AIM 2009 SQuaREs Proposal: Computations with Explicit Reduction Theories

Proposal Summary:

This is a proposal to gather researchers with distinct perspectives on the topic of explicit computational methods for reduction theories and their application to study geometry, cohomology, and modular forms. The purpose of our week-long collaboration is to:

1) Discuss/share current methods for using explicit reduction domains, and interesting open problems.
2) Lay the groundwork for creating a unified computational framework for explicit reduction theory computations in SAGE, and discuss how to combine existing specialized projects to this end.
3) Set concrete goals for future collaborations and software development.

We also intend to meet for a second week at a later date, after the first substantial steps have been taken.

Background:

Reduction theories have played an important role in mathematics, going back to Gauss's enumeration of positive definite binary quadratic forms $ax^2+bxy+cy^2$ over $\mathbf{Z}$ by using the conditions that every $\mathrm{SL}_2(\mathbf{Z})$-orbit has a *"reduced" representative* in the explicit reduction domain $0 < |b| \leq a \leq c$.

Attempts to extend this idea of a reduced representative for $\mathrm{SL}_n(\mathbf{Z})$-orbits of positive definite quadratic forms in $n > 2$ variables have led to many different notions of a reduced representative (Minkowski, Voronoi, Hermite, Korkin-Zolotarev, etc.), all of which agree when $n = 2$. These ideas allow one to prove both abstract and concrete results, though explicit computations become difficult very quickly as n grows. (Examples of these are the finiteness of the class number of forms with given discriminant, and the enumeration of all classes of forms with given discriminant.) Due to the increasing complexity of doing explicit computations, most applications of reduction theory are aimed at establishing "finiteness-type" results analogous to the finiteness of the class number for quadratic forms above. Siegel did this in his construction of Siegel domains for reduction theories of congruence subgroups of $\mathrm{Sp}_{2n}(\mathbf{Z})$, allowing him to show the existence of finite volume fundamental domains for their actions.

In recent years, particularly with the increasing usefulness of computers in mathematics, there has been interest from several directions to really push explicit reduction methods to their limits. Knowledge of explicit reduction domains for various arithmetic groups has applications to areas of mathematics ranging from compactifications of moduli spaces, existence of various Galois representations and cohomological modular forms, lattice theory, and the enumeration of reduced objects up to a given bound. However fulfilling this desire has required various researchers to invest a great deal of effort writing specialized one-time programs to accomplish a particular task. The goal of this proposal is to bring together several of these researchers to help establish a common platform for more easily performing computations of and with explicit reduction theories for specific arithmetic groups, and to help focus attention on the most interesting problems that can be solved with these methods.

Applications:

Compactifications of Moduli spaces – This is described in LNM812 by Namikawa and the book "Smooth compactification of locally symmetric varieties" by Ash, Mumford, Rapoport and Tai. The idea is that one can construct a very useful (but non-canonical) compactification of a locally symmetric space relative to a choice of fundamental domain reduction cone called the *toroidal compactification* which has much nicer boundary components than the more canonical Satake compactification. This can then be used to compute interesting geometric quantities (like intersection numbers and cohomology groups) which have boundary contributions.

Cohomological Modular forms – This application (pioneered by Avner Ash) is described in detail in the appendix by Paul Gunnells of William Stein's book "Modular forms, a computational approach." The idea is that one can compute cusp forms for congruence subgroups Γ of an arithmetic group by looking instead at the group cohomology of Γ with various coefficient systems, and this can be computed by essentially constructing an explicit fundamental domain for the quotient $\Gamma\backslash S$ if the symmetric space S of G, which is done via an explicit reduction theory. (To simplify computations, one actually works with a lower-dimensional retract of this quotient.) The action of Hecke operators can then be computed by an extension of the usual modular symbols algorithm used for

$\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, but now for non-top-dimensional cohomology (called the *Sharbly complex*). Here the reduction algorithm for Sharblies is a very time consuming step of the process, and would benefit from parallelization.

Lattice Theory (Sphere packing and covering) – These applications are described in the books "Sphere packings, Lattices and Groups" by Conway and Sloane and "Perfect Lattices in Euclidean Space" by Jacques Martinet. A *perfect lattice* is a positive definite quadratic lattice which is uniquely determined by its minimal vectors (and their common value). Perfect lattices play a crucial role in actually constructing explicit reduction theories for SL_n because of Voronoi's observation that the reduction domain for positive definite quadratic forms can be described as a polyhedral cone whose facets are indexed by perfect lattices. Voronoi gave two polyhedral reduction theories (resp. based on enumerating perfect forms, and "types" of Delone subdivisions) that can be used to explicitly solve the lattice sphere packing problem (done for $n \leq 8$) and the lattice sphere covering problem (done for $n \leq 5$). A generalization of the first polyhedral reduction theory in the setting of a number field was described by Koecher in 1960.

Enumeration of reduced objects with fixed invariants – Aside from Gauss's enumeration of integral binary forms, knowledge of explicit reduction inequalities have been used by Brandt-Intrau, Townes, and Nipp to make tables of ternary and quaternary positive definite quadratic forms. However even the most recent table (of Nipp in 1991) is quite old by computational standards, and would do well to be checked and extended to the setting of Hermitian forms over a number field.

K-theory –The Sharby method used by Gunnells and Yasaki to compute cohomological modular forms have their origins in the 1978 paper of Lee and Szczarba to compute $K_3(\mathbf{Z})$. Recently, shortly after Rognes had shown that $K_4(\mathbf{Z})$ is trivial (2000), Elbaz-Vincent, Gangl and Soulé computed the Voronoi cell complex for certain modular groups which enabled them as a corollary to prove that $K_5(\mathbf{Z}) = \mathbf{Z}$ and $K_6(\mathbf{Z})$ has only 3-torsion (2002), as well as a recent similar result for $K_7(\mathbf{Z})$ (no p-torsion for $p > 7$). Important progress by Voevodsky and Rost, still unpublished, would imply the same (in fact many more) results from a completely different angle but their methods do not apply for $K_{4n}(\mathbf{Z})$. Hence it would be desirable to push the Voronoi method further for the case of $K_8(\mathbf{Z})$ which incidentally has implications for the famous Kummer–Vandiver Conjecture. This would require extensive computational work on reduction theories which could also be used to compute $K_n(O_F)$ for small n, where O_F is the ring of integers of a small degree number field F.

Computing special values of zeta functions – It is a very useful computational fact that the special values of the Riemann zeta function $\zeta(s)$ and its quadratic twists can be described explicitly in terms of Bernoulli numbers. For arithmetic applications it would be very desirable to be able to find special values of more general Dedekind zeta functions $\zeta_F(s)$ in a similar way. This can be done via a more general formula for the special values of Shintani zeta functions, which are defined relative to a product of linear forms and a rational polyhedral cone, which could be computed explicitly given the reduction domain for the action of the units of F on the product of all archimedean completions of F. This can be thought of concretely as giving a reduction theory of integral quadratic forms in one variable over F.

Researchers:

Dan Yasaki – was a student of Leslie Saper at Duke University whose thesis work was on spines for $\mathbf{Q}$-rank 1 groups. He has written several papers on computing the cohomology of the Picard modular group (over $\mathbf{Z}[i]$) and has also worked with Paul Gunnells to extend these techniques to locally symmetric spaces attached to GL_2 over real quadratic and totally complex quartic fields. He has written specialized software to use these ideas to compute the action of Hecke operators on these cohomology groups with the goal of explicitly exhibiting new cohomological modular forms. He is currently a tenure-track assistant professor at UNC Greensboro.

Paul Gunnells – is an expert in using the Sharbly method to compute interesting cohomological modular forms for higher rank groups, and has worked with Ash and McConnell to compute the cohomology of subgroups of $\mathrm{SL}_4(\mathbf{Z})$. He has written many papers on various aspects of computing cohomology of arithmetic groups other than $\mathrm{SL}_2(\mathbf{Z})$ and is particularly interested in using explicit (toroidal) compactifications and reduction theories to compute the action of Hecke operators on cohomology. He is an associate professor at the University of Massachusetts at Amherst.

Achill Schürmann – is an expert in explicit computations with several different reduction theories for positive definite quadratic forms (see his new AMS book "Computational Geometry of Positive Definite Quadratic Forms – Polyhedral Reduction Theories, Algorithms, and Applications"). Together with Dutour-Sikirić and Vallentin, he designed specialized C++ software (e.g. the "secondary cone cruiser") to compute Voronoi reduction domains which was used to classify all perfect lattices in dimensions $n \leq 8$, which were used to find the best known covering lattices in many cases. He is currently an assistant professor at the Delft University of Technology in the Netherlands.

Mathieu Dutour-Sikirić – Together with Schürmann and Vallentin, he designed specialized C++ software (e.g. the "secondary cone cruiser") to compute Voronoi reduction domains which was used to classify all perfect lattices in dimensions $n \leq 8$, which were used to find the best known covering lattices in many cases. He has also authored the GAP package "polyhedral", which performs many useful polyhedral cone related computations (e.g. computations of dual descriptions, face-lattices, automorphism groups, volumes, Wythoff constructions and group resolutions). He is currently a Researcher of Mathematics at Institut Rudjer Boskovic in Zagreb.

Jonathan Hanke – is interested in explicit computations with quadratic forms and with automorphic forms on algebraic groups. He has experience developing software in C++ and in the freely available open-source SAGE computer algebra system. He has developed algorithms and code to compute the exact numbers represented by a positive definite quadratic form in $n \geq 4$ variables, and has worked with Bhargava to prove Conway's 290-Conjecture. He is currently a tenure-track assistant professor at the University of Georgia at Athens (in the US).

Herbert Gangl – has worked with Soulé and Elbaz-Vincent to find the homology ranks of certain Voronoi complexes, which computes $K_5(\mathbf{Z})$ and $K_6(\mathbf{Z})$ as a consequence. He is interested in using computational methods to understand questions related to polylogarithms and K-theory, especially the homology of the general linear group, configuration and moduli spaces, combinatorial Hopf algebras, multiple zeta values or (quasi-)modular forms and has written code in Pari/GP to explore these topics. He is currently a lecturer at Durham University in the UK.