Arithmetic of critical orbits and recurrence

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Fix:

- E an elliptic curve over $\mathbb{Q}$,
- $\alpha \in E(K)$ a non-torsion point.

Question: For which natural numbers n does there exist a prime $\mathfrak{p}$ of $\mathcal{O}_K$ such that after reduction modulo $\mathfrak{p}$, α becomes a point of exact order n?

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Exact order: $[n]\alpha \equiv \mathcal{O} \mod \mathfrak{p}$, $[k]\alpha \not\equiv \mathcal{O} \mod \mathfrak{p}$ for all $k < n$.

In coordinates: if E is in Weierstrass form, $[n]\alpha$ coincides with $\mathcal{O}$ if and only if $\mathfrak{p}$ divides the denominator of $x([n]\alpha)$.

Alternative notation

Definition (Primitive prime divisor)

Let $\{a_n\}$ be a sequence of ideals in $\mathcal{O}_K$. We say a prime ideal p of $\mathcal{O}_K$ is a *primitive prime divisor* of a_n if

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Theorem (Silverman)

The Zsigmondy set associated to the sequence of denominators of $x([n]\alpha)$ is finite.

Reduction mod p

Corollary

For all but finitely many n, there exists a prime p of $\mathcal{O}_K$ such that after reduction mod p, α has exact order n.

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More examples of sequences with finite Zsigmondy sets:

1. Bang-Zsigmondy sequences $a^n - b^n$, $a > b > 0$ coprime with $\frac{a}{b}$ not a root of unity. (Bang, Zsigmondy)
2. Fibonacci sequence (Carmichael) and its generalizations
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a_n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
$2^n - 1$	1	3	7	$3 \cdot 5$	31	$3^2 \cdot 7$	127	$3 \cdot 5 \cdot 17$	$7 \cdot 73$	$3 \cdot 11 \cdot 31$	$23 \cdot 89$	$3^2 \cdot 5 \cdot 7 \cdot 13$
F_n	1	1	2	3	5	2^3	13	$3 \cdot 7$	$2 \cdot 17$	$5 \cdot 11$	89	$2^4 \cdot 3^2$
w_n	1	1	1	-1	-2	-3	-1	7	11	$2^2 \cdot 5$	-19	$3 \cdot -29$

These are all examples of *strong divisibility sequences*:

$$\gcd(a_m, a_n) = a_{\gcd(m,n)}.$$

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Key idea: Strong divisibility + rapid growth $\Rightarrow$ finite Zsigmondy set.

Coinciding with periodic points

Dynamical analogue of this elliptic curve example? The identity is fixed point.

Let $f(z) = z^d, d \geq 2$, K a number field. Let $\alpha \in K$ be a point with infinite forward f-orbit, and $\zeta \in \bar{\mathbb{Q}}$ a non-zero f-periodic point.

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This generalizes broadly:

Theorem (Ingram-Silverman 2007)

Let $\phi(z) \in \mathbb{Q}(z)$ with degree $d \geq 2$, $\alpha \in \mathbb{Q}$ a point with infinite forward orbit, and $\gamma \in \mathbb{Q}$ periodic for ϕ. Assume that γ is not totally ramified. Then the numerator sequence associated to $\{\phi^n(\alpha) - \gamma\}$ has finite Zsigmondy set.

Reducing to period n

Broad idea in arithmetic dynamics: torsion points on elliptic curves and preperiodic points of dynamical systems have similar properties.

So given $f(z) \in K(z)$ and $\alpha \in K$ with infinite forward orbit, we ask: for which $n \in \mathbb{N}$ does there exist a prime p of $\mathcal{O}_K$ such that α has exact *period* n after reduction mod p?

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unless... we consider the *critical* orbits of polynomials.

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Theorem (K.)

Let $f(z) = z^d + c, c \in \mathbb{Q}$. Then the Zsigmondy set of $\{f^n(0)\}$ has at most 8 elements, and there is an effectively computable bound depending on c on the maximal element.

A computational question

Fix d, n. Does there exist $c \in \mathbb{Q}$ such that $f(z) = z^d + c$ has n in the Zsigmondy set of the critical orbit?

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Reduces to existence of integer points on a finite number of Thue curves:
$(d, n) = (d, 2) : x + y = \pm 1$ (not really Thue)
$(d, n) = (2, 3) : x^3 + 2x^2y + xy^2 + y^3 = \pm 1$
$(d, n) = (4, 3) : x^3(x^3 + y^3)^4 + y^{15} = \pm 1$
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Remark: $f(z) = z^2 - \frac{7}{4}$

Critical recurrence

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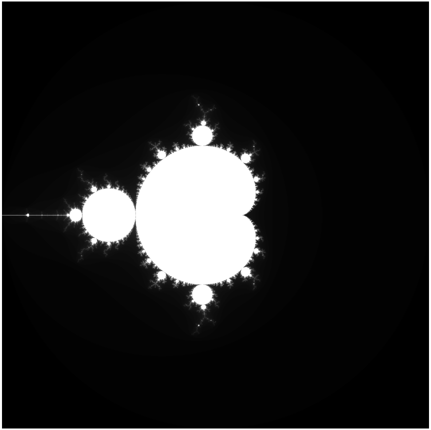
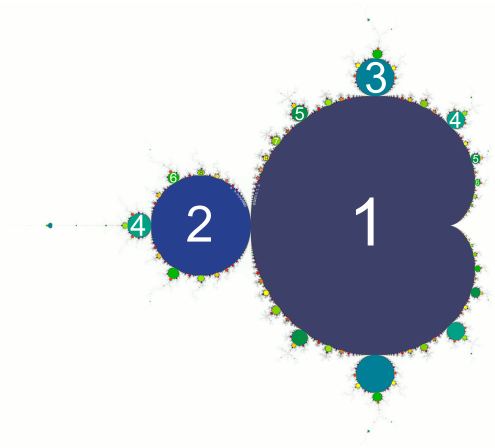


Better results through complex dynamics

Consider the case of $d = 2$. The *hyperbolic components* of the Mandelbrot set are the loci where the critical orbit is in the basin of attraction of some attracting periodic cycle of fixed period n.



On the boundary of the Mandelbrot set

Let $\rho_n := \min\left\{\frac{1}{4}, \frac{1}{2^{2^{n-2}}}\right\}$. Define $D(n)$ to be the set of complex parameters c such that 0 lies in the basin of attraction of a point of period n with multiplier less than ρ_n.

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Note:

- This is the best possible bound, as expected!
- This tells us where to look for possible higher values of n in Zsigmondy sets: when c is a good rational approximation of a center of a hyperbolic component.

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Break: explain this! White board time.

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For $n \leq 8$, the first 22 convergents of the real centers of hyperbolic components with attracting cycles of period n all have $\mathcal{Z}_f \subset \{1, 2\}$, except for $c = -\frac{7}{4}$, with $3 \in \mathcal{Z}_f$.

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Better convergents seem to do worse:

Example: convergents of the center of the $n = 4$ hyperbolic component closest to -2.

c	$\frac{-31}{16}$	$\frac{-33}{17}$	$\frac{-295}{152}$	$\frac{1213}{625}$	$\frac{14851}{7652}$	$\frac{16064}{8277}$	$\frac{1428483}{736028}$	$\frac{5729996}{2952389}$
D	19	19	33	43	27	58	...?	...?

Here D is the number of digits of the largest primitive prime factor!

Existence of powers in orbits

Question: Fix $f(z) \in K(z)$ and $\alpha \in K$ a point of infinite forward orbit. What can we say about the set of indices n which have $f^n(z) - z = y^m$ for some $y \in K$, $m > 1$?

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Not always finite, obviously (e.g. $f(z) = g(z)^2, \alpha = 0$).

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Faltings' theorem says it suffices to bound m.

Diophantine results for polynomials

Theorem (Schinzel, Tijdeman)

Let $f(z) \in \mathbb{Q}[z]$. There exists an effectively computable bound M such that for $m \geq M$,

$$f(x) = y^m$$

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This can be easily extended to S-integers in a number field, so long as y is not a root of unity. S-units in orbits are finite, so this is ok, and with some work we get:

Corollary

The conjecture holds for polynomials.

Connection to recurrence

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Away from primes of bad reduction or primes less than the degree, if p is a prime divisor of $f^n(0) = y^m$, then there exists a point of small norm in $\mathbb{C}_p$ which is p-adically attracting of small multiplier norm.

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Theorem (Benedetto-Ingram-Jones-Levy)

If f is a rational PCF map, the answer is yes, away from a finite set of primes.

The moral of the conjecture

BIJL doesn't help us with the question of powers in critical orbits; why do we expect the stronger question to be true generally?

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Thanks!