Counting points on (more general) curves

Jan Tuitman, KU Leuven

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Zeta functions

Suppose that

- $\mathbf{F}_q$ finite field of cardinality $q = p^n$.
- $X/\mathbf{F}_q$ a smooth proper algebraic curve of genus g.

Recall that the zeta function of X is defined as

$$Z(X, T) = \sum_{i=1}^{\infty} |X(\mathbf{F}_{q^i})| \frac{T^i}{i}.$$

It follows from the Weil conjectures that $Z(X, T)$ is of the form

$$\frac{\chi(T)}{(1-T)(1-qT)},$$

where $\chi(T) \in \mathbf{Z}[T]$ of degree $2g$, with inverse roots that

- have absolute value $q^{\frac{1}{2}}$
- are permuted by the map $x \to 1/(qx)$.

Computing zeta functions

Problem

How to compute $Z(X, T)$ (efficiently)?

Note that this problem has cryptographic applications when X is a (hyper)elliptic curve.

Theorem

Let F_p denote the pth power Frobenius map and $H^*_{rig}(X)$ the rigid cohomology. Then

$$\chi(T) = \det(1 - T\,\mathrm{F}_p^n \,|\, H^1_{rig}(X)).$$

Hyperelliptic curves

A hyperelliptic curve X is given by an (affine) equation of the form

$$y^2 = Q(x),$$

with $Q \in \mathbf{F}_q[x]$ a monic polynomial of degree $2g+1$ with $\gcd(Q, Q') = 1$.

To define $H^1_{\mathrm{rig}}(X)$, we start by lifting Q to characteristic 0:

Let $\mathcal{Q} \in \mathbf{Z}_q[x]$ denote a monic lift of Q of degree $2g+1$.

Some rings

We define a ring $\mathbf{Z}_q\langle x, y, y^{-1}\rangle^\dagger$ of overconvergent functions:

$$\{\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty} a_{i,j}x^i y^j | a_{i,j} \in \mathbf{Z}_q, \exists \rho > 1 : \lim_{i+|j|\to\infty} |a_{i,j}|\rho^i = 0\}.$$

Moreover, we denote

$$\mathcal{R} = \mathbf{Z}_q[x, y, y^{-1}]/(\mathcal{Q}), \quad \mathcal{R}^\dagger = \mathbf{Z}_q\langle x, y, y^{-1}\rangle^\dagger/(\mathcal{Q})$$

$$\mathcal{U} = \operatorname{Spec}\mathcal{R}, \qquad \mathbb{U} = \mathcal{U} \otimes \mathbf{Q}_q, \qquad U = \mathcal{U} \otimes \mathbf{F}_q.$$

Rigid cohomology

We define the overconvergent Kähler differentials

$$\Omega^1_{\mathcal{R}^\dagger} = \frac{R^\dagger dx \oplus R^\dagger dy}{(2ydy - \mathcal{Q}'dx)}$$

and the overconvergent De Rham complex

$$\Omega^\bullet_{\mathcal{R}^\dagger} : \quad 0 \longrightarrow \mathcal{R}^\dagger \xrightarrow{d} \Omega_{\mathcal{R}^\dagger} \longrightarrow 0.$$

We then have

$$H^1_{\text{rig}}(U) = H^1(\Omega^\bullet_{\mathcal{R}^\dagger} \otimes \mathbf{Q}_q) = \text{coker}(d) \otimes \mathbf{Q}_q.$$

Frobenius lift

The pth power Frobenius map on $\mathcal{R} \otimes \mathbf{F}_q$ can be lifted to $\mathcal{R}$.

If $\sigma \in \mathrm{Gal}(\mathbf{Q}_q/\mathbf{Q}_p)$ denotes the unique lift of the pth power Frobenius map on $\mathbf{F}_q$, then

$$F_p(y)^2 = \mathcal{Q}^\sigma(F_p(x)).$$

So we define

$$F_p(x) = x^p,$$
$$F_p(y) = \mathcal{Q}^\sigma(x^p)^{\frac{1}{2}} = y^p\left(1 + \frac{\mathcal{Q}^\sigma(x^p) - \mathcal{Q}(x)^p}{y^{2p}}\right)^{\frac{1}{2}}.$$

The square root can be computed efficiently by Hensel lifting.

Computing in the cohomology

We can write any 1-form $\omega \in \Omega_{\mathcal{R}^\dagger}$ as

$$\sum_{i=-\infty}^{\infty} \frac{a_i(x)}{y^i} dx,$$

with $a_i \in \mathbf{Z}_q[x]$ of degree $< 2g+1$ for all $i \in \mathbf{Z}$. Writing $B(x) = A_1(x)Q(x) + A_2(x)Q'(x)$, we have

$$B(x)\frac{dx}{y^i} \equiv \left(A_1(x) + \frac{2A_2'(x)}{(i-2)}\right)\frac{dx}{y^{i-2}}.$$

This allows us to eliminate all terms with $i > 2$. We can do something similar for the terms with $i \leq 0$.

A basis for the cohomology

As a consequence, one can show that:

Theorem

A basis for $H^1_{rig}(U)$ *is given by*

$$[x^0 \frac{dx}{y}, \ldots, x^{2g-1} \frac{dx}{y}, x^0 \frac{dx}{y^2}, \ldots, x^{2g} \frac{dx}{y^2}]$$

and the first $2g$ *vectors form a basis for the subspace* $H^1_{rig}(X)$.

Kedlaya's algorithm

Roughly:

- Compute $F_p(\frac{1}{y})$ and $F_p(x^i \frac{dx}{y}) = px^{ip+p-1} F_p(\frac{1}{y})dx$.
- Reduce back to the basis $[x^0 \frac{dx}{y}, \ldots, x^{2g-1} \frac{dx}{y}]$ and read off the matrix $\mathfrak{F}$ of F_p on $H^1_{\mathrm{rig}}(X)$.
- Compute the matrix $\mathfrak{F}^{(n)} = \mathfrak{F}^{\sigma^{n-1}} \ldots \mathfrak{F}^{\sigma} \mathfrak{F}$ of F_p^n on $H^1_{\mathrm{rig}}(X)$.
- Determine $\chi(T) = \det(1 - \mathrm{F}_p^n \, T | H^1_{\mathrm{rig}}(X))$.

The polynomial $\chi(T) = \sum_{i=0}^{2g} \chi_i T^i \in \mathbf{Z}[T]$ is determined exactly if known to high enough p-adic precision, since there are explicit bounds for the size of its coefficients.

More general curves

We let $X/\mathbf{F}_q$ denote the smooth projective curve given by the (affine) equation

$$Q(x, y) = y^d + Q_{d-1}(x)y^{d-1} + \ldots + Q_0 = 0,$$

where $Q(x, y)$ is irreducible and $Q_i(x) \in \mathbf{F}_q[x]$ for all i.

We let $\mathcal{Q} \in \mathbf{Z}_q[x]$ denote a lift of Q that is monic of degree d in y.

Assumption

The zero locus of $\mathcal{Q}$ in $\mathbf{A}^2_{\mathbf{Z}_q}$ is smooth over $\mathbf{Z}_q$.

Proposition

The $\mathbf{Z}_q[x]$-module $\mathbf{Z}_q[x,y]/(\mathcal{Q})$ is free with basis $[1, y, \ldots, y^{d-1}]$.

Proposition

There exists $r(x) \neq 0 \in \mathbf{Z}_q[x]$, squarefree in $\mathbf{Q}_q[x]$, such that the element $s = r/\frac{\partial \mathcal{Q}}{\partial y}$ of $\mathbf{Q}_q(x,y)$ is contained in $\mathbf{Z}_q[x,y]/(\mathcal{Q})$.

r can be taken to divide the resultant Δ in y of $\mathcal{Q}$ and $\frac{\partial \mathcal{Q}}{\partial y}$, hence can be easily computed. We denote:

$$\mathcal{S} = \mathbf{Z}_q[x, \frac{1}{r}], \qquad \mathcal{R} = \mathbf{Z}_q[x, \frac{1}{r}, y]/(\mathcal{Q}),$$

$$\mathcal{S}^\dagger = \mathbf{Z}_q\langle x, \frac{1}{r}\rangle^\dagger, \qquad \mathcal{R}^\dagger = \mathbf{Z}_q\langle x, \frac{1}{r}, y\rangle^\dagger/(\mathcal{Q}).$$

So we invert r instead of y.

Frobenius lift

Define sequences $(\alpha_i)_{i\geq 0}$, $(\beta_i)_{i\geq 0}$, with $\alpha_i \in S^\dagger$ and $\beta_i \in \mathcal{R}^\dagger$, by the following recursion:

$$\alpha_0 = \frac{1}{r^p},$$
$$\beta_0 = y^p,$$
$$\alpha_{i+1} = \alpha_i(2 - \alpha_i r^\sigma(x^p)) \pmod{p^{2^{i+1}}},$$
$$\beta_{i+1} = \beta_i - \mathcal{Q}^\sigma(x^p, b_i)s^\sigma(x^p, \beta_i)\alpha_i \pmod{p^{2^{i+1}}}.$$

Then one easily checks that the σ-semilinear ringhomomorphism $\mathrm{F}_p : \mathcal{R}^\dagger \to \mathcal{R}^\dagger$ defined by

$$\mathrm{F}_p(x) = x^p, \qquad \mathrm{F}_p(\frac{1}{r}) = \lim_{i\to\infty} \alpha_i, \qquad \mathrm{F}_p(y) = \lim_{i\to\infty} \beta_i,$$

is a Frobenius lift.

The connection matrix

We take

$$\mathcal{U} = \operatorname{Spec} \mathcal{R}, \quad \mathbb{U} = \mathcal{U} \otimes \mathbf{Q}_q, \quad U = \mathcal{U} \otimes \mathbf{F}_q,$$
$$\mathcal{X}/\mathbf{Z}_q \text{ a smooth proper curve containing } \mathcal{U},$$

and let $M \in M_{d \times d}(\mathbf{Z}_q[x])$ denote the matrix for which

$$d(y^j) = jy^{j-1}dy = -jy^{j-1}\frac{s}{r}\frac{\partial Q}{\partial x}dx = \sum_{i=0}^{d-1}\left(\frac{M_{ij}}{r}\right)y^i dx,$$

for all $0 \leq j \leq d-1$ as 1-forms on $\mathcal{U}$.

Some more assumptions

It is necessary to assume:

Assumption

- *The zero locus of r on $\mathbf{A}^1_{\mathbf{Z}_q}$ is smooth over $\mathbf{Z}_q$ and does not contain 0.*
- *The zero locus of r on $\mathbf{A}^2_{\mathbf{Z}_q} \cap \mathcal{X}$ is smooth over $\mathbf{Z}_q$.*
- *The ramification indices e_P at all points $P \in \mathcal{X} \setminus \mathcal{U}$ are not divisible by p.*

To simplify the exposition, we also assume:

Assumption

$\deg(M) < \deg(r)$.

Effective convergence bounds

Proposition

Let $N \in \mathbf{N}$. Then modulo p^N:

1. $\mathrm{F}_p(1/r)$ *is congruent to* $\sum_{i=p}^{pN} \frac{\rho_i(x)}{r^i}$, *where for all* $p \leq i \leq pN$ *the polynomial* $\rho_i \in \mathbf{Z}_q[x]$ *satisfies* $\deg(\rho_i) < \deg(r)$.
2. $\mathrm{F}_p(y^i)$ *is congruent to* $\sum_{j=0}^{d-1} \phi_{i,j}(x) y^j$, *where*

$$\phi_{i,j} = \sum_{k=0}^{p(N-1)+1} \frac{\phi_{i,j,k}(x)}{r^k},$$

for all $0 \leq i, j \leq d-1$ *and* $\phi_{i,j,k} \in \mathbf{Z}_q[x]$ *satisfies* $\deg(\phi_{i,j,0}) < p(d-1)(-\mu)$ *and* $\deg(\phi_{i,j,k}) < \deg(r)$, *for all* $0 \leq i, j \leq d-1$ *and* $1 \leq k \leq p(N-1)+1$.

Computing in the cohomology I

Proposition

For all $\ell \in \mathbf{N}$ *and every vector* $u \in \mathbf{Q}_q[x]^{\oplus d}$*, there exist (unique) vectors* $v, w \in \mathbf{Q}_q[x]^{\oplus d}$ *with* $\deg(v) < \deg(r)$*, such that*

$$\frac{\sum_{i=0}^{d-1} u_i y^i}{r^\ell} \frac{dx}{r} = d\left(\frac{\sum_{i=0}^{d-1} v_i y^i}{r^\ell}\right) + \frac{\sum_{i=0}^{d-1} w_i y^i}{r^{\ell-1}} \frac{dx}{r}$$

as 1*-forms on* $\mathbb{U}$*.*

Sketch of the proof: r is separable, so r' is invertible in $\mathbf{Q}_q[x]/(r)$. v has to satisfy $\left(\frac{M}{r'} - \ell I\right) v \equiv \frac{u}{r'} \pmod{r}$ over $\mathbf{Q}_q[x]/(r)$. We show that the finite exponents of $(M/r)dx$ are contained in $[0, 1)$, hence $\det(\ell I - M/r')$ is invertible in $\mathbf{Q}_q[x]/(r)$, so there is a unique solution v.

Computing in the cohomology II

Proposition

For every vector $u \in \mathbf{Q}_q[x]^{\oplus d}$ *with* $\deg(u) \geq \deg(r)$, *there exist vectors* $v, w \in \mathbf{Q}_q[x]^{\oplus d}$ *with* $\deg(w) < \deg(u)$, *such that*

$$(\sum_{i=0}^{d-1} u_i y^i)\frac{dx}{r} = d(\sum_{i=0}^{d-1} v_i y^i) + (\sum_{i=0}^{d-1} w_i y^i)\frac{dx}{r}$$

as 1*-forms on* $\mathbb{U}$.

Sketch of the proof: We denote $t = 1/x$. Since $\deg(M) < \deg(r)$, we can expand $\frac{M}{r}\,dx = \left(\frac{M_{-1}}{t} + M_0 + \dots\right)dt$, where $M_i \in M_{d\times d}(\mathbf{Q}_q)$ for all i. Similarly, if $k = \deg(u) - \deg(r) + 2$, then we can write $(\sum_{i=0}^{d-1} u_i y^i)\frac{dx}{r} = \left(\frac{b_{-k}}{t^k} + \frac{b_{-(k-1)}}{t^{k-1}} + \dots\right)dt$, where $b_i \in (\mathbf{Q}_q)^{\oplus d}$ for all i. We show that the infinite exponents of $(M/r)dx$ are ≤ 0, so the linear system $(M_{-1} - (k-1)I)c = b_{-k}$ has a unique solution $c \in (\mathbf{Q}_q)^{\oplus d}$. We now take $v = cx^{k-1}$ and $w = u - (Mv + r\frac{dv}{dx})$.

Precision loss I

Proposition

Let $\omega \in \Omega^1_{\mathcal{U}}$ *be of the form*

$$\omega = \frac{\sum_{i=0}^{d-1} w_i(x) y^i}{r^\ell} \frac{dx}{r},$$

where $\ell \in \mathbf{N}$ *and* $w_i \in \mathbf{Z}_q[x]$ *satisfies* $\deg(w_i) < \deg(r)$ *for all* $0 \leq i \leq d-1$. *We define* $e = \max\{e_P | P \in \mathcal{X} \setminus \mathcal{U}, x(P) \neq \infty\}$. *If we represent the class of* ω *in* $H^1_{rig}(U)$ *as in the Theorem using the Proposition, then*

$$p^{\lfloor \log_p(\ell e) \rfloor} u_i(x) \in \mathbf{Z}_q[x]$$

for all $0 \leq i \leq d-1$.

Precision loss II

Proposition

Let $\omega \in \Omega^1_{\mathcal{U}}$ *be of the form*

$$\omega = (\sum_{i=0}^{d-1} w_i(x) y^i) \frac{dx}{r},$$

where $w_i \in \mathbf{Z}_q[x]$ *for all* $0 \leq i \leq d-1$ *and* $\deg(w_i) \geq \deg(r)$ *for some* $0 \leq i \leq d-1$. *We define* $m = (\deg(w) - \deg(r) + 1)$. *If we represent the class of* ω *in* $H^1_{rig}(U)$ *as in the Theorem using the Proposition, then*

$$p^{\lfloor \log_p(m) \rfloor} u_i(x) \in \mathbf{Z}_q[x]$$

for all $0 \leq i \leq d-1$.

A basis for the cohomology

First, let E denote the $\mathbf{Q}_q$-vector space of 1-forms

$$\omega = (\sum_{i=0}^{d-1} u_i(x)y^i)\frac{dx}{r},$$

where $u_i \in \mathbf{Q}_q[x]$ satisfies $\deg(u_i) < \deg(r)$ for all $0 \leq i \leq d-1$. Now, let E_1 denote the kernel of the map that sends $\omega \in E$ to the element $\frac{\partial Q}{\partial y}\sum_{i=0}^{d-1} u_i y^i$ of $\mathbf{Q}_q[x,y]/(\mathcal{Q}, r)$. Finally, let E_2 denote the subspace of E_1 generated by the elements $d(y^i)$ for all $0 \leq i \leq d-1$.

Theorem

We have isomorphisms:

$$H^1_{rig}(U) \cong E/E_2, \qquad H^1_{rig}(X - x^{-1}(\infty)) \cong E_1/E_2.$$

Final remarks

- This allows us to compute $Z(X - x^{-1}(\infty), T)$, from which $Z(X, T)$ can be easily obtained.
- The assumption $\deg(M) < \deg(r)$ can be removed by temporarily using a basis for the $\mathbf{Z}_q[x]$-module $\mathbf{Z}_q[x, y]/(\mathcal{Q})$ with respect to which it is satisfied, when carrying out the reductions at ∞ in the cohomology.
- The assumption that $\mathcal{Q} = 0$ is smooth over $\mathbf{Z}_q$ can probably be removed if we know a basis for the integral closure of $\mathbf{Z}_q[x]$ in the function field $\mathbf{Q}_q(x, y)$.
- The way we compute in the cohomology is taken from work of Lauder (and his student Walker) on the fibration method.

  Alan G.B. Lauder, "A recursive method for computing zeta functions of varieties"