text{if } i = j \\ -1/n & \text{if } i \neq j \end{cases}$$

If the maximum electric potential were to occur at a vertex other than i, then the current from this vertex would be positive into the rest of the graph, a contradiction. Therefore, the maximum electric potential must occur at the ith vertex. Due to our requirement that the electric potentials on the boundary sum to zero, (2), the electric potential at the ith vertex must be positive. □

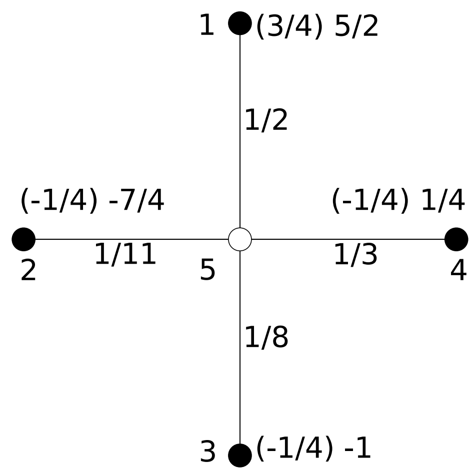

Figure 1: Electrical network of a $\bigstar_4$ graph with vertex labels to the left of/below the vertices (boundary vertices are solid black while the interior vertex is white), current in parentheses, electric potential to the right of current, and conductivities to the left of/below the edges connecting the vertices.

2.5 Not Necessarily Kirchhoff

Remark 2.5. The Neumann to Dirichlet Map is not necessarily a Kirchhoff matrix.

Proof. If the Neumann to Dirichlet Map, H, were a Kirchhoff matrix, then

$$H_{ij} > 0 \text{ for } i = j$$
$$H_{ij} \leq 0 \text{ for } i \neq j$$

Note that extending the previous arguement to show that the off-diagonal entries of the Neumann to Dirichlet Map are negative is impossible. Consider the following counterexample.

In Figure 1 the currents, denoted in parentheses, are consistent with the previously prescribed method while the electric potentials do not observe the required sign conventions of a Kirchhoff matrix i.e. the electric potential on vertex four is positive. Thus, in general, H is not a Kirchhoff matrix. $\square$

3 Entry Relationships

3.1 Relationships on the Four Star Network

By the use of mixed boundary data problems certain relationships among entries of H, the Neumann to Dirichlet Map, have been discovered for star graphs (specifically for $\bigstar_4$, although generalization to $\bigstar_n$ is trivial).

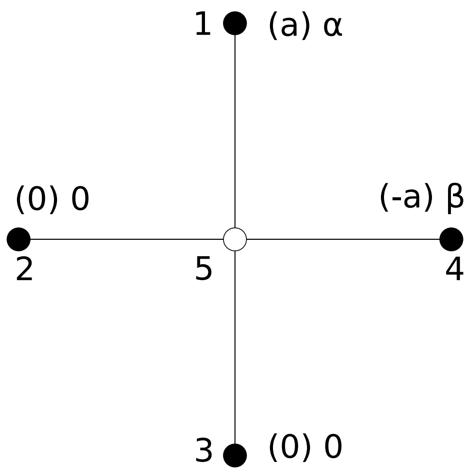

Figure 2: Electrical network of a $\bigstar_4$ graph with the same labeling conventions as Figure 1, except conductivities are absent for purposes of their arbitrariness.

Let H be the Neumann to Dirichlet Map for the electrical network represented in Figure 2. Notice that prescribing a current of a to vertex one, an electric potential and current of zero to vertex two, and a current of zero to vertex three fully determines the remaining data (trivial scaling will be necessary to coincide with our requirement that the sum of the electric potential on the boundary vertices should be zero, (2)). Now we have that

$$H \begin{bmatrix} a \\ 0 \\ 0 \\ -a \end{bmatrix} = \begin{bmatrix} * \\ b \\ b \\ * \end{bmatrix} \tag{5}$$

where b is the electric potential at vertices two and three due to scaling. Thus (5) has a non-trivial solution ($a \neq 0$) i.e. $a\eta_{21} - a\eta_{24} = a\eta_{31} - a\eta_{34}$ has a non-trivial solution ($a \neq 0$). Therefore by dividing through by a and rearranging, $\eta_{21} + \eta_{34} = \eta_{31} + \eta_{24}$, a relationship in H. This argument and resulting relationship is simply generalizable to any of the vertices.

3.2 Generalization of Relationships

Definition 3.1. There exists a *generalized path*, $p \leftrightarrow q$, between two boundary vertices p and q if and only if there exists a sequence of vertices in G whose edges join p to q.

Definition 3.2. If $P = (p_1, p_2, ..., p_k)$ and $Q = (q_1, q_2, ..., q_k)$ are sets of boundary vertices, then there exist a *generalized k-connection* from P to Q if and only if there exists a permutation, σ, such that there exist generalized paths $\{p_i \leftrightarrow q_{\sigma(i)}\}$ which are vertex disjoint.

Please notice that most literature refer to paths as being sequences of interior vertices, and subsequently, k-connections are defined in terms of these paths. Note that generalized paths do not distinguish between interior and boundary vertices, and that generalized k-connections are defined in terms of generalized paths.

Theorem 3.3. *If the entry in the Neumann to Dirichlet Map, η_{pq}, is not equal to zero, then there exists a generalized path between p and q, $p \leftrightarrow q$.*

Proof by contrapositive. Assume that there does not exist a generalized path between p and q. That is, the connected component containing p and that containing q are distinct. Observe that η_{pq} reflects the effect of the current at q on the voltage at p. Since p and q lie in distinct connected components, they are effectively independent of one another; it follows that $\eta_{pq} = 0$. □

Conjecture 3.4. *If there does not exists a generalized 2-connection between the sets of boundary vertices, $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then*

$$\eta_{p_1q_1} + \eta_{p_2q_2} - \eta_{p_1q_2} - \eta_{p_2q_1} = 0.$$

Now the pending question is do these relationships, symmetry, and row/column sums equal to zero provide a characterization of the Neumann to Dirichlet Map.

4 Duality

Due to the analogies between the Dirichlet to Neumann Map and the Neumann to Dirichlet Map and a graph and its dual graph, we will explore relationships of duality.

Definition 4.1. The *Adjacidence Tensor* also known as the *Incidacency Matrix* is

In order to ensure the uniqueness of the solution, we will require that

$$e^T x = 0 \tag{6}$$

That is to say that the sum of the electric potential around the boundary is equal to zero. While distinguishing an electric ground might seem more natural in a physical situation and would also ensure uniqueness, the motivation for the specified requirement is to retain symmetry, and this symmetry will be seen to be rather convenient and useful.